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Combinatorics of generalized q-Euler numbers

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ABSTRACT

New enumerating functions for the Euler numbers are considered. Several of the relevant generating functions appear in connection to entries in Ramanujan's Lost Notebook. The results presented here are, in part, a response to a conjecture made by M.E.H. Ismail and C. Zhang about the symmetry of polynomials in Ramanujan's expansion for a generalization of the Rogers–Ramanujan series. Related generating functions appear in the work of H. Prodinger and L.L. Cristea in their study of geometrically distributed random variables. An elementary combinatorial interpretation for each of these enumerating functions is given in terms of a related set of statistics.

1. Introduction

The Euler numbers E_n are the integers defined by

$$\sum_{n=0}^{\infty} \frac{E_n x^n}{n!} = \sec x + \tan x. \tag{1.1}$$

In 1879, D. André [1,2] gave a combinatorial interpretation for the Euler numbers E_n. These numbers count the number of permutations $\pi = \pi_1\pi_2\cdots\pi_n$ of elements in the set $[n] := \{1, 2, \ldots, n\}$ such that the sign of $\pi_i - \pi_{i+1}$ equals $(-1)^i$, $1 \leqslant i < n$. Such permutations are called alternating or up-down permutations. Alternating permutations have rich combinatorial structure and have been studied extensively over the last century [7–11,13,16,22,30]. Particular emphasis has been placed upon the enumeration of alternating permutations by various weights and conditions.

In this paper, we undertake a combinatorial analysis of several new q-analogues of the Euler numbers. The resulting expressions provide new enumerations for alternating permutations. The asso-

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ciated generating functions are quotients of basic hypergeometric series and arise in several contexts related to the work of S. Ramanujan [17–19,29]. In particular, the generating functions from Section 4 appear in the expansions of Ramanujan's Hadamard product for the generalized Rogers–Ramanujan series from page 57 of his Lost Notebook [25], [4, Chapter 13]:

$$\sum_{n=0}^{\infty} \frac{q^{n^2} z^n}{(q;q)_n} = \prod_{n=1}^{\infty} \left(1 + \frac{zq^{2n-1}}{1 - \sum_{j=1}^{\infty} q^{jn} y_j}\right), \tag{1.2}$$

where

$$y_1 = \frac{1}{(1-q)\psi^2(q)}, \qquad y_2 = 0, \qquad y_3 = \frac{q+q^3}{(q;q)_3 \psi^2(q)} - \frac{\sum_{n=1}^{\infty} \frac{(2n-1)q^{2n-1}}{1-q^{2n-1}}}{(1-q)^3 \psi^6(q)},$$
$$y_4 = y_1 y_3. \tag{1.3}$$

The functions $\psi(q)$ and $(\alpha; q)_n$ appearing in (1.2) and (1.3) are defined by

$$\psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2}, \qquad (\alpha; q)_n := \prod_{j=0}^{n-1} 1 - \alpha q^j, \quad |q| < 1. \tag{1.4}$$

In [20], Ismail and Zhang observed that the polynomials appearing in the expansion (1.2), as the coefficients of $(q;q)_j^{-1} \psi^{-2}(q)$ in y_j, are symmetric about the middle coefficient(s). We explain this symmetry in Sections 4, 5, 7, and 8 by unraveling the combinatorial significance of these polynomials.

The series appearing in this paper arise in an entirely different setting in the work of Prodinger and Cristea [23,24]. These authors employ generating functions to determine the probability that a random word over the infinite alphabet $\{1, 2, 3, \ldots,\}$ satisfies certain inequality conditions. They assume that, within a word, each letter j occurs with (geometric) probability pq^{j-1}, independently, for $0 < q < 1$ and $p = 1 - q$. In Sections 2–3, we derive direct combinatorial interpretations for certain generating functions from [23,24]. Quotients of the series considered in the present paper also have beautiful continued fraction representations [15,23,24].

For nonnegative integers A, B, C, D, consider the following q-analogue of $\tan x$:

$$\sum_{n=0}^{\infty} \frac{f_{2n+1}(q) x^{2n+1}}{(q;q)_{2n+1}} = \frac{\sum_{n=0}^{\infty} \frac{(-1)^n q^{An^2+Bn} x^{2n+1}}{(q;q)_{2n+1}}}{\sum_{n=0}^{\infty} \frac{(-1)^n q^{Cn^2+Dn} x^{2n}}{(q;q)_{2n}}}. \tag{1.5}$$

When $(A, B, C, D) = (0, 0, 0, 0)$, $f_{2n+1}(q)$ is the q-tangent number $T_{2n+1}(q)$ of F.H. Jackson [21]. In [18], Huber proves that the coefficients $T^o_{2n+1}(q)$ of $(q;q)_{2n+1}^{-1} \psi^{-2}(q)$ in y_{2n+1} in (1.2) are $f_{2n+1}(q)$ for $(A, B, C, D) = (1, 1, 1, 0)$. In this paper, we discuss q-tangent numbers corresponding to (A, B, C, D) given in the following table. Let $\tau^{\alpha\beta}_{2n+1}$ represent the probability that a word from $\{1, 2, 3, \ldots\}$ of length $2n + 1$ defined in the preceeding paragraph satisfies the inequality conditions $\alpha\beta$.

f_{2n+1}	(A, B)	(C, D)	Probability
T_{2n+1}	$(0, 0)$	$(0, 0)$	$\tau^{\leqslant >}_{2n+1}, \tau^{\geqslant <}_{2n+1}$
T^o_{2n+1}	$(1, 1)$	$(1, 0)$	$\tau^{<>}_{2n+1}$
T^e_{2n+1}	$(1, 0)$	$(1, 0)$	$\tau^{><}_{2n+1}$

The column on the right contains the numbers considered by Prodinger for which the series (1.5) is an associated generating function [23, Theorem 2.2].

For each value of A, B, C and D, the quotient (1.5) induces a corresponding recursion relation for the function $f_{2n+1}(q)$. From these formulas, we obtain the following related polynomials. For a polynomial $p(q)$, let $\hat{p}(q)$ denote the *dual* of $p(q)$ (see [23, Remark 3.3]). Several of the dual polynomials occur in connection with probabilities from [23].

(A,B)	(C,D)	f_{2n+1}	Relevant relations	Probability
$(0,1)$	$(0,1)$	T^{des}_{2n+1}	$T^{des}_{2n+1}(q)=q^nT_{2n+1}(q)$	–
$(2,1)$	$(2,-1)$	$\hat{T}_{2n+1}$	$\hat{T}_{2n+1}(q)=T_{2n+1}(q)$	–
$(2,0)$	$(2,-2)$	$\hat{T}^{des}_{2n+1}$	$\hat{T}^{des}_{2n+1}(q)=q^{-2n}T^{des}_{2n+1}(q)$	–
$(1,0)$	$(1,-1)$	$\hat{T}^{o}_{2n+1}$	$\hat{T}^{o}_{2n+1}(q)=q^{-n}T^{o}_{2n+1}(q)$	$\tau^{\leqslant\geqslant}_{2n+1}$
$(1,1)$	$(1,-1)$	$\hat{T}^{e}_{2n+1}$	$\hat{T}^{e}_{2n+1}(q)=q^{-n-1}T^{e}_{2n+1}(q)$	$\tau^{\geqslant\leqslant}_{2n+1}$

In Section 2, a well-known arithmetic interpretation of the classical q-tangent numbers $T_{2n+1}(q)$ is discussed. We provide an elementary proof of this interpretation that demonstrates fundamental ideas used throughout the paper. A new q-analogue $T^{des}_{2n+1}(q)$ is also discussed in the same section. In Section 3, we deduce combinatorial interpretations for new q-analogues of the secant numbers appearing in [23,24] defined by

$$\sum_{n=0}^{\infty}\frac{g_{2n}(q)x^{2n}}{(q;q)_{2n}}=\left(\sum_{n=0}^{\infty}\frac{(-1)^nq^{Cn^2+Dn}x^{2n}}{(q;q)_{2n}}\right)^{-1}.$$

(C,D)	g_{2n}	Relevant relations	Probability
$(0,0)$	S_{2n}		$\sigma^{\leqslant>}_{2n}$
$(0,1)$	qS^{des}_{2n}	$S^{des}_{2n}(q)=q^{n-1}S_{2n}(q)$	–
$(1,0)$	qS^{o}_{2n}, S^{e}_{2n}		$\sigma^{<>}_{2n}$
$(2,-1)$	$\hat{S}_{2n}$	$\hat{S}_{2n}(q)=q^{n(2n-1)}S_{2n}(q^{-1})$	$\sigma^{<\geqslant}_{2n}$
$(2,-2)$	$\hat{S}^{des}_{2n}$	$\hat{S}_{2n}(q)=q^{n(2n-1)-1}S^{des}_{2n}(q^{-1})$	–
$(1,-1)$	$\hat{S}^{o}_{2n}$	$\hat{S}^{o}_{2n}(q)=q^{1-n}S^{o}_{2n}(q)$	$\sigma^{\leqslant\geqslant}_{2n}$

The values $\sigma^{\alpha\beta}_{2n}$ denote the probability that a given word of length $2n$, under the aforementioned hypotheses, satisfies the inequality conditions $\alpha\beta$. It should be noted that the generating function for the polynomials $S^{o}_{2n}(q)$ appearing in Section 3 differs by a factor of q^{-1} from the corresponding generating function for $\sigma^{<>}_{2n}$ in [23].

In Sections 4 and 5, we describe combinatorics of the new q-tangent numbers $T^{o}_{2n+1}(q)$ and $T^{e}_{2n+1}(q)$. Our arithmetic interpretations explain the symmetry arising among the coefficients of these polynomials.

In Section 6, we deduce arithmetic interpretations for second-order tangent numbers $T^{(2)}_{2n+1}(q)$, $T^{o\,(2)}_{2n+1}(q)$, $T^{e\,(2)}_{2n+1}(q)$ obtained by squaring the generating functions discussed in the previous sections. We include similar interpretations for second order q-secant numbers. In [17,18], it is shown that the coefficients of $(q;q)^{-1}_{2n+1}\psi^{-4}(q)$ in y_{2n} of (1.2) are scalar multiples of $T^{o\,(2)}_{2n+1}(q)$.

We indicate in Section 7 how closed formulas for the q-Euler numbers may be derived in terms of the Bell polynomials. We also comment on an application of our results to a conjecture made by Ismail and Zhang [20, Conjecture 4.3] concerning a more general class of polynomials appearing in (1.2) within corresponding expansions of y_j, $j\geqslant 1$.

Before proceeding, we introduce some necessary definitions and notation. A pair (π_i,π_j) is called an inversion of the permutation $\pi=\pi_1\pi_2\cdots\pi_n$ if $i<j$ and $\pi_i>\pi_j$. We denote by $inv(\pi)$ the number of inversions of the permutation π. The *descent* set $D(\pi)$ is defined by $\{i \mid \pi_i>\pi_{i+1}\}$, and $des(\pi)$ denotes the size of $D(\pi)$. For a permutation π, we define

$$\pi_o=\pi_1\pi_3\pi_5\cdots \quad\text{and}\quad \pi_e=\pi_0\pi_2\pi_4\cdots,$$

where $\pi_0=\infty$. Two kinds of *half descents* of π are defined by $des(\pi_o)$ and $des(\pi_e)$.

Define the q-binomial coefficient by

$$\begin{bmatrix} n \\ k \end{bmatrix} := \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}, & \text{if } 0 \leqslant n \leqslant k, \\ 0, & \text{otherwise.} \end{cases}$$

The following lemma is one of several combinatorial interpretations for the q-binomial coefficient. We will refer to this lemma often in the remainder of the paper. For a proof of the lemma, see [28, p. 132].

Lemma 1.1.

$$\sum_{\pi} q^{inv(\pi)} = \begin{bmatrix} n \\ k \end{bmatrix},$$

where the sum is over all permutations π with $D(\pi) \subset \{k\}$.

A more instructive view of Lemma 1.1 follows by defining $P_n^{(k)}$, for a given n and $k \leqslant n$, to be the set of all permutations π on $[n]$ such that

$$\pi_1 < \pi_2 < \pi_3 < \cdots < \pi_{k-1} < \pi_k, \qquad \pi_{k+1} < \pi_{k+2} < \cdots < \pi_{n-1} < \pi_n.$$

Then it follows from Lemma 1.1 that

$$\sum_{\pi \in P_n^{(k)}} q^{inv(\pi)} = \begin{bmatrix} n \\ k \end{bmatrix}.$$

We denote by A_n the set of up-down alternating permutations $\pi = \pi_1 \pi_2 \cdots \pi_n$ on the set $[n]$ with $\pi_1 < \pi_2 > \pi_3 < \cdots$, and we denote by $\bar{A}_n$ the set of down-up alternating permutations π on the set $[n]$ with $\pi_1 > \pi_2 < \pi_3 > \cdots$.

2. The classical q-Euler numbers

Jackson's q-analogues of the sine and cosine functions [21] are

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(q;q)_{2n+1}/(1-q)^{2n+1}} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(q;q)_{2n}/(1-q)^{2n}}.$$

By considering quotients of these functions, we arrive at a q-analogue of the tangent numbers, $T_{2n+1}(q)$, defined by

$$\sum_{n=0}^{\infty} \frac{T_{2n+1}(q) x^{2n+1}}{(q;q)_{2n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(q;q)_{2n+1}} \Bigg/ \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(q;q)_{2n}}. \tag{2.1}$$

If we replace x by $x(1-q)$ in (2.1) and let $q \to 1^-$, the corresponding identity reduces to the relation obtained by equating odd parts on each side of Eq. (1.1). Multiplying both sides of (2.1) by the denominator on the right side and equating coefficients of x, we obtain

$$T_{2n+1}(q) = \sum_{k=1}^{n} \begin{bmatrix} 2n+1 \\ 2k \end{bmatrix} (-1)^{k-1} T_{2(n-k)+1}(q) + (-1)^n. \tag{2.2}$$

The following combinatorial interpretation of the polynomials $T_{2n+1}(q)$ is well known [6,14,27,28]. We include a proof based upon the recursion (2.2) as an aid to the reader, since later results in the paper have proofs that are similar in nature.

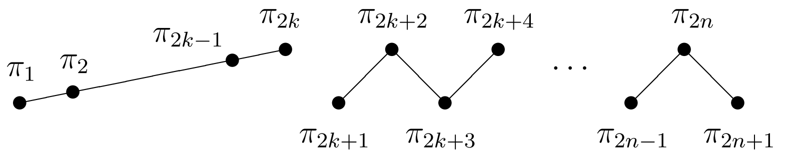

Fig. 1. $\mathcal{A}_{2n+1}^{(k)}$.

Theorem 2.1. *For a nonnegative integer n, we have*

$$T_{2n+1}(q) = \sum_{\pi \in A_{2n+1}} q^{inv(\pi)}.$$

Proof. Let

$$f_{2n+1}(q) = \sum_{\pi \in A_{2n+1}} q^{inv(\pi)}.$$

For $n = 0$, it is clear that

$$f_1(q) = 1 = T_1(q).$$

For any positive integer n, we will prove that $f_{2n+1}(q)$ satisfies the recurrence (2.2). For a positive integer $k \leqslant n$, let $\mathcal{A}_{2n+1}^{(k)}$ be the set of permutations π on $[2n+1]$ such that

$$\pi_1 < \pi_2 < \cdots < \pi_{2k-1} < \pi_{2k},$$

$$\pi_{2k+1} < \pi_{2k+2} > \pi_{2k+3} < \cdots < \pi_{2n} > \pi_{2n+1}.$$

Fig. 1 shows the conditions for $\pi \in \mathcal{A}_{2n+1}^{(k)}$.

From Lemma 1.1 and the definition of $f_{2n+1}(q)$, we see that

$$\sum_{\pi \in \mathcal{A}_{2n+1}^{(k)}} q^{inv(\pi)} = \begin{bmatrix} 2n+1 \\ 2k \end{bmatrix} f_{2(n-k)+1}(q). \tag{2.3}$$

For a positive integer $k \leqslant n$, we denote by $\mathcal{B}_{2n+1}^{(k)}$ the set of permutations π on $[2n+1]$ such that

$$\pi_1 < \pi_2 < \pi_3 < \cdots < \pi_{2k} > \pi_{2k+1} < \pi_{2k+2} > \pi_{2k+3} < \cdots < \pi_{2n} > \pi_{2n+1}.$$

Fig. 2 shows the conditions for $\pi \in \mathcal{B}_{2n+1}^{(k)}$.

We now compute the generating function for permutations $\pi \in \mathcal{B}_{2n+1}^{(k)}$

$$\sum_{\pi \in \mathcal{B}_{2n+1}^{(k)}} q^{inv(\pi)}.$$

From the definitions of $\mathcal{A}_{2n+1}^{(k)}$ and $\mathcal{B}_{2n+1}^{(k)}$, we see that for any k, $1 \leqslant k \leqslant n$,

$$\mathcal{A}_{2n+1}^{(k)} = \mathcal{B}_{2n+1}^{(k)} \cup \mathcal{B}_{2n+1}^{(k+1)},$$

where $\mathcal{B}_{2n+1}^{(n+1)} = \{\pi \mid \pi_1 < \pi_2 < \cdots < \pi_{2n} < \pi_{2n+1}\}$. Thus

$$\sum_{\pi \in \mathcal{B}_{2n+1}^{(k)}} q^{inv(\pi)} = \sum_{\pi \in \mathcal{A}_{2n+1}^{(k)}} q^{inv(\pi)} - \sum_{\pi \in \mathcal{B}_{2n+1}^{(k+1)}} q^{inv(\pi)}. \tag{2.4}$$

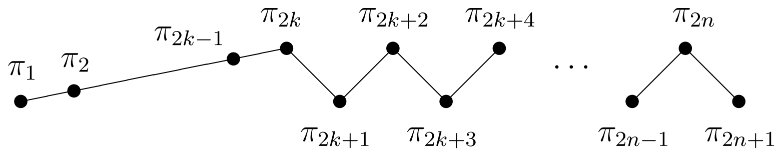

Fig. 2. $\mathcal{B}_{2n+1}^{(k)}$.

By iterating (2.4), we deduce

$$\sum_{\pi\in\mathcal{B}_{2n+1}^{(1)}} q^{inv(\pi)} = \sum_{k=1}^{n}(-1)^{k-1}\sum_{\pi\in\mathcal{A}_{2n+1}^{(k)}} q^{inv(\pi)} + (-1)^n \sum_{\pi\in\mathcal{B}_{2n+1}^{(n+1)}} q^{inv(\pi)}. \tag{2.5}$$

Note that $\mathcal{B}_{2n+1}^{(1)} = A_{2n+1}$. Therefore, it follows from (2.3) that (2.5) is equivalent to

$$f_{2n+1}(q) = \sum_{k=1}^{n}\begin{bmatrix}2n+1\\2k\end{bmatrix}(-1)^{k-1} f_{2(n-k)+1}(q) + (-1)^n,$$

which completes the proof. □

The following theorem gives the generating function for alternating permutations π in A_{2n+1} by weight $inv(\pi) + des(\pi)$.

Theorem 2.2. *Define*

$$\sum_{n=0}^{\infty}\frac{T_{2n+1}^{des}(q)x^{2n+1}}{(q;q)_{2n+1}} = \frac{\sum_{n=0}^{\infty}\frac{(-1)^n q^n x^{2n+1}}{(q;q)_{2n+1}}}{\sum_{n=0}^{\infty}\frac{(-1)^n q^n x^{2n}}{(q;q)_{2n}}}.$$

Then

$$T_{2n+1}^{des}(q) = \sum_{\pi\in A_{2n+1}} q^{inv(\pi)+des(\pi)}.$$

Proof. Recalling the definition of $T_{2n+1}(q)$ given by (2.1) and noting that $des(\pi) = n$ for any $\pi \in A_{2n+1}$, we have

$$\sum_{\pi\in A_{2n+1}} q^{inv(\pi)+des(\pi)} = q^n T_{2n+1}(q).$$

By comparing the recurrence relations for $T_{2n+1}^{des}(q)$ and $T_{2n+1}(q)$, we see that $T_{2n+1}^{des}(q) = q^n T_{2n+1}(q)$. □

The dual functions corresponding to $T_{2n+1}(q)$ and $T_{2n+1}^{des}(q)$ are discussed in the following theorem.

Theorem 2.3. *Define*

$$\sum_{n=0}^{\infty}\frac{\hat{T}_{2n+1}(q)x^{2n+1}}{(q;q)_{2n+1}} = \frac{\sum_{n=0}^{\infty}\frac{(-1)^n q^{2n^2+n} x^{2n+1}}{(q;q)_{2n+1}}}{\sum_{n=0}^{\infty}\frac{(-1)^n q^{2n^2-n} x^{2n}}{(q;q)_{2n}}},$$

and

$$\sum_{n=0}^{\infty}\frac{\hat{T}^{des}_{2n+1}(q)x^{2n+1}}{(q;q)_{2n+1}}=\frac{\sum_{n=0}^{\infty}\frac{(-1)^n q^{2n^2}x^{2n+1}}{(q;q)_{2n+1}}}{\sum_{n=0}^{\infty}\frac{(-1)^n q^{2n^2-2n}x^{2n}}{(q;q)_{2n}}}.$$

Then we have

$$\hat{T}_{2n+1}(q)=T_{2n+1}(q)\quad \textit{and}\quad \hat{T}^{des}_{2n+1}(q)=q^{-2n}T^{des}_{2n+1}(q).$$

Proof. We first note that the polynomials $\hat{T}_{2n+1}(q)$ satisfy $\hat{T}_1(q)=1$ and

$$\hat{T}_{2n+1}(q)=\sum_{k=0}^{n-1}\begin{bmatrix}2n+1\\2k+1\end{bmatrix}(-1)^{n-k-1}q^{2(n-k)^2-(n-k)}\hat{T}_{2k+1}(q)+(-1)^n q^{2n^2+n}. \tag{2.6}$$

For any alternating permutation $\pi\in A_{2n+1}$, define $\bar{\pi}$ by

$$\bar{\pi}=\pi_{2n+1}\pi_{2n}\pi_{2n-1}\cdots\pi_2\pi_1.$$

Then $\bar{\pi}$ is clearly an up-down alternating permutation. Furthermore

$$inv(\pi)+inv(\bar{\pi})=n(2n+1).$$

Thus

$$\begin{aligned}\sum_{\pi\in A_{2n+1}}q^{inv(\pi)}&=\sum_{\pi\in A_{2n+1}}q^{n(2n+1)-inv(\bar{\pi})}\\&=q^{n(2n+1)}T_{2n+1}\left(q^{-1}\right).\end{aligned}$$

Therefore, it suffices to show that

$$\hat{T}_{2n+1}(q)=q^{n(2n+1)}T_{2n+1}\left(q^{-1}\right).$$

We now show that $q^{n(2n+1)}T_{2n+1}(q^{-1})$ satisfies the same recursion as $\hat{T}_{2n+1}(q)$. Substitute $q^{k(2k+1)}T_{2k+1}(q^{-1})$ for $\hat{T}_{2k+1}$ in (2.6). Then

$$\begin{aligned}&\sum_{k=0}^{n-1}\begin{bmatrix}2n+1\\2k+1\end{bmatrix}_q(-1)^{n-k-1}q^{2(n-k)^2-(n-k)}q^{k(2k+1)}T_{2k+1}\left(q^{-1}\right)+(-1)^n q^{2n^2+n}\\&\quad=\sum_{k=0}^{n-1}\begin{bmatrix}2n+1\\2k+1\end{bmatrix}_{q^{-1}}(-1)^{n-k-1}q^{2n^2+n}T_{2k+1}\left(q^{-1}\right)+(-1)^n q^{2n^2+n}\\&\quad=q^{n(2n+1)}T_{2n+1}\left(q^{-1}\right),\end{aligned}$$

where the last equality follows from the recursion formula (2.2) for $T_{2n+1}(q)$.

It follows from the recurrences for $\hat{T}_{2n+1}$ and $\hat{T}^{des}_{2n+1}$ that $\hat{T}_{2n+1}(q)=q^n\hat{T}^{des}_{2n+1}(q)$. This identity is equivalent to $\hat{T}^{des}_{2n+1}(q)=q^{-2n}T^{des}_{2n+1}(q)$ since $\hat{T}_{2n+1}(q)=T_{2n+1}(q)=q^{-n}T^{des}(q)$. □

By equating coefficients of x^{2n+1} in the generating functions for $T_{2n+1}(q)$ and $\hat{T}_{2n+1}(q)$, we obtain a special case of a formula due to Gauss [3, p. 37].

Corollary 2.4. *For any nonnegative integer n,*

$$\sum_{j=0}^{n}q^{2j^2+j}\begin{bmatrix}2n+1\\2j+1\end{bmatrix}=\sum_{j=0}^{n}q^{2j^2-j}\begin{bmatrix}2n+1\\2j\end{bmatrix}.$$

The classical q-secant numbers enumerate alternating permutations on the set $[2n]$ for $n \geqslant 1$ by the number of inversions.

Theorem 2.5. *Define*

$$\sum_{n=0}^{\infty} \frac{S_{2n}(q)x^{2n}}{(q;q)_{2n}} = \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(q;q)_{2n}} \right)^{-1}. \tag{2.7}$$

Then

$$S_{2n}(q) = \sum_{\pi \in A_{2n}} q^{inv(\pi)}.$$

For a proof of Theorem 2.5, see [5,26–28]. The following theorem gives the generating function for alternating permutations π in A_{2n} by weight $inv(\pi) + des(\pi)$.

Theorem 2.6. *Define*

$$\sum_{n=0}^{\infty} \frac{S^{des}_{2n}(q)x^{2n}}{(q;q)_{2n}} = \left(\sum_{n=0}^{\infty} \frac{(-1)^n q^{n+1} x^{2n}}{(q;q)_{2n}} \right)^{-1}.$$

Then

$$S^{des}_{2n}(q) = \sum_{\pi \in A_{2n}} q^{inv(\pi)+des(\pi)}.$$

Proof. Recalling the definition of $S_{2n}(q)$ given by (2.7) and noting that $des(\pi) = n-1$ for any $\pi \in A_{2n}$, we have

$$\sum_{\pi \in A_{2n}} q^{inv(\pi)+des(\pi)} = q^{n-1} S_{2n}(q).$$

By comparing the recursions for $S^{des}_{2n}(q)$ and $S_{2n}(q)$, we see that $S^{des}_{2n}(q) = q^{n-1} S_{2n}(q)$. □

The dual functions of $S_{2n}(q)$ and $S^{des}_{2n}(q)$ are discussed in the following theorem.

Theorem 2.7. *Let*

$$\sum_{n=0}^{\infty} \frac{\hat{S}_{2n}(q)x^{2n}}{(q;q)_{2n}} = \left(\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(2n-1)} x^{2n}}{(q;q)_{2n}} \right)^{-1}$$

and

$$\sum_{n=0}^{\infty} \frac{\hat{S}^{des}_{2n}(q)x^{2n}}{(q;q)_{2n}} = \left(\sum_{n=0}^{\infty} \frac{(-1)^n q^{2n(n-1)} x^{2n}}{(q;q)_{2n}} \right)^{-1}.$$

Then

$$\hat{S}_{2n}(q) = \sum_{\pi \in \bar{A}_{2n}} q^{inv(\pi)} \quad \text{and} \quad \hat{S}^{des}_{2n}(q) = \sum_{\pi \in \bar{A}_{2n}} q^{inv(\pi)-des(\pi)}.$$

Proof. We first note that the polynomials $\hat{S}_{2n}(q)$ satisfy $\hat{S}_0(q) = 1$ and

$$\hat{S}_{2n}(q) = \sum_{k=0}^{n-1} \begin{bmatrix} 2n \\ 2k \end{bmatrix} (-1)^{n-k-1} q^{(n-k)(2(n-k)-1)} \hat{S}_{2k}(q), \quad \text{for } n \geqslant 1. \tag{2.8}$$

For any up-down alternating permutation $\pi \in A_{2n}$, define $\bar{\pi}$ by

$$\bar{\pi} = \pi_{2n}\pi_{2n-1}\pi_{2n-2} \cdots \pi_2\pi_1.$$

Then $\bar{\pi}$ is clearly a down-up alternating permutation. Furthermore

$$inv(\pi) + inv(\bar{\pi}) = n(2n-1).$$

Thus

$$\begin{aligned}\sum_{\pi \in \bar{A}_{2n}} q^{inv(\pi)} &= \sum_{\pi \in A_{2n}} q^{n(2n-1)-inv(\pi)} \\ &= q^{n(2n-1)} S_{2n}\big(q^{-1}\big).\end{aligned}$$

The first statement in the theorem is equivalent to

$$\hat{S}_{2n}(q) = q^{n(2n-1)} S_{2n}\big(q^{-1}\big). \tag{2.9}$$

We now show that $q^{n(2n-1)} S_{2n}(q^{-1})$ satisfies the same recursion as $\hat{S}_{2n}(q)$. Substitute $q^{k(2k-1)} S_{2k}(q^{-1})$ for $\hat{S}_{2k}$ in (2.8). Then

$$\begin{aligned}&\sum_{k=0}^{n-1} \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q (-1)^{n-k-1} q^{(n-k)(2(n-k)-1)} q^{k(2k-1)} S_{2k}\big(q^{-1}\big) \\ &\quad = \sum_{k=0}^{n-1} \begin{bmatrix} 2n \\ 2k \end{bmatrix}_{q^{-1}} (-1)^{n-k-1} q^{2n^2-n} S_{2k}\big(q^{-1}\big) \\ &\quad = q^{n(2n-1)} S_{2n}\big(q^{-1}\big),\end{aligned}$$

where the last equality follows from the recursion formula for $S_{2n}(q)$.

We now show that

$$\hat{S}^{des}_{2n}(q) = \sum_{\pi \in \bar{A}_{2n}} q^{inv(\pi)-des(\pi)},$$

which is equivalent to

$$\hat{S}^{des}_{2n}(q) = q^{2n(n-1)} S_{2n}\big(q^{-1}\big).$$

since $des(\pi) = n$ for a permutation $\pi \in \bar{A}_{2n}$ and $\hat{S}_{2n}(q) = q^{n(2n-1)} S_{2n}(q^{-1})$ by (2.9). We now show that $q^{2n(n-1)} S_{2n}(q^{-1})$ satisfies the same recursion as $\hat{S}^{des}_{2n}(q)$, namely

$$\hat{S}^{des}_{2n}(q) = \sum_{k=0}^{n-1} \begin{bmatrix} 2n \\ 2k \end{bmatrix} (-1)^{n-k-1} q^{2(n-k)(n-k-1)} \hat{S}^{des}_{2k}(q), \quad \text{for } n \geqslant 1. \tag{2.10}$$

Substitute $q^{2k(k-1)} S_{2k}(q^{-1})$ for $\hat{S}^{des}_{2k}$ in (2.10). Then

$$\begin{aligned}\sum_{k=0}^{n-1} \begin{bmatrix} 2n \\ 2k \end{bmatrix}_q (-1)^{n-k-1} q^{2(n-k)(n-k-1)} q^{2k(k-1)} S_{2k}\left(q^{-1}\right) \\ = \sum_{k=0}^{n-1} \begin{bmatrix} 2n \\ 2k \end{bmatrix}_{q^{-1}} (-1)^{n-k-1} q^{2n(n-1)} S_{2k}\left(q^{-1}\right) \\ = q^{2n(n-1)} S_{2n}\left(q^{-1}\right),\end{aligned}$$

where the last equality follows from the recursion formula for $S_{2n}(q)$. □

3. New *q*-secant numbers

The following theorem provides a combinatorial interpretation for a new class of secant numbers. Recall the definition of half descent $des(\pi_o)$ for a permutation π.

Theorem 3.1. *Define*

$$\sum_{n=0}^{\infty} \frac{S_{2n}^o(q) x^{2n}}{(q;q)_{2n}} = \left(\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2+1} x^{2n}}{(q;q)_{2n}} \right)^{-1}.$$

Then, for $n \geqslant 1$,

$$S_{2n}^o(q) = \sum_{\pi \in A_{2n}} q^{inv(\pi)+des(\pi_o)}.$$

Proof. Let

$$g_{2n}(q) = \sum_{\pi \in A_{2n}} q^{inv(\pi)+des(\pi_o)}.$$

From the definitions of $inv(\pi)$ and $des(\pi_o)$, it is clear that

$$g_2(q) = 1 = S_2^o(q).$$

The polynomials $S_{2n}^o(q)$ satisfy $S_0^o(q) = q^{-1}$ and

$$S_{2n}^o(q) = \sum_{k=0}^{n-1} \begin{bmatrix} 2n \\ 2k \end{bmatrix} (-1)^{n-k-1} q^{(n-k)^2} S_{2k}^o(q), \quad \text{for } n \geqslant 1. \tag{3.1}$$

We define $g_0(q) = q^{-1}$. We will show that the polynomials $g_{2n}(q)$ satisfy (3.1) for $n > 1$.

For a positive integer $k \leqslant n$, let $\mathcal{A}_{2n}^{(k)}$ be the set of permutations π on $[2n]$ such that

$$\pi_1 < \pi_2 > \pi_3 < \cdots > \pi_{2k-1} < \pi_{2k},$$

$$\pi_{2k+2} > \pi_{2k+4} > \cdots > \pi_{2n} > \pi_{2n-1} > \pi_{2n-3} > \cdots > \pi_{2k+1}.$$

Fig. 3 shows the conditions for $\pi \in \mathcal{A}_{2n}^{(k)}$.

Note that there are $(n-k)^2 - (n-k)$ inversions in $\pi_{2k+1}\pi_{2k+2}\cdots\pi_{2n}$ for the permutation $\pi \in \mathcal{A}_{2n}^{(k)}$. Thus, from Lemma 1.1 and the definition of g_{2n}, we see that

$$\sum_{\pi \in \mathcal{A}_{2n}^{(k)}} q^{inv(\pi)+des(\pi_1\pi_3\cdots\pi_{2k-1})+(n-k)} = \begin{bmatrix} 2n \\ 2k \end{bmatrix} q^{(n-k)^2} g_{2k}(q). \tag{3.2}$$

For a positive integer $k < n$, we decompose $\mathcal{A}_{2n}^{(k)}$ into disjoint subsets as follows:

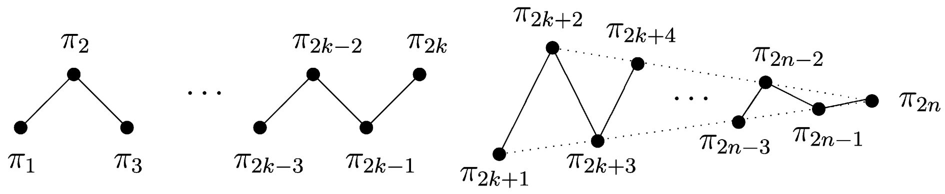

Fig. 3. $\mathcal{A}_{2n}^{(k)}$.

$$\begin{aligned}
\mathcal{A}_{2n}^{(k)} &= \{\pi \mid \pi_{2k} > \pi_{2k+2}\} \cup \{\pi \mid \pi_{2k} < \pi_{2k+2}\} \\
&= \{\pi \mid \pi_{2k} > \pi_{2k+2} > \pi_{2k-1}\} \cup \{\pi \mid \pi_{2k} > \pi_{2k-1} > \pi_{2k+2}\} \cup \{\pi \mid \pi_{2k+2} > \pi_{2k} > \pi_{2k-1}\} \\
&= \{\pi \mid \pi_{2k} > \pi_{2k+2} > \pi_{2k+1} > \pi_{2k-1}\} \cup \{\pi \mid \pi_{2k} > \pi_{2k+2} > \pi_{2k-1} > \pi_{2k+1}\} \\
&\quad \cup \{\pi \mid \pi_{2k} > \pi_{2k-1} > \pi_{2k+2} > \pi_{2k+1}\} \cup \{\pi \mid \pi_{2k+2} > \pi_{2k} > \pi_{2k-1} > \pi_{2k+1}\} \\
&\quad \cup \{\pi \mid \pi_{2k+2} > \pi_{2k} > \pi_{2k+1} > \pi_{2k-1}\} \cup \{\pi \mid \pi_{2k+2} > \pi_{2k+1} > \pi_{2k} > \pi_{2k-1}\} \\
&= \{\pi \mid \pi_{2k} > \pi_{2k+2} > \pi_{2k+1} > \pi_{2k-1}\} \cup \{\pi \mid \pi_{2k} > \pi_{2k-1} > \pi_{2k+1}\} \\
&\quad \cup \{\pi \mid \pi_{2k+2} > \pi_{2k} > \pi_{2k+1} > \pi_{2k-1}\} \cup \{\pi \mid \pi_{2k+2} > \pi_{2k+1} > \pi_{2k} > \pi_{2k-1}\} \\
&=: \mathcal{B}_{2n}^{(k)} \cup \mathcal{C}_{2n}^{(k)} \cup \mathcal{D}_{2n}^{(k)} \cup \mathcal{E}_{2n}^{(k)}.
\end{aligned} \tag{3.3}$$

Note that $\mathcal{B}_{2n}^{(k)}$ is the set of alternating permutations π on $[2n]$ such that

$$\pi_1 < \pi_2 > \pi_3 < \cdots > \pi_{2k-1} < \pi_{2k},$$
$$\pi_{2k} > \pi_{2k+2} > \cdots > \pi_{2n} > \pi_{2n-1} > \pi_{2n-3} > \cdots > \pi_{2k+1} > \pi_{2k-1},$$

from which it is clear that $\mathcal{B}_{2n}^{(k)}$ is a subset of $\mathcal{A}_{2n}^{(k-1)}$. Fig. 4 shows the conditions for $\pi \in \mathcal{B}_{2n}^{(k)}$.

We define $\mathcal{B}_{2n}^{(n)} = A_{2n}$. For a permutation $\pi \in \mathcal{B}_{2n}^{(k)}$ with $1 < k \leqslant n$, if $\pi_{2k-3} > \pi_{2k-1}$, then

$$\pi_{2k-2} > \pi_{2k-3} > \pi_{2k-1},$$

which shows that such π satisfy the conditions of $\mathcal{C}_{2n}^{(k-1)}$. Thus

$$\begin{aligned}
\mathcal{B}_{2n}^{(k)} &= \{\pi \mid \pi_{2k-3} > \pi_{2k-1}\} \cup \{\pi \mid \pi_{2k-3} < \pi_{2k-1}\} \\
&= \mathcal{C}_{2n}^{(k-1)} \cup \{\pi \mid \pi_{2k-3} < \pi_{2k-1}\}.
\end{aligned} \tag{3.4}$$

We now compute the generating function for permutations $\pi \in \mathcal{B}_{2n}^{(k)}$

$$\sum_{\pi \in \mathcal{B}_{2n}^{(k)}} q^{inv(\pi)+des(\pi_1\pi_3\cdots\pi_{2k-1})+(n-k)}.$$

Let $\pi \in \mathcal{B}_{2n}^{(k)}$. If $\pi_{2k-3} > \pi_{2k-1}$, then

$$inv(\pi) + des(\pi_1\pi_3\cdots\pi_{2k-1}) + (n-k) = inv(\pi) + des(\pi_1\pi_3\cdots\pi_{2k-3}) + (n-k+1).$$

However, if $\pi_{2k-3} < \pi_{2k-1}$, then

$$inv(\pi) + des(\pi_1\pi_3\cdots\pi_{2k-1}) + (n-k) = inv(\pi) + des(\pi_1\pi_3\cdots\pi_{2k-3}) + (n-k+1) - 1.$$

In this case, we look for a permutation σ such that

$$inv(\pi) + des(\pi_1\pi_3\cdots\pi_{2k-1}) = inv(\sigma) + des(\sigma_1\sigma_3\cdots\sigma_{2k-3}) + 1.$$

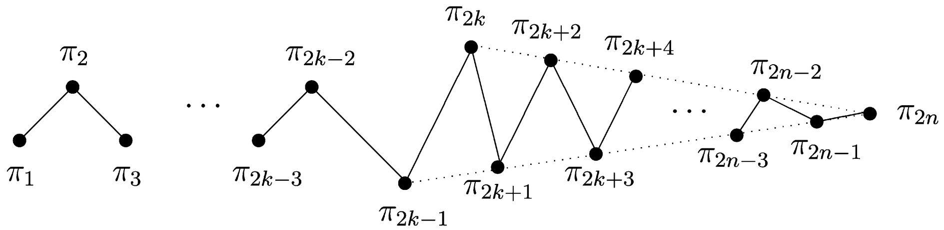

Fig. 4. $\mathcal{B}_{2n}^{(k)}$.

Let m be defined by

$$\pi_{2k+m} = \max\{\pi_{2k+j} \mid \pi_{2k+j} < \pi_{2k-2},\ j \geqslant -1\}.$$

There exists such an m since π is an alternating permutation, so that $\pi_{2k-1} < \pi_{2k-2}$. It follows that $\pi_{2k-1} \leqslant \pi_{2k+m} < \pi_{2k-2}$. We switch π_{2k-2} and π_{2k+m}, and denote the resulting partition by $\bar{\pi}$. Switching π_{2k-2} with π_{2k+m} results in a decrease of the inversion number, namely

$$inv(\pi) = inv(\bar{\pi}) + 1.$$

Moreover, since $\bar{\pi}_{2i+1} = \pi_{2i+1}$ for $i < k-1$ and $\pi_{2k-3} < \pi_{2k-1}$,

$$des(\pi_1\pi_3\cdots\pi_{2k-1}) = des(\bar{\pi}_1\bar{\pi}_3\cdots\bar{\pi}_{2k-3}).$$

Thus

$$inv(\pi) + des(\pi_1\pi_3\cdots\pi_{2k-1}) = inv(\bar{\pi}) + des(\bar{\pi}_1\bar{\pi}_3\cdots\bar{\pi}_{2k-3}) + 1.$$

If π_{2k-2} were switched with π_{2k+m} for $m \geqslant 0$, then $\pi_{2k-1} < \pi_{2k+m}$ and $\pi_{2k} \geqslant \pi_{2k+m}$. Hence, from the definition of $\bar{\pi}$, we see that

$$\begin{aligned}
&\bar{\pi}_1 < \bar{\pi}_2 > \bar{\pi}_3 < \cdots < \bar{\pi}_{2k-2},\\
&\bar{\pi}_{2k} > \bar{\pi}_{2k+2} > \cdots > \bar{\pi}_{2n} > \bar{\pi}_{2n+1} > \bar{\pi}_{2n-1} > \cdots > \bar{\pi}_{2k+1} > \bar{\pi}_{2k-1},\\
&\bar{\pi}_{2k} > \bar{\pi}_{2k-2} > \bar{\pi}_{2k-1} > \bar{\pi}_{2k-3},
\end{aligned}$$

which shows $\bar{\pi} \in \mathcal{D}_{2n+1}^{(k-1)}$. If π_{2k-2} and π_{2k-1} were switched, namely $m = -1$, then $\pi_{2k-2} < \pi_i$ for $i \geqslant 2k$. Hence, from the definition of $\bar{\pi}$, we see that

$$\begin{aligned}
&\bar{\pi}_1 < \bar{\pi}_2 > \bar{\pi}_3 < \cdots < \bar{\pi}_{2k-2},\\
&\bar{\pi}_{2k} > \bar{\pi}_{2k+2} > \cdots > \bar{\pi}_{2n} > \bar{\pi}_{2n+1} > \bar{\pi}_{2n-1} > \cdots > \bar{\pi}_{2k+1} > \bar{\pi}_{2k-1},\\
&\bar{\pi}_{2k} > \bar{\pi}_{2k-1} > \bar{\pi}_{2k-2} > \bar{\pi}_{2k-3},
\end{aligned}$$

which shows $\bar{\pi} \in \mathcal{E}_{2n+1}^{(k-1)}$. Thus, by (3.3) and (3.4), for any k, $1 < k \leqslant n$,

$$\begin{aligned}
&\left\{\pi \;\middle|\; \pi \in \mathcal{B}_{2n}^{(k)}, \pi_{2k-3} > \pi_{2k-1}\right\} \cup \left\{\bar{\pi} \;\middle|\; \pi \in \mathcal{B}_{2n}^{(k)}, \pi_{2k-3} < \pi_{2k-1}\right\}\\
&\quad = \mathcal{C}_{2n}^{(k-1)} \cup \mathcal{D}_{2n}^{(k-1)} \cup \mathcal{E}_{2n}^{(k-1)}\\
&\quad = \mathcal{A}_{2n}^{(k-1)} \setminus \mathcal{B}_{2n}^{(k-1)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sum_{\pi\in\mathcal{B}_{2n}^{(k)}} q^{inv(\pi)+des(\pi_1\pi_3\cdots\pi_{2k-1})+(n-k)}\\
&\quad=\sum_{\substack{\pi\in\mathcal{B}_{2n}^{(k)}\\ \pi_{2k-3}>\pi_{2k-1}}} q^{inv(\pi)+des(\pi_1\pi_3\cdots\pi_{2k-3})+(n-k+1)}+\sum_{\substack{\pi\in\mathcal{B}_{2n}^{(k)}\\ \pi_{2k-3}<\pi_{2k-1}}} q^{inv(\pi)+des(\pi_1\pi_3\cdots\pi_{2k-3})+(n-k)}\\
&\quad=\sum_{\substack{\pi\in\mathcal{B}_{2n}^{(k)}\\ \pi_{2k-3}>\pi_{2k-1}}} q^{inv(\pi)+des(\pi_1\pi_3\cdots\pi_{2k-3})+(n-k+1)}+\sum_{\substack{\pi\in\mathcal{B}_{2n}^{(k)}\\ \pi_{2k-3}<\pi_{2k-1}}} q^{inv(\bar\pi)+des(\bar\pi_1\bar\pi_3\cdots\bar\pi_{2k-3})+(n-k+1)}\\
&\quad=\sum_{\pi\in\mathcal{A}_{2n}^{(k-1)}} q^{inv(\pi)+des(\pi_1\pi_3\cdots\pi_{2k-3})+(n-k+1)}-\sum_{\pi\in\mathcal{B}_{2n}^{(k-1)}} q^{inv(\pi)+des(\pi_1\pi_3\cdots\pi_{2k-3})+(n-k+1)}. \qquad (3.5)
\end{aligned}$$

By iterating (3.5), for any $n>1$, we deduce

$$\begin{aligned}
\sum_{\pi\in\mathcal{B}_{2n}^{(n)}} q^{inv(\pi)+des(\pi_o)} &= \sum_{k=1}^{n-1}(-1)^{n-k-1}\left(\sum_{\pi\in\mathcal{A}_{2n}^{(k)}} q^{inv(\pi)+des(\pi_1\pi_3\cdots\pi_{2k-1})+(n-k)}\right)\\
&\quad+(-1)^{n-1}\sum_{\pi\in\mathcal{B}_{2n}^{(1)}} q^{inv(\pi)+n-1},
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
g_{2n}(q) &= \sum_{k=1}^{n-1}\begin{bmatrix}2n\\2k\end{bmatrix}(-1)^{n-k-1}q^{(n-k)^2}g_{2k}(q)+(-1)^{n-1}q^{n^2-1}\\
&= \sum_{k=0}^{n-1}\begin{bmatrix}2n\\2k\end{bmatrix}(-1)^{n-k-1}q^{(n-k)^2}g_{2k}(q),
\end{aligned}$$

where the second equality holds since $g_0(q)=q^{-1}$. $\square$

Theorem 3.2. *Define*

$$\sum_{n=0}^{\infty}\frac{\hat S_{2n}^{o}(q)x^{2n}}{(q;q)_{2n}}=\left(\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n-1)}x^{2n}}{(q;q)_{2n}}\right)^{-1}.$$

Then, for $n\geqslant 1$,

$$\hat S_{2n}^{o}(q)=q^{1-n}S_{2n}^{o}(q).$$

Proof. Using the recurrences satisfied by $S_{2n}^{o}(q)$ and $\hat S_{2n}^{o}(q)$, we can easily prove that $\hat S_{2n}^{o}(q)=q^{1-n}S_{2n}^{o}(q)$ for $n\geqslant 1$. We omit the details. $\square$

The other half descent $des(\pi_e)$ is discussed in the following theorem.

Theorem 3.3. *Define*

$$S_{2n}^{e}(q)=\sum_{\pi\in A_{2n}} q^{inv(\pi)+des(\pi_e)}. \qquad (3.6)$$

Then we have

$$S_{2n}^{e}(q)=qS_{2n}^{o}(q).$$

Proof. For a permutation $\pi \in A_{2n}$, define the map σ by $\sigma(\pi) = (2n+1-\pi_{2n})(2n+1-\pi_{2n-1})\cdots(2n+1-\pi_2)(2n+1-\pi_1)$. Then $\sigma(\pi)$ is an alternating permutation in A_{2n} with

$$inv(\pi) = inv\big(\sigma(\pi)\big) \quad \text{and} \quad des(\pi_o) + 1 = des\big(\sigma(\pi)_e\big),$$

since $\sigma(\pi)_e = (\infty)(2n+1-\pi_{2n-1})(2n+1-\pi_{2n-3})\cdots(2n+1-\pi_3)(2n+1-\pi_1)$. Therefore, it follows that $S^e_{2n}(q) = qS^o_{2n}(q)$. $\square$

It follows from Theorems 3.1 and 3.3 that

$$\sum_{n=0}^{\infty} \frac{S^e_{2n}(q)x^{2n}}{(q;q)_{2n}} = \left(\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2} x^{2n}}{(q;q)_{2n}} \right)^{-1}.$$

4. New q-tangent numbers associated with odd indices

Define $T^o_{2n+1}(q)$ by

$$\sum_{n=0}^{\infty} \frac{T^o_{2n+1}(q)x^{2n+1}}{(q;q)_{2n+1}} = \frac{\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2+n} x^{2n+1}}{(q;q)_{2n+1}}}{\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2} x^{2n}}{(q;q)_{2n}}}.$$

Then, by the definition of $S^o_{2n}(q)$ in Theorem 3.1,

$$\sum_{n=0}^{\infty} \frac{T^o_{2n+1}(q)x^{2n+1}}{(q;q)_{2n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2+n} x^{2n+1}}{(q;q)_{2n+1}} \sum_{n=0}^{\infty} \frac{qS^o_{2n}(q)x^{2n}}{(q;q)_{2n}}.$$

Thus

$$T^o_{2n+1}(q) = \sum_{k=0}^{n} \begin{bmatrix} 2n+1 \\ 2k \end{bmatrix} (-1)^{n-k} q^{(n-k)^2+(n-k)+1} S^o_{2k}(q). \tag{4.1}$$

In [18], Huber proves that the coefficients $(q;q)^{-1}_{2n+1}\psi^{-2}(q)$ in y_{2n+1} in (1.2) are $T^o_{2n+1}(q)$, whose combinatorial interpretation is given in the following theorem.

Theorem 4.1. *For each nonnegative integer n, we have*

$$T^o_{2n+1}(q) = \sum_{\pi \in A_{2n+1}} q^{inv(\pi)+des(\pi_o)}.$$

Proof. Let

$$f_{2n+1}(q) = \sum_{\pi \in A_{2n+1}} q^{inv(\pi)+des(\pi_o)}.$$

For $n = 0$, it is clear that

$$f_1(q) = 1 = T^o_1(q).$$

For any positive integer n, we will show that $f_{2n+1}(q)$ satisfies Eq. (4.1).

For a positive integer $k \leqslant n$, let $\mathcal{A}^{(k)}_{2n+1}$ be the set of permutations π on $[2n+1]$ such that

$$\begin{aligned} &\pi_1 < \pi_2 > \pi_3 < \cdots > \pi_{2k-1} < \pi_{2k}, \\ &\pi_{2k+2} > \pi_{2k+4} > \cdots > \pi_{2n} > \pi_{2n+1} > \pi_{2n-1} > \cdots > \pi_{2k+1}. \end{aligned} \tag{4.2}$$

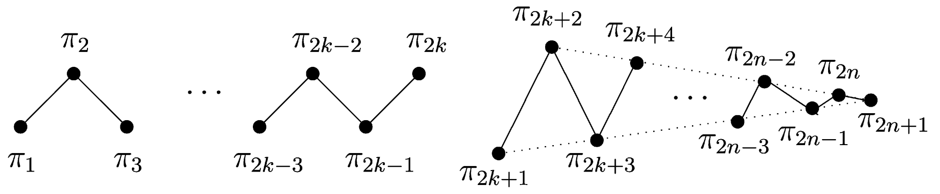

Fig. 5. $\mathcal{A}^{(k)}_{2n+1}$.

Fig. 5 shows the conditions for $\pi \in \mathcal{A}^{(k)}_{2n+1}$.

From Lemma 1.1 and Theorem 3.1, we see that for $k \leqslant n$,

$$\sum_{\pi \in \mathcal{A}^{(k)}_{2n+1}} q^{inv(\pi)+des(\pi_1\pi_3\cdots\pi_{2k-1})} = \begin{bmatrix} 2n+1 \\ 2k \end{bmatrix} q^{(n-k)^2} S^o_{2k}(q). \tag{4.3}$$

We decompose $\mathcal{A}^{(n)}_{2n+1}$ into disjoint subsets as follows:

$$\begin{aligned} \mathcal{A}^{(n)}_{2n+1} &= \{\pi \mid \pi_{2n} > \pi_{2n+1}\} \cup \{\pi \mid \pi_{2n} < \pi_{2n+1}\} \\ &= \{\pi \mid \pi_{2n} > \pi_{2n+1} > \pi_{2n-1}\} \cup \{\pi \mid \pi_{2n} > \pi_{2n-1} > \pi_{2n+1}\} \cup \{\pi \mid \pi_{2n} < \pi_{2n+1}\} \\ &=: \mathcal{B}^{(n)}_{2n+1} \cup \mathcal{D}^{(n)}_{2n+1} \cup \mathcal{E}^{(n)}_{2n+1}. \end{aligned} \tag{4.4}$$

Note that $A_{2n+1} = \mathcal{B}^{(n)}_{2n+1} \cup \mathcal{D}^{(n)}_{2n+1}$. For $k < n$, we decompose $\mathcal{A}^{(k)}_{2n+1}$ into disjoint subsets as follows:

$$\begin{aligned} \mathcal{A}^{(k)}_{2n+1} &= \{\pi \mid \pi_{2k} > \pi_{2k+1}\} \cup \{\pi \mid \pi_{2k} < \pi_{2k+1}\} \\ &= \{\pi \mid \pi_{2k} > \pi_{2k+1} > \pi_{2k-1}\} \cup \{\pi \mid \pi_{2k} > \pi_{2k-1} > \pi_{2k+1}\} \cup \{\pi \mid \pi_{2k} < \pi_{2k+1}\} \\ &= \{\pi \mid \pi_{2k} > \pi_{2k+2} > \pi_{2k+1} > \pi_{2k-1}\} \cup \{\pi \mid \pi_{2k+2} > \pi_{2k} > \pi_{2k+1} > \pi_{2k-1}\} \\ &\quad \cup \{\pi \mid \pi_{2k} > \pi_{2k-1} > \pi_{2k+1}\} \cup \{\pi \mid \pi_{2k} < \pi_{2k+1}\} \\ &=: \mathcal{B}^{(k)}_{2n+1} \cup \mathcal{C}^{(k)}_{2n+1} \cup \mathcal{D}^{(k)}_{2n+1} \cup \mathcal{E}^{(k)}_{2n+1}. \end{aligned} \tag{4.5}$$

Note that, since the permutations in $\mathcal{B}^{(k)}_{2n+1}$ satisfy the conditions of (4.2), $\mathcal{B}^{(k)}_{2n+1}$ is the set of alternating permutations π on $[2n+1]$ such that

$$\pi_1 < \pi_2 > \pi_3 < \cdots < \pi_{2k},$$
$$\pi_{2k} > \pi_{2k+2} > \cdots > \pi_{2n} > \pi_{2n+1} > \pi_{2n-1} > \cdots > \pi_{2k-1}.$$

Fig. 6 shows the conditions for $\pi \in \mathcal{B}^{(k)}_{2n+1}$.

Furthermore, for any positive k, $1 < k \leqslant n+1$,

$$\mathcal{B}^{(k)}_{2n+1} = \{\pi \mid \pi_{2k-3} > \pi_{2k-1}\} \cup \{\pi \mid \pi_{2k-3} < \pi_{2k-1}\} = \mathcal{D}^{(k-1)}_{2n+1} \cup \{\pi \mid \pi_{2k-3} < \pi_{2k-1}\},$$

where $\mathcal{B}^{(n+1)} = A_{2n+1}$. For $k > 1$, we now compute the generating function for permutations $\pi \in \mathcal{B}^{(k)}_{2n+1}$

$$\sum_{\pi \in \mathcal{B}^{(k)}_{2n+1}} q^{inv(\pi)+des(\pi_1\pi_3\cdots\pi_{2k-1})}.$$

Let $\pi \in \mathcal{B}^{(k)}_{2n+1}$. If $\pi_{2k-3} > \pi_{2k-1}$, namely $\pi \in \mathcal{D}^{(k-1)}_{2n+1}$, then

$$inv(\pi) + des(\pi_1\pi_3\cdots\pi_{2k-1}) = inv(\pi) + des(\pi_1\pi_3\cdots\pi_{2k-3}) + 1.$$

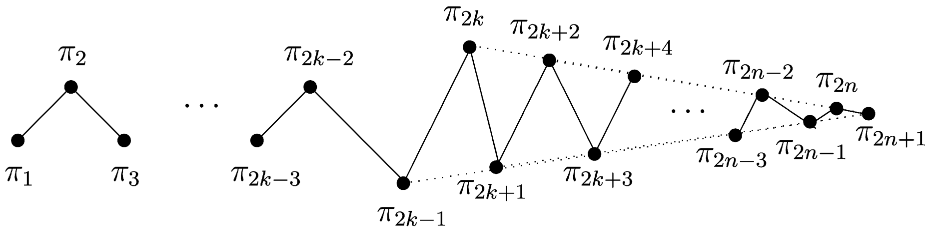

Fig. 6. $\mathcal{B}_{2n+1}^{(k)}$.

However, if $\pi_{2k-3} < \pi_{2k-1}$, then

$$inv(\pi) + des(\pi_1\pi_3\cdots\pi_{2k-1}) = inv(\pi) + des(\pi_1\pi_3\cdots\pi_{2k-3}).$$

In this case, we look for a permutation σ such that

$$inv(\pi) + des(\pi_1\pi_3\cdots\pi_{2k-1}) = inv(\sigma) + des(\sigma_1\sigma_3\cdots\sigma_{2k-3}) + 1.$$

Let the positive integer m be defined by

$$\pi_{2k+m} = \max\{\pi_{2k+j} \mid \pi_{2k+j} < \pi_{2k-2},\ j \geqslant -1\}.$$

Since $\pi_{2k-1} < \pi_{2k-2}$, there exists such an m. We switch π_{2k+m} with π_{2k-2} and denote the resulting partition by $\bar{\pi}$. Switching π_{2k-2} with π_{2k+m} results in a decrease of the inversion number, namely

$$inv(\pi) = inv(\bar{\pi}) + 1.$$

Moreover, if π_{2k-2} was switched with π_{2k+m} for $m > -1$, it is trivial that

$$des(\pi_1\pi_3\cdots\pi_{2k-1}) = des(\bar{\pi}_1\bar{\pi}_3\cdots\bar{\pi}_{2k-3}),$$

since $\bar{\pi}_{2i+1} = \pi_{2i+1}$ for $i < k$ and $\pi_{2k-3} < \pi_{2k-1}$. If π_{2k-2} was switched with π_{2k-1}, since $\pi_{2k-3} < \pi_{2k-1} < \pi_{2k-2}$,

$$des(\pi_1\pi_3\cdots\pi_{2k-1}) = des(\bar{\pi}_1\bar{\pi}_3\cdots\bar{\pi}_{2k-3}).$$

Thus, in either case,

$$inv(\pi) + des(\pi_1\pi_3\cdots\pi_{2k-1}) = inv(\bar{\pi}) + des(\bar{\pi}_1\bar{\pi}_3\cdots\bar{\pi}_{2k-3}) + 1.$$

From the definition of $\bar{\pi}$, if π_{2k-2} was switched with π_{2k-1}, then

$$\bar{\pi}_1 < \bar{\pi}_2 > \bar{\pi}_3 < \cdots < \bar{\pi}_{2k-4} > \bar{\pi}_{2k-3} < \bar{\pi}_{2k-2},$$
$$\bar{\pi}_{2k} > \bar{\pi}_{2k+2} > \cdots > \bar{\pi}_{2n} > \bar{\pi}_{2n+1} > \bar{\pi}_{2n-1} > \cdots > \bar{\pi}_{2k-1},$$
$$\bar{\pi}_{2k-2} < \bar{\pi}_{2k-1},$$

which shows that $\bar{\pi} \in \mathcal{E}_{2n+1}^{(k-1)}$. If π_{2k-2} was switched with π_{2k+m} for $m > -1$, then it follows from the maximality of π_{2k+m} that

$$\bar{\pi}_1 < \bar{\pi}_2 > \bar{\pi}_3 < \cdots < \bar{\pi}_{2k-4} > \bar{\pi}_{2k-3} < \bar{\pi}_{2k-2},$$
$$\bar{\pi}_{2k} > \bar{\pi}_{2k+2} > \cdots > \bar{\pi}_{2n} > \bar{\pi}_{2n+1} > \bar{\pi}_{2n-1} > \cdots > \bar{\pi}_{2k-1},$$
$$\bar{\pi}_{2k} > \bar{\pi}_{2k-2} > \bar{\pi}_{2k-1} > \bar{\pi}_{2k-3},$$

which shows $\bar{\pi} \in \mathcal{C}_{2n+1}^{(k-1)}$. Thus, for any k, $1 < k \leqslant n+1$,

$$\begin{aligned}
&\left\{\pi \;\middle|\; \pi \in \mathcal{B}_{2n+1}^{(k)}, \pi_{2k-3} > \pi_{2k-1}\right\} \cup \left\{\bar{\pi} \;\middle|\; \pi \in \mathcal{B}_{2n+1}^{(k)}, \pi_{2k-3} < \pi_{2k-1}\right\}\\
&= \mathcal{C}_{2n+1}^{(k-1)} \cup \mathcal{D}_{2n+1}^{(k-1)} \cup \mathcal{E}_{2n+1}^{(k-1)}\\
&= \mathcal{A}_{2n+1}^{(k-1)} \setminus \mathcal{B}_{2n+1}^{(k-1)},
\end{aligned} \tag{4.6}$$

where the last equality of (4.6) follows from (4.5). Therefore,

$$\begin{aligned}
&\sum_{\pi \in \mathcal{B}_{2n+1}^{(k)}} q^{inv(\pi)+des(\pi_1\pi_3\cdots\pi_{2k-1})}\\
&= \sum_{\substack{\pi \in \mathcal{B}_{2n+1}^{(k)}\\ \pi_{2k-3} > \pi_{2k-1}}} q^{inv(\pi)+des(\pi_1\pi_3\cdots\pi_{2k-3})+1} + \sum_{\substack{\pi \in \mathcal{B}_{2n+1}^{(k)}\\ \pi_{2k-3} < \pi_{2k-1}}} q^{inv(\pi)+des(\pi_1\pi_3\cdots\pi_{2k-3})}\\
&= \sum_{\substack{\pi \in \mathcal{B}_{2n+1}^{(k)}\\ \pi_{2k-3} > \pi_{2k-1}}} q^{inv(\pi)+des(\pi_1\pi_3\cdots\pi_{2k-3})+1} + \sum_{\substack{\pi \in \mathcal{B}_{2n+1}^{(k)}\\ \pi_{2k-3} < \pi_{2k-1}}} q^{inv(\bar{\pi})+des(\bar{\pi}_1\bar{\pi}_3\cdots\bar{\pi}_{2k-3})+1}\\
&= q\left(\sum_{\pi \in \mathcal{A}_{2n+1}^{(k-1)}} q^{inv(\pi)+des(\pi_1\pi_3\cdots\pi_{2k-3})} - \sum_{\pi \in \mathcal{B}_{2n+1}^{(k-1)}} q^{inv(\pi)+des(\pi_1\pi_3\cdots\pi_{2k-3})}\right).
\end{aligned} \tag{4.7}$$

By iterating (4.7), we deduce

$$\begin{aligned}
&\sum_{\pi \in \mathcal{B}_{2n+1}^{(n+1)}} q^{inv(\pi)+des(\pi_o)}\\
&= \sum_{k=1}^{n} (-1)^{n-k} \left(\sum_{\pi \in \mathcal{A}_{2n+1}^{(k)}} q^{inv(\pi)+des(\pi_1\pi_3\cdots\pi_{2k-1})+(n-k)+1}\right) + (-1)^n \sum_{\pi \in \mathcal{B}_{2n+1}^{(1)}} q^{inv(\pi)+n},
\end{aligned}$$

which is equivalent to

$$f_{2n+1}(q) = \sum_{k=0}^{n} \begin{bmatrix} 2n+1 \\ 2k \end{bmatrix} (-1)^{n-k} q^{(n-k)^2+(n-k)+1} S_{2k}^{o}(q)$$

by (4.3) and the definitions of $\mathcal{B}_{2n+1}^{(n+1)}$, $\mathcal{B}_{2n+1}^{(1)}$, and $S_0^o(q) = q^{-1}$. $\square$

Define $\hat{T}_{2n+1}^{o}(q)$ by

$$\sum_{n=0}^{\infty} \frac{\hat{T}_{2n+1}^{o}(q) x^{2n+1}}{(q;q)_{2n+1}} = \frac{\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2} x^{2n+1}}{(q;q)_{2n+1}}}{\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2-n} x^{2n}}{(q;q)_{2n}}}.$$

Theorem 4.2. *For a nonnegative integer n, we have*

$$\hat{T}_{2n+1}^{o}(q) = q^{-n} T_{2n+1}^{o}(q).$$

Proof. The theorem follows from the recursions satisfied by $T_{2n+1}^{o}(q)$ and $\hat{T}_{2n+1}^{o}(q)$. $\square$

5. New *q*-tangent numbers associated with even indices

Let $T_{2n+1}^{e}(q)$ be the polynomial satisfying

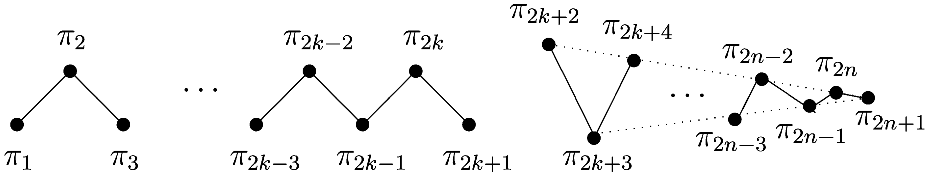

Fig. 7. $\mathcal{A}_{2n+1}^{(k)}$.

$$\sum_{n=0}^{\infty} \frac{T_{2n+1}^{e}(q)x^{2n+1}}{(q;q)_{2n+1}} = \frac{\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2} x^{2n+1}}{(q;q)_{2n+1}}}{\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2} x^{2n}}{(q;q)_{2n}}}.$$

From the generating function for $T_{2n+1}^{e}(q)$, it follows that

$$T_{2n+1}^{e}(q) = (-1)^n q^{n^2} + \sum_{k=0}^{n-1} \begin{bmatrix} 2n+1 \\ 2k+1 \end{bmatrix} (-1)^{n-k-1} q^{(n-k)^2} T_{2k+1}^{e}(q). \tag{5.1}$$

Theorem 5.1. *For a nonnegative integer n,*

$$T_{2n+1}^{e}(q) = \sum_{\pi \in A_{2n+1}} q^{inv(\pi)+des(\pi_e)}.$$

Proof. Let

$$f_{2n+1}(q) = \sum_{\pi \in A_{2n+1}} q^{inv(\pi)+des(\pi_e)}.$$

For $n = 0$, it is clear that

$$f_1(q) = 1 = T_1^{e}(q).$$

For any positive integer n, we will show that $f_{2n+1}(q)$ satisfies the recurrence (5.1).

For a nonnegative integer $k \leqslant n$, let $\mathcal{A}_{2n+1}^{(k)}$ be the set of permutations π on $[2n+1]$ such that

$$\pi_1 < \pi_2 > \pi_3 < \cdots < \pi_{2k} > \pi_{2k+1},$$

$$\pi_{2k+2} > \pi_{2k+4} > \cdots > \pi_{2n} > \pi_{2n+1} > \pi_{2n-1} > \cdots > \pi_{2k+3}.$$

Fig. 7 shows the conditions for $\pi \in \mathcal{A}_{2n+1}^{(k)}$.

From Lemma 1.1 and the definition of f_{2n+1}, we see that

$$\sum_{\pi \in \mathcal{A}_{2n+1}^{(k)}} q^{inv(\pi)+des(\pi_0\pi_2\pi_4\cdots\pi_{2k})} = \begin{bmatrix} 2n+1 \\ 2k+1 \end{bmatrix} q^{(n-k)^2} f_{2k+1}(q). \tag{5.2}$$

For $k < n$, we decompose $\mathcal{A}_{2n+1}^{(k)}$ into disjoint subsets as follows:

$$\begin{aligned} \mathcal{A}_{2n+1}^{(k)} &= \{\pi \mid \pi_{2k+1} < \pi_{2k+2}\} \cup \{\pi \mid \pi_{2k+1} > \pi_{2k+2}\} \\ &= \{\pi \mid \pi_{2k+1} < \pi_{2k+2} < \pi_{2k}\} \cup \{\pi \mid \pi_{2k+1} < \pi_{2k} < \pi_{2k+2}\} \cup \{\pi \mid \pi_{2k+1} > \pi_{2k+2}\} \\ &= \{\pi \mid \pi_{2k+1} < \pi_{2k+3} < \pi_{2k+2} < \pi_{2k}\} \cup \{\pi \mid \pi_{2k+3} < \pi_{2k+1} < \pi_{2k+2} < \pi_{2k}\} \\ &\quad \cup \{\pi \mid \pi_{2k+1} < \pi_{2k} < \pi_{2k+2}\} \cup \{\pi \mid \pi_{2k+1} > \pi_{2k+2}\} \\ &=: \mathcal{B}_{2n+1}^{(k)} \cup \mathcal{C}_{2n+1}^{(k)} \cup \mathcal{D}_{2n+1}^{(k)} \cup \mathcal{E}_{2n+1}^{(k)}. \end{aligned} \tag{5.3}$$

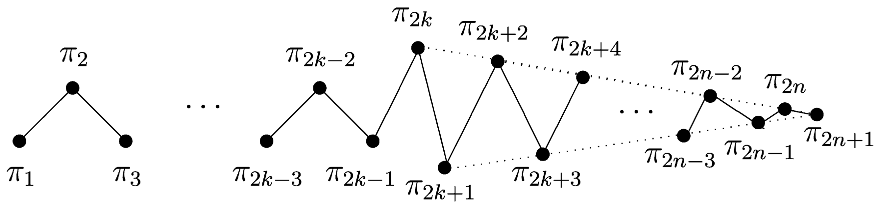

Fig. 8. $\mathcal{B}^{(k)}_{2n+1}$.

Note that $\mathcal{B}^{(k)}_{2n+1}$ is the set of alternating permutations π on $[2n+1]$ such that

$$\pi_1 < \pi_2 > \pi_3 < \cdots < \pi_{2k} > \pi_{2k+1},$$
$$\pi_{2k} > \pi_{2k+2} > \cdots > \pi_{2n} > \pi_{2n+1} > \pi_{2n-1} > \cdots > \pi_{2k+3} > \pi_{2k+1}.$$

Fig. 8 shows the conditions for $\pi \in \mathcal{B}^{(k)}_{2n+1}$.

We define $\mathcal{B}^{(n)}_{2n+1} = A_{2n+1}$. For a permutation $\pi \in \mathcal{B}^{(k)}_{2n+1}$ with $1 \leqslant k \leqslant n$, if $\pi_{2k-2} < \pi_{2k}$, then

$$\pi_{2k-1} < \pi_{2k-2} < \pi_{2k},$$

which shows that such π satisfies the conditions of $\mathcal{D}^{(k-1)}_{2n+1}$. Thus

$$\begin{aligned}\mathcal{B}^{(k)}_{2n+1} &= \{\pi \mid \pi_{2k-2} < \pi_{2k}\} \cup \{\pi \mid \pi_{2k-2} > \pi_{2k}\} \\ &= \mathcal{D}^{(k-1)}_{2n+1} \cup \{\pi \mid \pi_{2k-2} > \pi_{2k}\}.\end{aligned}$$

We now compute the generating function for permutations $\pi \in \mathcal{B}^{(k)}_{2n+1}$

$$\sum_{\pi \in \mathcal{B}^{(k)}_{2n+1}} q^{inv(\pi)+des(\pi_0\pi_2\pi_4\cdots\pi_{2k})}.$$

Let $\pi \in \mathcal{B}^{(k)}_{2n+1}$. If $\pi_{2k-2} < \pi_{2k}$, then

$$inv(\pi) + des(\pi_0\pi_2\pi_4\cdots\pi_{2k}) = inv(\pi) + des(\pi_0\pi_2\pi_4\cdots\pi_{2k-2}).$$

However, if $\pi_{2k-2} > \pi_{2k}$, then

$$inv(\pi) + des(\pi_0\pi_2\pi_4\cdots\pi_{2k}) = inv(\pi) + des(\pi_0\pi_2\pi_4\cdots\pi_{2k-2}) + 1.$$

In this case, we look for a permutation σ such that

$$inv(\pi) + des(\pi_0\pi_2\pi_4\cdots\pi_{2k}) = inv(\sigma) + des(\sigma_0\sigma_2\sigma_4\cdots\sigma_{2k-2}).$$

Let m be defined by

$$\pi_{2k+m} = \min\{\pi_{2k+j} \mid \pi_{2k+j} > \pi_{2k-1},\ 0 \leqslant j \leqslant 2n-2k+1\}.$$

There exists such an m since π is an alternating permutation, namely $\pi_{2k-1} < \pi_{2k}$. So, $\pi_{2k+m} \leqslant \pi_{2k} < \pi_{2k-2}$. We switch π_{2k+m} with π_{2k-1} and denote the resulting partition by $\bar{\pi}$. Switching π_{2k-1} with π_{2k+m} results in increasing of an inversion, namely

$$inv(\pi) + 1 = inv(\bar{\pi}).$$

Moreover, since $\pi_{2i} = \bar{\pi}_{2i}$ for $i < k$,

$$des(\pi_0\pi_2\pi_4\cdots\pi_{2k-2}) = des(\bar{\pi}_0\bar{\pi}_2\bar{\pi}_4\cdots\bar{\pi}_{2k-2}).$$

Thus

$$\begin{aligned}
inv(\pi)+des(\pi_2\pi_4\cdots\pi_{2k}) &= inv(\pi)+des(\pi_0\pi_2\pi_4\cdots\pi_{2k-2})+1\\
&= inv(\bar\pi)+des(\bar\pi_0\bar\pi_2\bar\pi_4\cdots\bar\pi_{2k-2}).
\end{aligned}$$

If π_{2k-1} was switched with π_{2k+m} for $m>0$, then $\pi_{2k+j}<\pi_{2k}$. Hence, from the definition of $\bar\pi$, we see that

$$\begin{aligned}
&\bar\pi_1<\bar\pi_2>\bar\pi_3<\cdots<\bar\pi_{2k-2}>\bar\pi_{2k-1},\\
&\bar\pi_{2k}>\bar\pi_{2k+2}>\cdots>\bar\pi_{2n}>\bar\pi_{2n+1}>\bar\pi_{2n-1}>\cdots>\bar\pi_{2k+1},\\
&\bar\pi_{2k-2}>\bar\pi_{2k}>\bar\pi_{2k-1}>\bar\pi_{2k+1},
\end{aligned}$$

which shows, from (5.3), that $\bar\pi\in\mathcal{C}_{2n+1}^{(k-1)}$. If π_{2k-1} and π_{2k} were switched, namely $m=0$, then $\pi_{2k-1}>\pi_i$ for $i>2k$. Hence, from the definition of $\bar\pi$, we see that

$$\begin{aligned}
&\bar\pi_1<\bar\pi_2>\bar\pi_3<\cdots<\bar\pi_{2k-2}>\bar\pi_{2k-1},\\
&\bar\pi_{2k}>\bar\pi_{2k+2}>\cdots>\bar\pi_{2n}>\bar\pi_{2n+1}>\bar\pi_{2n-1}>\cdots>\bar\pi_{2k+1},\\
&\bar\pi_{2k-2}>\bar\pi_{2k-1}>\bar\pi_{2k}>\bar\pi_{2k+1}.
\end{aligned}\tag{5.4}$$

Note that, since $\mathcal{E}_{2n+1}^{(k-1)}\subseteq\mathcal{A}_{2n+1}^{(k-1)}$, we see that

$$\mathcal{E}_{2n+1}^{(k-1)}=\{\pi\mid\pi_{2k-2}>\pi_{2k-1}>\pi_{2k}>\pi_{2k+1}\}.$$

Therefore, (5.4) implies $\bar\pi\in\mathcal{E}_{2n+1}^{(k-1)}$. Hence, by (5.3), for any k, $1\leqslant k\leqslant n$

$$\begin{aligned}
&\bigl\{\pi\bigm|\pi\in\mathcal{B}_{2n+1}^{(k)},\pi_{2k-2}<\pi_{2k}\bigr\}\cup\bigl\{\bar\pi\bigm|\pi\in\mathcal{B}_{2n+1}^{(k)},\pi_{2k-2}>\pi_{2k}\bigr\}\\
&\quad=\mathcal{C}_{2n+1}^{(k-1)}\cup\mathcal{D}_{2n+1}^{(k-1)}\cup\mathcal{E}_{2n+1}^{(k-1)}\\
&\quad=\mathcal{A}_{2n+1}^{(k-1)}\setminus\mathcal{B}_{2n+1}^{(k-1)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sum_{\pi\in\mathcal{B}_{2n+1}^{(k)}}q^{inv(\pi)+des(\pi_0\pi_2\pi_4\cdots\pi_{2k})}\\
&\quad=\sum_{\substack{\pi\in\mathcal{B}_{2n+1}^{(k)}\\ \pi_{2k-2}<\pi_{2k}}}q^{inv(\pi)+des(\pi_0\pi_2\pi_4\cdots\pi_{2k-2})}+\sum_{\substack{\pi\in\mathcal{B}_{2n+1}^{(k)}\\ \pi_{2k-2}>\pi_{2k}}}q^{inv(\pi)+des(\pi_0\pi_2\pi_4\cdots\pi_{2k-2})+1}\\
&\quad=\sum_{\substack{\pi\in\mathcal{B}_{2n+1}^{(k)}\\ \pi_{2k-2}<\pi_{2k}}}q^{inv(\pi)+des(\pi_0\pi_2\pi_4\cdots\pi_{2k-2})}+\sum_{\substack{\pi\in\mathcal{B}_{2n+1}^{(k)}\\ \pi_{2k-2}>\pi_{2k}}}q^{inv(\bar\pi)+des(\bar\pi_0\bar\pi_2\bar\pi_4\cdots\bar\pi_{2k-2})}\\
&\quad=\sum_{\pi\in\mathcal{A}_{2n+1}^{(k-1)}}q^{inv(\pi)+des(\pi_0\pi_2\pi_4\cdots\pi_{2k-2})}-\sum_{\pi\in\mathcal{B}_{2n+1}^{(k-1)}}q^{inv(\pi)+des(\pi_0\pi_2\pi_4\cdots\pi_{2k-2})}.
\end{aligned}\tag{5.5}$$

By iterating (5.5), we deduce

$$\begin{aligned}
&\sum_{\pi\in\mathcal{B}_{2n+1}^{(n)}}q^{inv(\pi)+des(\pi_e)}\\
&\quad=\sum_{k=0}^{n-1}(-1)^{n-k-1}\Biggl(\sum_{\pi\in\mathcal{A}_{2n+1}^{(k)}}q^{inv(\pi)+des(\pi_0\pi_2\pi_4\cdots\pi_{2k})}\Biggr)+(-1)^n\sum_{\pi\in\mathcal{B}_{2n+1}^{(0)}}q^{inv(\pi)},
\end{aligned}$$

which is equivalent to

$$f_{2n+1}(q)=\sum_{k=0}^{n-1}\begin{bmatrix}2n+1\\2k+1\end{bmatrix}(-1)^{n-k-1}q^{(n-k)^2}f_{2k+1}(q)+(-1)^n q^{n^2}. \qquad \square$$

Theorem 5.2. *Define*

$$\sum_{n=0}^{\infty}\frac{\hat{T}^e_{2n+1}(q)x^{2n+1}}{(q;q)_{2n+1}}=\frac{\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2+n}x^{2n+1}}{(q;q)_{2n+1}}}{\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2-n}x^{2n}}{(q;q)_{2n}}}.$$

Then, for $n\geqslant 1$,

$$\hat{T}^e_{2n+1}(q)=q^{-n-1}T^e_{2n+1}(q).$$

Proof. Note that

$$\hat{T}^e_{2n+1}(q)=(-1)^n q^{n^2+n}+\sum_{k=0}^{n-1}\begin{bmatrix}2n+1\\2k+1\end{bmatrix}(-1)^{n-k-1}q^{(n-k)^2-(n-k)}\hat{T}^e_{2k+1}(q). \tag{5.6}$$

To prove Theorem 5.2, multiply both sides of (5.6) by q^{n+1} to obtain, for $n\geqslant 1$,

$$\begin{aligned}q^{n+1}\hat{T}^e_{2n+1}(q)&=(-1)^n q^{(n+1)^2}+\begin{bmatrix}2n+1\\1\end{bmatrix}(-1)^{n-1}q^{n^2+1}\\&\quad+\sum_{k=1}^{n-1}\begin{bmatrix}2n+1\\2k+1\end{bmatrix}(-1)^{n-k-1}q^{(n-k)^2+k+1}\hat{T}^e_{2k+1}(q).\end{aligned} \tag{5.7}$$

Note that

$$\begin{aligned}(-1)^n q^{(n+1)^2}+\begin{bmatrix}2n+1\\1\end{bmatrix}(-1)^{n-1}q^{n^2+1}&=(-1)^{n-1}q^{n^2+1}\big(1+q+\cdots+q^{2n-1}\big)\\&=(-1)^n q^{n^2}+\begin{bmatrix}2n+1\\1\end{bmatrix}(-1)^{n-1}q^{n^2}.\end{aligned} \tag{5.8}$$

Inserting (5.8) into (5.7), we see that the recursion (5.1) for $T^e_{2n+1}(q)$ is identical to the recursion for $q^{n+1}\hat{T}^e_{2n+1}(q)$ in (5.7). $\square$

6. Symmetries of the q-Euler numbers

In [20], Ismail and Zhang conjectured that the polynomials $T^o_{2n+1}(q)$ are symmetric about the middle coefficient(s). We prove their conjecture in the following theorem.

Theorem 6.1. *The polynomials $T_{2n+1}(q)$, $T^o_{2n+1}(q)$, and $T^e_{2n+1}(q)$ are symmetric about the middle coefficient(s).*

Proof. For each alternating permutation $\pi=\pi_1\pi_2\cdots\pi_{2n+1}$, the permutation $\overline{\pi}=\pi_{2n+1}\pi_{2n}\cdots\pi_1$ is also an alternating permutation. Recall that $des(\pi_e)=\pi_0\pi_2\cdots\pi_{2n}$ and $des(\overline{\pi}_e)=\pi_0\pi_{2n}\cdots\pi_2$. From the definition of $\overline{\pi}$, it follows that

$$\begin{aligned}n(2n+1)&=inv(\pi)+inv(\overline{\pi}),\\2n(n+1)&=inv(\pi)+des(\pi_o)+inv(\overline{\pi})+des(\overline{\pi}_o),\\2n(n+1)+1&=inv(\pi)+des(\pi_e)+inv(\overline{\pi})+des(\overline{\pi}_e).\end{aligned}$$

Therefore, the inversion map $\pi \to \overline{\pi}$ is a bijection between

$$\begin{aligned}&\big\{\pi \bigm| inv(\pi)=k\big\} \quad \text{and} \quad \big\{\pi \bigm| inv(\pi)=n(2n+1)-k\big\},\\ &\big\{\pi \bigm| inv(\pi)+des(\pi_o)=k\big\} \quad \text{and} \quad \big\{\pi \bigm| inv(\pi)+des(\pi_o)=2n(n+1)-k\big\},\\ &\big\{\pi \bigm| inv(\pi)+des(\pi_e)=k\big\} \quad \text{and} \quad \big\{\pi \bigm| inv(\pi)+des(\pi_e)=2n(n+1)+1-k\big\}.\end{aligned}$$

If $k=n(2n+1)/2$, then $\{\pi \mid inv(\pi)=k\}=\{\pi \mid inv(\pi)=n(2n+1)-k\}$. Otherwise, the two sets are disjoint. It follows that, if n is even, the coefficients of T_{2n+1} are symmetric about the term

$$\big|\big\{\pi \bigm| inv(\pi)=n(2n+1)/2\big\}\big| \cdot q^{n(2n+1)/2}.$$

If n is odd, the coefficients are symmetric about the terms corresponding to $q^{\lfloor n(2n+1)/2\rfloor}$ and $q^{\lfloor n(2n+1)/2\rfloor+1}$. The coefficients of $T^o_{2n+1}(q)$ and $T^e_{2n+1}(q)$ can similarly be seen to be symmetric about the middle term(s). □

The q-secant numbers discussed in Section 3 are symmetric about the middle coefficient.

Theorem 6.2. *The polynomials $S^o_{2n}(q)$ and $\hat{S}^o_{2n}(q)$ are symmetric about the middle coefficient.*

Proof. It suffices to show that $S^o_{2n}(q)$ is symmetric. First note that the alternating permutation with the least weight is

$$\pi = 1\,3\,2\,5\,4\,7\cdots(2n-1)\,(2n-2)\,2n,$$

with $inv(\pi)=n-1$ and $des(\pi_o)=0$; while the alternating permutation with the largest weight is

$$\pi=(2n-1)\,2n\,(2n-3)\,(2n-2)\,\cdots 1\,2,$$

with $inv(\pi)=2n^2-2n$ and $des(\pi_o)=n-1$. Thus

$$S^o_{2n}(q)=q^{n-1}+\cdots+q^{2n^2-n-1}.$$

It is clear that $S^o_2(q)=1$ is symmetric. Suppose $S^o_{2k}(q)$ is symmetric about q^{k^2-1} for $k<n$. Since the q-binomial coefficient $\left[{x \atop y}\right]$ is symmetric,

$$\begin{bmatrix}2n\\2k\end{bmatrix} q^{(n-k)^2} S^o_{2k}(q)$$

is symmetric. The exponent of the middle term is

$$(n-k)^2+\big(2nk-2k^2\big)+k^2-1=n^2-1.$$

Therefore, $S^o_{2n}(q)$ is symmetric about q^{n^2-1}. □

Note. It would be interesting to find a combinatorial proof of Theorem 6.2 analogous to that of Theorem 6.1. We note in passing that the polynomials $S_{2n}(q)$, $S^{des}_{2n}(q)$ and their dual polynomials $\hat{S}_{2n}(q)$, $\hat{S}^{des}_{2n}(q)$ appearing in Theorems 2.5, 2.6, and 2.7 are the only polynomials considered in this paper that are not symmetric about the middle coefficient(s).

Definition. A polynomial $p:\mathbb{C}\to\mathbb{C}$ of degree n is said to be reciprocal if

$$p(z)=\pm z^n p\left(\frac{1}{z}\right). \tag{6.1}$$

The following corollary follows from Theorems 2.3, 3.2 4.2, 5.2, and 6.1.

Corollary 6.3. *The polynomials $\hat{T}^{des}_{2n+1}(q)$, $\hat{T}^{o}_{2n+1}(q)$, $\hat{T}^{e}_{2n+1}(q)$, and $\hat{S}^{o}_{2n}(q)$ are reciprocal. More precisely, if $f(q) \in \{\hat{T}^{des}_{2n+1}(q), \hat{T}^{o}_{2n+1}(q), \hat{T}^{e}_{2n+1}(q), \hat{S}^{o}_{2n}(q) \mid n \geqslant 0\}$, and f has degree n, then*

$$f(1/q) = f(q)/q^n.$$

7. Higher order q-Euler numbers

The tangent numbers of order k are defined by the Taylor series coefficients in the expansion of $\tan^k z$ about $z = 0$. Since

$$\frac{d}{dz}\tan^2 z = \frac{d^2}{dz^2}\tan z,$$

we see that for $n \geqslant 1$, the numbers

$$\frac{d^{2n}}{dz^{2n}}\tan^2 z\bigg|_{z=0} = \frac{d^{2n+1}}{dz^{2n+1}}\tan z\bigg|_{z=0} \tag{7.1}$$

each enumerate the alternating permutations on $[2n+1]$. Equivalently, the first and second order tangent numbers are identical. The q-extensions of second order q-tangent numbers, in contrast, generate polynomials distinct from those of first order. In the following theorem, we offer a combinatorial interpretation for the second order q-tangent numbers arising in the previous sections.

Before we state the theorem, we define the permutation statistics α, β, and γ on A_{2n+1} by

$$\begin{aligned}
\alpha(\pi) &= inv(\pi) + \max(\pi) - 2n - 1,\\
\beta(\pi) &= inv(\pi) + des(\pi_o) + \max(\pi) - 2n - \frac{3 + sign(\pi_{\max(\pi)-1} - \pi_{\max(\pi)+1})}{2},\\
\gamma(\pi) &= inv(\pi) + des(\pi_e) + \max(\pi) - 2n - 2,
\end{aligned}$$

where $\max(\pi)$ denotes the index of $2n+1$ in π. Throughout the section, we denote $\pi_i^j = \pi_i\pi_{i+1}\cdots\pi_j$ for a given permutation π.

Theorem 7.1. *Let $T_{2n+1}(q)$, $T^{o}_{2n+1}(q)$, $T^{e}_{2n+1}(q)$ denote the q-analogues of the tangent numbers defined by Theorems* 2.1, 4.1, *and* 5.1, *respectively. Define $T^{(2)}_{2n}(q)$, $T^{o\,(2)}_{2n}(q)$, $T^{e\,(2)}_{2n}(q)$ by*

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{T^{(2)}_{2n}(q)z^{2n}}{(q;q)_{2n}} &= \left(\sum_{n=0}\frac{T_{2n+1}(q)z^{2n+1}}{(q;q)_{2n+1}}\right)^2,\\
\sum_{n=0}^{\infty}\frac{T^{o\,(2)}_{2n}(q)z^{2n}}{(q;q)_{2n}} &= \left(\sum_{n=0}\frac{T^{o}_{2n+1}(q)z^{2n+1}}{(q;q)_{2n+1}}\right)^2,\\
\sum_{n=0}^{\infty}\frac{T^{e\,(2)}_{2n}(q)z^{2n}}{(q;q)_{2n}} &= \left(\sum_{n=0}\frac{T^{e}_{2n}(q)z^{2n+1}}{(q;q)_{2n+1}}\right)^2.
\end{aligned}$$

Then

$$T^{(2)}_{2n}(q) = \sum_{\pi\in A_{2n+1}} q^{\alpha(\pi)}, \qquad T^{o\,(2)}_{2n}(q) = \sum_{\pi\in A_{2n+1}} q^{\beta(\pi)}, \qquad T^{e\,(2)}_{2n}(q) = \sum_{\pi\in A_{2n+1}} q^{\gamma(\pi)}.$$

Proof. For a permutation π in A_{2n+1}, let $\pi_{2k} = 2n+1$ for some k, $1 \leqslant k \leqslant n$. Then

$$\begin{aligned}
inv(\pi) &= \left|\left\{(i,j) \mid i < 2k < j \text{ and } \pi_i > \pi_j\right\}\right| + \left|\left\{(i,j) \mid i < j < 2k \text{ and } \pi_i > \pi_j\right\}\right|\\
&\quad + \left|\left\{(i,j) \mid 2k < i < j \text{ and } \pi_i > \pi_j\right\}\right| + \left|\left\{(2k,j) \mid 2k < j\right\}\right|\\
&= \left|\left\{(i,j) \mid i < 2k < j \text{ and } \pi_i > \pi_j\right\}\right| + inv\left(\pi_1^{2k-1}\right) + inv\left(\pi_{2k+1}^{2n+1}\right) + 2(n-k) + 1.
\end{aligned}$$

Thus,

$$\alpha(\pi) = \left|\left\{(i,j) \mid i < 2k < j \text{ and } \pi_i > \pi_j\right\}\right| + inv\left(\pi_1^{2k-1}\right) + inv\left(\pi_{2k+1}^{2n+1}\right).$$

We now show that $T_{2n}^{(2)}(q)$ is the generating function for permutations in A_{2n+1} with weight α. The polynomials $T_{2n}^{(2)}(q)$ satisfy

$$T_{2n}^{(2)}(q) = \sum_{k=1}^{n} \begin{bmatrix} 2n \\ 2k-1 \end{bmatrix} T_{2k-1}(q) T_{2(n-k)+1}(q). \tag{7.2}$$

By Lemma 1.1, the q-binomial coefficient $\left[\begin{smallmatrix} 2n \\ 2k-1 \end{smallmatrix}\right]$ in the summand on the right-hand side of (7.2) counts the inversions between the two sub-permutations π_1^{2k-1} and π_{2k+1}^{2n+1}, namely

$$\left|\left\{(i,j) \mid i < 2k < j \text{ and } \pi_i > \pi_j\right\}\right|;$$

$T_{2k-1}(q)$ and $T_{2(n-k)+1}(q)$ count the inversions of π_1^{2k-1} and π_{2k+1}^{2n+1}, respectively. Therefore, $T_{2n}^{(2)}(q)$ is the generating function for permutations π in A_{2n+1} with weight $\alpha(\pi)$. The arithmetic interpretations for the polynomials $T^{o\,(2)}_{2n}(q)$ and $T^{e\,(2)}_{2n}(q)$ are similarly derived. We omit the details. □

In the next theorem, we present a corresponding interpretation for the second order q-secant numbers studied in Sections 2 and 3. The proof for each interpretation is similar to that of Theorem 7.1.

Theorem 7.2. *Let $S_{2n}(q)$ and $S_{2n}^{o}(q)$ denote the q-analogues of the secant numbers defined by Theorems* 2.5 *and* 3.1, *respectively. Define $S_{2n}^{(2)}(q)$ and $S^{o\,(2)}_{2n}(q)$ by*

$$\sum_{n=0}^{\infty} \frac{S_{2n}^{(2)}(q) z^{2n}}{(q;q)_{2n}} = \left(\sum_{n=0} \frac{S_{2n}(q) z^{2n}}{(q;q)_{2n}}\right)^2,$$
$$\sum_{n=0}^{\infty} \frac{S^{o\,(2)}_{2n}(q) z^{2n}}{(q;q)_{2n}} = \left(\sum_{n=0} \frac{S^{o}_{2n}(q) z^{2n}}{(q;q)_{2n}}\right)^2.$$

Then

$$S_{2n}^{(2)}(q) = \sum_{\pi \in A_{2n+1}} q^{\delta(\pi)},$$
$$S^{o\,(2)}_{2n}(q) = \sum_{\pi \in A_{2n+1}} q^{\upsilon(\pi)} = \sum_{\pi \in A_{2n+1}} q^{\omega(\pi)},$$

where

$$\delta(\pi) = inv(\pi) - 2 \cdot inv\left(\pi_{\min(\pi)+1}^{2n+1}\right) + \binom{2n+1-\min(\pi)}{2} - \min(\pi) + 1,$$
$$\upsilon(\pi) = \delta(\pi) + des(\pi_o) - 2 \cdot des\left(\left(\pi_{\min(\pi)+1}^{2n+1}\right)_o\right) + n - \frac{\min(\pi)+3}{2},$$
$$\begin{aligned}\omega(\pi) = {}& \delta(\pi) + des(\pi_e) - 2 \cdot des\left(\left(\pi_{\min(\pi)+1}^{2n+1}\right)_e\right) \\ & + n - \frac{\min(\pi) + 2 + sign(\pi_{\max(\pi)-1} - \pi_{\max(\pi)+1})}{2} \\ & + sign(\pi_{2n+1} - 1) + sign(\pi_1 - 1) - 2.\end{aligned}$$

Proof. For a permutation π in A_{2n+1}, let $\pi_{2k+1} = 1$ for some k, $0 \leqslant k \leqslant n$. Then

$$\begin{aligned} inv(\pi) &= \left|\left\{(i,j) \mid i < 2k+1 < j \text{ and } \pi_i > \pi_j\right\}\right| + \left|\left\{(i,j) \mid i < j < 2k+1 \text{ and } \pi_i > \pi_j\right\}\right| \\ &\quad + \left|\left\{(i,j) \mid 2k+1 < i < j \text{ and } \pi_i > \pi_j\right\}\right| + \left|\left\{(i,2k+1) \mid i < 2k+1\right\}\right| \\ &= \left|\left\{(i,j) \mid i < 2k+1 < j \text{ and } \pi_i > \pi_j\right\}\right| + inv\left(\pi_1^{2k}\right) + inv\left(\pi_{2k+2}^{2n+1}\right) + 2k. \end{aligned}$$

Thus,

$$\begin{aligned} \delta(\pi) &= \left|\left\{(i,j) \mid i < 2k+1 < j \text{ and } \pi_i > \pi_j\right\}\right| + inv\left(\pi_1^{2k}\right) - inv\left(\pi_{2k+2}^{2n+1}\right) + \binom{2(n-k)}{2} \\ &= \left|\left\{(i,j) \mid i < 2k+1 < j \text{ and } \pi_i > \pi_j\right\}\right| + inv\left(\pi_1^{2k}\right) + inv\left(\overline{\pi_{2k+2}^{2n+1}}\right), \end{aligned} \tag{7.3}$$

where $\overline{\pi_{2k+2}^{2n+1}} = \pi_{2n+1}\pi_{2n}\cdots\pi_{2k+3}\pi_{2k+2}$.

We now show that $S_{2n}^{(2)}(q)$ is the generating function of permutations in A_{2n+1} with weight δ. The polynomials $S_{2n}^{(2)}(q)$ satisfy

$$S_{2n}^{(2)}(q) = \sum_{k=0}^{n} \begin{bmatrix} 2n \\ 2k \end{bmatrix} S_{2k}(q) S_{2(n-k)}(q). \tag{7.4}$$

By Lemma 1.1, the q-binomial coefficient $\left[\begin{smallmatrix} 2n \\ 2k \end{smallmatrix}\right]$ in the summand on the right-hand side of (7.4) counts the inversions between the two sub-permutations π_1^{2k} and $\overline{\pi_{2k+2}^{2n+1}}$, namely

$$\left|\left\{(i,j) \mid i < 2k+1 < j \text{ and } \pi_i > \pi_j\right\}\right|;$$

$S_{2k}(q)$ and $S_{2(n-k)}(q)$ count the inversions of π_1^{2k} and $\overline{\pi_{2k+1}^{2n+1}}$, respectively, since $\pi_1^{2k} \in A_{2k}$ and $\overline{\pi_{2k+1}^{2n+1}} \in A_{2(n-k)}$. Therefore, $S_{2n}^{(2)}(q)$ is the generating function for permutations π in A_{2n+1} with weight $\delta(\pi)$.

To obtain the claimed arithmetic interpretation for the polynomials $S^{o}{}_{2n}^{(2)}(q)$, note that, with the convention that $des(\emptyset) = 0$, where $\emptyset$ denotes the empty permutation,

$$\begin{aligned} &des(\pi_{2n+1}\pi_{2n-1}\ldots\pi_{\min(\pi)+2}) + sign(\pi_{2n+1} - 1) - 1 \\ &\quad = \frac{2n - \min(\pi) - 1}{2} - des(\pi_{\min(\pi)+2}\pi_{\min(\pi)+4}\cdots\pi_{2n+1}). \end{aligned}$$

Hence, by (7.3), we find that

$$\begin{aligned} \upsilon(\pi) &= \delta(\pi) + des(\pi_o) - 2 \cdot des\left(\left(\pi_{\min(\pi)+2}^{2n+1}\right)_o\right) + n - \frac{\min(\pi)+3}{2} \\ &= \delta(\pi) + des(\pi_1\pi_3\cdots\pi_{\min(\pi)-2}) - des(\pi_{\min(\pi)+2}\pi_{\min(\pi)+3}\cdots\pi_{2n+1}) \\ &\quad + \frac{2n - \min(\pi) - 1}{2} \\ &= \delta(\pi) + des(\pi_1\pi_3\cdots\pi_{\min(\pi)-2}) + des(\pi_{2n+1}\pi_{2n-1}\cdots\pi_{\min(\pi)+2}) \\ &\quad + sign(\pi_{2n+1} - 1) - 1 \\ &= \left|\left\{(i,j) \mid i < \min(\pi) < j \text{ and } \pi_i > \pi_j\right\}\right| + inv\left(\pi_1^{\min(\pi)-1}\right) + des\left(\left(\pi_1^{\min(\pi)-1}\right)_o\right) \\ &\quad + inv\left(\overline{\pi_{\min(\pi)+1}^{2n+1}}\right) + des\left(\left(\overline{\pi_{\min(\pi)+1}^{2n+1}}\right)_o\right) + sign(\pi_{2n+1} - 1) - 1, \end{aligned}$$

where $\overline{\pi_{\min(\pi)+1}^{2n+1}} = \pi_{2n+1}\pi_{2n}\cdots\pi_{2k+3}\pi_{\min(\pi)+1}$. Therefore, since $S_0^o(q) = q^{-1}$, by Lemma 1.1, Theorem 3.1, and the definition of ${S^o}_{2n}^{(2)}(q)$, we see that

$$ {S^o}_{2n}^{(2)}(q) = \sum_{\pi \in A_{2n+1}} q^{\upsilon(\pi)}. $$

By squaring the generating function for $S_{2n}^e(q)$ defined by (3.6) and using the fact that $S_{2n}^e(q) = S_{2n}^o(q)$, we readily observe that the weight $\omega(\pi)$ corresponds to the same enumeration for A_{2n+1} as $\upsilon(\pi)$. We omit the details. □

8. Concluding remarks

Define the Bell polynomials $B_{n,k}(x_1, x_2, \ldots, x_{n-k+1})$ via the generating function [12]

$$ \exp\left(u\sum_{m=1}^{\infty}\frac{x_m z^m}{m!}\right) = 1 + \sum_{n=1}^{\infty}\sum_{k=1}^{n} B_{n,k}(x_1, x_2, x_{n-k+1})\frac{u^k z^n}{n!}. $$

Then, by Faá Di Bruno's formula [12, p. 137] and Theorem 2.5,

$$ S_{2n}(q) = \frac{(q;q)_{2n}}{(2n)!}\left(1 + \sum_{v=1}^{2n}(-1)^v v! B_{2n,v}(\alpha_1, \ldots, \alpha_{2n-v+1})\right), \tag{8.1} $$

where

$$ \alpha_k = \begin{cases} (-1)^{k/2} k!/(q;q)_k, & \text{if } k \text{ is even}, \\ 0, & \text{if } k \text{ is odd}. \end{cases} $$

From (2.1) and (2.7), we obtain

$$ T_{2n+1}(q) = \sum_{j=0}^{n} \begin{bmatrix} 2n+1 \\ 2j \end{bmatrix} (-1)^{n-j} S_{2j}(q). \tag{8.2} $$

Closed formulas and relations for the other q-Euler numbers can be similarly derived.

As mentioned in the Introduction, the generalized tangent numbers $T_{2n+1}^o(q)$ are the polynomials arising in the coefficient of $\psi^{-2}(q)$ in $(q;q)_{2n+1} y_{2n+1}$ of (1.2). Constant multiples of the second order extensions ${T^o}_{2n+1}^{(2)}(q)$ appear in (1.2) as the coefficient of $\psi^{-4}(q)$ in the corresponding expansion of $(q;q)_{2n} y_{2n}$ for $n \geqslant 2$ [18, Theorem 3.4]. Ismail and Zhang [20, Theorem 4.1] prove that each y_j can be expressed as a polynomial in certain elliptic parameters over the field of rational functions in q. The authors of [20] suggest that polynomials appearing in the numerators of these expansions, denoted by $D_{r,s,t}(q)$, have interesting combinatorial properties. Our study of $T_{2n+1}^o(q)$ explicitly addresses the combinatorics of the polynomials $D_{r,0,1}(q)$ and $D_{r,0,2}(q)$. Recursion formulas for y_j appearing in [17,18] show that, in general, the polynomials $D_{r,s,t}(q)$ arise as linear combinations of finite products $\prod_j (T_{2n_j+1}^o(q))^{m_j}$. The results of the present paper explain the symmetry of the polynomials appearing in expansions for y_j observed by the authors of [20].

9. Addendum

A point of clarification is necessary to reconcile the polynomials appearing in [20] with those of the present paper. The polynomials $D_{r,s,t}(q)$ arise as coefficients in the series expansions for the zeros of

$$ R(x,q) := \sum_{n=0}^{\infty} \frac{q^{n^2} x^n}{(q;q)_n}. \tag{9.1} $$

Due to minor misprints, the polynomials $D_{r,0,t}$ appearing in [20, p. 374] are not in accordance with the polynomials $T^{o}_{2n+1}(q)$ studied here. For completeness, we include a corrected version of the relevant results in [20, Sections 3 and 4]. A more detailed discussion appears in [18, §4].

Theorem 9.1. *Let*

$$0 < i_1(q) < i_2(q) < \cdots < i_n(q) < \cdots \tag{9.2}$$

be the zeros of $R(-x,q)$, *and define* $\xi_n(q)$, X, u *such that*

$$i_n(q) := q^{-2n+1}\xi_n(q), \qquad X = q^n, \qquad u = X/\sqrt{\xi_n(q)}. \tag{9.3}$$

Let $\psi(q)$ *be defined by* (1.4) *and* $\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}$. *Denote*

$$h(x) := \frac{x}{(1-q)\psi^2(q)} \frac{{}_1\phi_1(0;q^3;q^2,q^2x^2)}{{}_1\phi_1(0;q;q^2,qx^2)} \tag{9.4}$$

and

$$g(x) := \sum_{m=0}^{\infty} \varphi^{4m}(q) \frac{(-1)^m x^{2m+1}}{2m+1} {}_2F_1\left(\begin{matrix} -m, 1/2 \\ 1 \end{matrix} \,\middle|\, \frac{\varphi^4(q) - 16q\psi^4(q^2)}{\varphi^4(q)}\right). \tag{9.5}$$

Then, provided the aforementioned series converge,

$$X = u\exp(-g\circ h\circ u). \tag{9.6}$$

Moreover, if we denote by $u(X)$ *the inverse function* $X \mapsto u$ *determined by* (9.6), *then*

$$\xi_n(q) = \exp\bigl(-2g\circ h\circ u(X)\bigr). \tag{9.7}$$

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