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Applied Mathematics Letters 16 (2003) 365–368

Applied Mathematics Letters

www.elsevier.com/locate/aml

A Note on the Bernoulli and Euler Polynomials

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(Received November 2001; accepted April 2002)

Abstract—In this paper, we obtain a simple property of the Bernoulli polynomials $B_n(x)$ and the Euler polynomials $E_n(x)$. As a consequence, the relationship between two polynomials is obtained from

$$B_n(x) = \sum_{\substack{k=0 \\ k\neq 1}}^{n} \binom{n}{k} B_k(0) E_{n-k}(x).$$

Keywords—Bernoulli polynomials, Euler polynomials.

We begin our study with classical two polynomials, Bernoulli polynomials $B_n(x)$ and Euler polynomials $E_n(x)$ having the following exponential generating functions:

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \qquad \text{and} \qquad \frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}.$$

Thus, the first four such polynomials, respectively, are

$$B_0(x) = 1, \quad B_1(x) = x - \frac{1}{2}, \quad B_2(x) = x^2 - x + \frac{1}{6}, \quad B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x,$$
$$E_0(x) = 1, \quad E_1(x) = x - \frac{1}{2}, \quad E_2(x) = x^2 - x, \quad E_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{4}.$$

It is known (see [1,2]) that there are explicit formulas for $B_n(x)$ and $E_n(x)$, respectively,

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}, \tag{1}$$

$$E_n(x) = \frac{1}{n+1} \sum_{k=1}^{n+1} \left(2 - 2^{k+1}\right) \binom{n+1}{k} B_k x^{n+1-k}, \tag{2}$$

where $B_k := B_k(0)$ is the Bernoulli number for each $k = 0, 1, \ldots, n$.

This work was supported by the Daejin University research grants in 2001.

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PII: S0893-9659(02)00206-9

These two polynomials have many similar properties (see [1,2]). One of these properties is

$$B_n(x+1) - B_n(x) = nx^{n-1}, \tag{3}$$

$$E_n(x+1) + E_n(x) = 2x^n. \tag{4}$$

The purpose of this paper is to obtain interesting properties of the Bernoulli and Euler polynomials, and the relationship between such polynomials. This has probably not been realised before.

First, we need the following identity:

$$\binom{n}{k}\binom{n-k}{j} = \binom{n}{j+k}\binom{j+k}{k}. \tag{5}$$

THEOREM 1. *For any integer $n \geq 0$, we have*

(a) $B_n(x+1) = \sum_{k=0}^{n} \binom{n}{k} B_k(x)$;
(b) $E_n(x+1) = \sum_{k=0}^{n} \binom{n}{k} E_k(x)$.

PROOF. Applying (1) and (5),

$$\begin{aligned} B_n(x+1) &= \sum_{k=0}^{n} \binom{n}{k} B_k (x+1)^{n-k} \\ &= \sum_{k=0}^{n} B_k \sum_{j=0}^{n-k} \binom{n}{k}\binom{n-k}{j} x^j \\ &= \sum_{k=0}^{n} B_k \sum_{j=0}^{n-k} \binom{n}{j+k}\binom{j+k}{k} x^j. \end{aligned}$$

Expanding the last expression gives

$$\begin{aligned} &\binom{n}{0}\left\{\binom{0}{0}B_0\right\} + \binom{n}{1}\left\{\binom{1}{0}B_0 x + \binom{1}{1}B_1\right\} + \cdots \\ &\quad + \binom{n}{n}\left\{\binom{n}{0}B_0 x^n + \binom{n}{1}B_1 x^{n-1} + \cdots + \binom{n}{n}B_n\right\} \\ &= \sum_{k=0}^{n} \binom{n}{k} \sum_{j=0}^{k} \binom{k}{j} B_j x^{k-j} \\ &= \sum_{k=0}^{n} \binom{n}{k} B_k(x), \end{aligned}$$

which proves (a). By similar arguments, we can prove (b) easily. ∎

From (3),(4) and Theorem 1, we obtain for any integer $n \geq 0$,

$$\sum_{k=0}^{n} \binom{n+1}{k} B_k(x) = (n+1)x^n, \tag{6}$$

$$\sum_{k=0}^{n} \binom{n}{k} E_k(x) + E_n(x) = 2x^n. \tag{7}$$

In many contexts (see [3–5]), a number of interesting and useful identities for combinatorial numbers are obtained from a matrix representation of a particular counting sequence, for example, the Pascal numbers and the Stirling numbers, etc. Such a matrix representation provides a powerful computational tool for deriving identities and an explicit formula for a given sequence.

Let $B(x)$, $E(x)$, and $X(x)$ be the $(n+1)\times 1$ matrices defined by

$$B(x) = \begin{bmatrix} B_0(x) \\ B_1(x) \\ \vdots \\ B_n(x) \end{bmatrix}, \qquad E(x) = \begin{bmatrix} E_0(x) \\ E_1(x) \\ \vdots \\ E_n(x) \end{bmatrix}, \qquad X(x) = \begin{bmatrix} 1 \\ x \\ \vdots \\ x^n \end{bmatrix},$$

and let P_{n+1} and Q_{n+1} be the $(n+1)\times(n+1)$ lower triangular matrices defined by

$$[P_{n+1}]_{ij} = \begin{cases} \binom{i-1}{j-1}, & \text{if } i \geq j, \\ 0, & \text{otherwise,} \end{cases}$$

$$[Q_{n+1}]_{ij} = \begin{cases} \frac{1}{i}\binom{i}{j-1}, & \text{if } i \geq j, \\ 0, & \text{otherwise.} \end{cases}$$

Then (6) and (7) can be represented as matrix systems of equations for each $n = 0, 1, \dots,$ respectively,

$$Q_{n+1}B(x) = X(x), \tag{8}$$

$$\frac{1}{2}\left(P_{n+1} + I_{n+1}\right)E(x) = X(x), \tag{9}$$

where I_{n+1} is the identity matrix of order $n+1$.

Noticing (8) and (9), the Bernoulli and Euler polynomials can be obtained simply from the following matrix equations, respectively:

$$B(x) = Q_{n+1}^{-1}X(x),$$
$$E(x) = 2(P_{n+1} + I_{n+1})^{-1}X(x),$$

and moreover, two matrices obtained from both

$$Q_{n+1}\left[B(0) \quad B(1) \quad \cdots \quad B(n)\right] \qquad \text{and} \qquad \frac{1}{2}\left(P_{n+1} + I_{n+1}\right)\left[E(0) \quad E(1) \quad \cdots \quad E(n)\right],$$

by deleting the first row and first column, represent the Vandermonde matrix

$$\begin{bmatrix} 1 & 2 & \cdots & n \\ 1^2 & 2^2 & \cdots & n^2 \\ \vdots & \vdots & \vdots & \vdots \\ 1^n & 2^n & \cdots & n^n \end{bmatrix}.$$

This suggests that the sum of powers of the first n positive integers can be expressed in terms of the Bernoulli and Euler polynomials, respectively.

Now, we obtain some relationships between the Bernoulli and Euler polynomials.

Let $\hat{Q}_n$ be the $n\times n$ matrix obtained from Q_n^{-1} by replacing its $(2,1)$,$(3,2)$,$\dots$,$(n,n-1)$ entries with zeros.

LEMMA 2.

$$\frac{1}{2}(P_n + I_n) = Q_n\hat{Q}_n. \tag{10}$$

PROOF. First, note that from (1) the (i,j)-entry of Q_n^{-1} is $\binom{i-1}{j-1}B_{i-j}$ for integers i and j with $i \geq j$. Applying identity (5) gives

$$\begin{aligned}\left[Q_n^{-1}(P_n + I_n)\right]_{ij} &= \sum_{t=j}^{i} \binom{i-1}{t-1}\binom{t-1}{j-1} B_{i-t} + \binom{i-1}{j-1} B_{i-j} \\ &= \sum_{t=j+1}^{i} \binom{i-1}{t-1}\binom{t-1}{j-1} B_{i-t} + 2\binom{i-1}{j-1} B_{i-j} \\ &= \sum_{t=j+1}^{i} \binom{i-1}{j-1}\binom{i-j}{t-j} B_{i-t} + 2\binom{i-1}{j-1} B_{i-j} \\ &= \binom{i-1}{j-1} \sum_{t=j+1}^{i} \binom{i-j}{i-t} B_{i-t} + 2\binom{i-1}{j-1} B_{i-j} \\ &= \binom{i-1}{j-1} \sum_{k=0}^{i-j-1} \binom{i-j}{k} B_k + 2\binom{i-1}{j-1} B_{i-j}.\end{aligned} \tag{11}$$

Substituting $x = 0$ in (6) obtains

$$\sum_{k=0}^{i-j-1} \binom{i-j}{k} B_k = \begin{cases} B_0, & \text{if } i = j+1, \\ 0, & \text{if } i \neq j+1. \end{cases}$$

From (11), we have

$$\left[Q_n^{-1}(P_n + I_n)\right]_{ij} = \begin{cases} j(B_0 + 2B_1) = 0, & \text{if } i = j+1, \\ 2\binom{i-1}{j-1} B_{ij}, & \text{if } i \neq j+1. \end{cases}$$

It implies that

$$\left[Q_n^{-1}(P_n + I_n)\right]_{ij} = 2\left[\hat{Q}_n\right]_{ij},$$

which completes the proof. ∎

From (8)–(10), we have

$$B(x) = \hat{Q}_{n+1} E(x). \tag{12}$$

The following theorem is an immediate consequence of (12).

THEOREM 3. *The Bernoulli polynomials can be expressed by the Euler polynomials as*

$$B_n(x) = \sum_{\substack{k=0 \\ k\neq 1}}^{n} \binom{n}{k} B_k E_{n-k}(x). \tag{13}$$

Comparing (1) with (13), perhaps more interesting is the fact that the Bernoulli polynomials $B_n(x)$ are the same as the polynomials obtained by replacing x^k in $B_n(x)$ with the Euler polynomials $E_k(x)$ for each $k = 0, 1, \ldots, n$ $(k \neq n-1)$, and by replacing x^{n-1} with 0.

For example,

$$\begin{aligned} B_3(x) &= x^3 - \frac{3}{2}x^2 + \frac{1}{2}x \\ &= E_3(x) - \frac{3}{2}(0)^2 + \frac{1}{2}E_1(x) \\ &= \left(x^3 - \frac{3}{2}x^2 + \frac{1}{4}\right) + \frac{1}{2}\left(x - \frac{1}{2}\right). \end{aligned}$$

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