



# Evaluations of Some Variant Euler Sums

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## Abstract

In this note we present some elementary methods for the summation of certain Euler sums with terms involving  $1 + 1/3 + 1/5 + \cdots + 1/(2k - 1)$ .

## 1 Introduction

In the last decade, based on extensive experimentation with computer algebraic systems, a large class of Euler sums have been explicitly evaluated in terms of the Riemann zeta function  $\zeta(k)$ . For example, let

$$H_k = 1 + \frac{1}{2} + \cdots + \frac{1}{k}.$$

Then

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{1}{k^2} H_k &= 2\zeta(3), \\ \sum_{k=1}^{\infty} \frac{1}{2^k k^2} H_k &= \zeta(3) - \frac{\pi^2}{12} \ln 2, \\ \sum_{k=1}^{\infty} \frac{1}{k^2} H_k^2 &= \frac{17}{4} \zeta(4), \end{aligned}$$

and

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2} H_k = \frac{5}{8} \zeta(3).$$

More details can be found in [1, 2, 3, 4] In particular, Borwein and Bradley [3] collected 32 beautiful proofs of the first sum above.

Motivated by the above results, in this note, replacing  $H_k$  by

$$h_k = H_{2k} - \frac{1}{2} H_k = 1 + \frac{1}{3} + \cdots + \frac{1}{2k-1}, \quad (1)$$

we study the following variant Euler sums

$$\sum_{k=1}^{\infty} a_k h_k$$

where the  $a_k$  are relatively simple function of  $k$ .

## 2 The Main Results

We begin to derive some series involving  $h_k$ . Since

$$-\ln(1-x) = \int_0^x \frac{dt}{1-t} = \sum_{k=1}^{\infty} \frac{x^k}{k},$$

replacing  $x$  by  $-x$  gives

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^k}{k}.$$

Averaging these two series gives us

$$\frac{1}{2} \ln \left( \frac{1+x}{1-x} \right) = \sum_{k=1}^{\infty} \frac{1}{2k-1} x^{2k-1}. \quad (2)$$

In term of the Cauchy product and partial fractions, we have

$$\begin{aligned} \frac{1}{4} \ln^2 \left( \frac{1+x}{1-x} \right) &= \sum_{k=1}^{\infty} \left( \frac{1}{(2k-1) \cdot 1} + \frac{1}{(2k-3) \cdot 3} + \cdots + \frac{1}{1 \cdot (2k-1)} \right) x^{2k} \\ &= \sum_{k=1}^{\infty} \frac{1}{2k} \left[ \left( \frac{1}{2k-1} + \frac{1}{1} \right) + \left( \frac{1}{2k-3} + \frac{1}{3} \right) + \cdots + \left( \frac{1}{1} + \frac{1}{2k-1} \right) \right] x^{2k} \\ &= \sum_{k=1}^{\infty} \left( 1 + \frac{1}{3} + \cdots + \frac{1}{2k-1} \right) \frac{x^{2k}}{k}. \end{aligned}$$

Noting that  $h_k$  is given by (1), we have

$$\sum_{k=1}^{\infty} \frac{h_k}{k} x^{2k} = \frac{1}{4} \ln^2 \left( \frac{1+x}{1-x} \right). \quad (3)$$

This enables us to evaluate a wide variety of interesting series via specialization, differentiation and integration.

First, setting  $x = 1/2$ , we find

$$\sum_{k=1}^{\infty} \frac{h_k}{2^{2k} k} = \frac{1}{4} \ln^2 3. \quad (4)$$

For  $x = \sqrt{2}/2$ ,

$$\sum_{k=1}^{\infty} \frac{h_k}{2^k k} = \frac{1}{4} \ln^2(3 + 2\sqrt{2}). \quad (5)$$

Putting  $x = (\sqrt{5} - 1)/2 = \phi$ , the golden ratio, we get

$$\sum_{k=1}^{\infty} \frac{h_k}{k} \phi^{2k} = \frac{1}{4} \ln^2(2 + \sqrt{5}). \quad (6)$$

Furthermore, for any  $\alpha \geq 2$ , putting  $x = (\sqrt{5}+1)/2\alpha$  and  $x = (\sqrt{5}-1)/2\alpha$  in (3) respectively, we get

$$\sum_{k=1}^{\infty} \frac{h_k}{\alpha^{2k} k} \left( \frac{\sqrt{5}+1}{2} \right)^{2k} = \frac{1}{4} \ln^2 \left( \frac{(2\alpha+1) + \sqrt{5}}{(2\alpha-1) - \sqrt{5}} \right) \quad (7)$$

and

$$\sum_{k=1}^{\infty} \frac{h_k}{\alpha^{2k} k} \left( \frac{\sqrt{5}-1}{2} \right)^{2k} = \frac{1}{4} \ln^2 \left( \frac{(2\alpha-1) + \sqrt{5}}{(2\alpha+1) - \sqrt{5}} \right). \quad (8)$$

Recalling the Fibonacci numbers which are defined by

$$F_1 = 1, F_2 = 1, F_k = F_{k-1} + F_{k-2} \text{ for } k \geq 2$$

and Binet's formula

$$F_k = \frac{1}{\sqrt{5}} \left( \left( \frac{\sqrt{5}+1}{2} \right)^k - \left( \frac{1-\sqrt{5}}{2} \right)^k \right),$$

combining (7) and (8), we find

$$\sum_{k=1}^{\infty} \frac{h_k}{\alpha^{2k} k} F_{2k} = \frac{\sqrt{5}}{20} \ln \left( \frac{\alpha^2 + \alpha - 1}{\alpha^2 - \alpha - 1} \right) \ln \left( \frac{\alpha^2 + \alpha\sqrt{5} + 1}{\alpha^2 - \alpha\sqrt{5} + 1} \right). \quad (9)$$

In particular, for  $\alpha = 2$

$$\sum_{k=1}^{\infty} \frac{h_k}{2^{2k} k} F_{2k} = \frac{\sqrt{5}}{4} \ln 5 \ln(9 + 4\sqrt{5}). \quad (10)$$

Another step along this path is to change variables. Setting  $x = \cos \theta$  in (3) leads to

$$\sum_{k=1}^{\infty} \frac{h_k}{k} \cos^{2k} \theta = \ln^2 (\cot(x/2)). \quad (11)$$

Integrating both sides from 0 to  $\pi$ , and using

$$\int_0^{\pi} \cos^{2k} \theta \, d\theta = \frac{\pi}{2^{2k}} \binom{2k}{k}$$

and

$$\int_0^{\pi} \ln^2 (\cot(x/2)) \, d\theta = \frac{\pi^3}{4},$$

we find

$$\sum_{k=1}^{\infty} \frac{h_k}{2^{2k} k} \binom{2k}{k} = \frac{\pi^2}{4}. \quad (12)$$

This adds another interesting series to Lehmer's list [6].

Next, for  $0 < x < 1$ , differentiating (3), then multiplying both sides by  $x$ , we obtain

$$\sum_{k=1}^{\infty} h_k x^{2k} = \frac{x}{2(1-x^2)} \ln \left( \frac{1+x}{1-x} \right). \quad (13)$$

Setting  $x = 1/2$ , we get

$$\sum_{k=1}^{\infty} \frac{h_k}{2^{2k}} = \frac{1}{3} \ln 3. \quad (14)$$

For  $x = \sqrt{2}/2$ ,

$$\sum_{k=1}^{\infty} \frac{h_k}{2^k} = \frac{\sqrt{2}}{2} \ln(3 + 2\sqrt{2}). \quad (15)$$

Similar to (10), we have

$$\sum_{k=1}^{\infty} \frac{h_k}{2^{2k}} F_{2k} = \frac{\sqrt{5}}{50} (10 \ln(5 + 2\sqrt{5}) + 3\sqrt{5} \ln 5 - 5 \ln 5). \quad (16)$$

Finally, for  $0 < x \leq 1$ , dividing both sides of (3) by  $x$  and integrating from 0 to  $x$ , we obtain

$$\sum_{k=1}^{\infty} \frac{h_k}{k^2} x^{2k} = \frac{1}{2} \int_0^x \frac{1}{t} \ln^2 \left( \frac{1+t}{1-t} \right) dt. \quad (17)$$

Using the substitution  $u = (1-x)/(1+x)$  and integration by parts, we get

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{h_k}{k^2} x^{2k} &= \int_{(1-x)/(1+x)}^1 \frac{\ln^2 u}{1-u^2} du \\ &= \frac{1}{2} \ln x \ln^2 \left( \frac{1-x}{1+x} \right) + \int_{(1-x)/(1+x)}^1 \frac{\ln u}{u} \ln \left( \frac{1-u}{1+u} \right) du. \end{aligned}$$

In view of (2), we have

$$\int_{(1-x)/(1+x)}^1 \frac{\ln u}{u} \ln \left( \frac{1-u}{1+u} \right) du = -2 \sum_{k=0}^{\infty} \frac{1}{2k+1} \int_{(1-x)/(1+x)}^1 u^{2k} \ln u du.$$

Since

$$\int u^{2k} \ln u du = \frac{1}{2k+1} u^{2k+1} \ln u - \frac{1}{(2k+1)^2} u^{2k+1} + C,$$

we find

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{h_k}{k^2} x^{2k} &= \frac{1}{2} \ln x \ln^2 \left( \frac{1-x}{1+x} \right) + 2 \ln \left( \frac{1-x}{1+x} \right) \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \left( \frac{1-x}{1+x} \right)^{2k+1} \\ &\quad + 2 \sum_{k=0}^{\infty} \frac{1}{(2k+1)^3} - 2 \sum_{k=0}^{\infty} \frac{1}{(2k+1)^3} \left( \frac{1-x}{1+x} \right)^{2k+1}. \end{aligned} \quad (18)$$

In terms of the polylogarithm function [5]

$$Li_n(x) = \sum_{k=1}^{\infty} \frac{x^n}{k^n},$$

and noting that

$$\sum_{k=0}^{\infty} \frac{x^n}{(2k+1)^n} = \frac{1}{2} (Li_n(x) - Li_n(-x))$$

and

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^3} = \sum_{k=0}^{\infty} \frac{1}{k^3} - \sum_{k=0}^{\infty} \frac{1}{(2k)^3} = \frac{7}{8} \zeta(3),$$

we finally obtain

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{h_k}{k^2} x^{2k} &= \frac{7}{4} \zeta(3) + \frac{1}{2} \ln x \ln^2 \left( \frac{1-x}{1+x} \right) \\ &\quad + \ln \left( \frac{1-x}{1+x} \right) \left( Li_2 \left( \frac{1-x}{1+x} \right) - Li_2 \left( \frac{x-1}{1+x} \right) \right) - \left( Li_3 \left( \frac{1-x}{1+x} \right) - Li_3 \left( \frac{x-1}{1+x} \right) \right). \end{aligned}$$

Setting  $x = 1$ , we get

$$\sum_{k=1}^{\infty} \frac{h_k}{k^2} = \frac{7}{4} \zeta(3). \quad (19)$$

For  $x = 1/3$ ,

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{h_k}{3^{2k} k^2} &= \frac{7}{8} \zeta(3) - \frac{1}{2} \ln 3 \ln^3 2 \\ &\quad + \frac{1}{3} \ln^3 2 + \ln 2 Li_2(-1/2) + Li_3(-1/2), \end{aligned}$$

where we have used

$$\begin{aligned} Li_2(1/2) &= \frac{\pi^2}{12} - \frac{1}{2} \ln^2 2; \\ Li_3(1/2) &= \frac{7}{8} \zeta(3) + \frac{1}{6} \ln^3 2 - \frac{\pi^2}{12} \ln 2. \end{aligned}$$

Moreover, noting that

$$h_k = \sum_{i=1}^k \int_0^1 x^{2(i-1)} dt = \int_0^1 \left( \sum_{i=1}^k x^{2(i-1)} \right) dt = \int_0^1 \frac{1-x^{2k}}{1-x^2} dx$$

and rewriting (8) as

$$\sum_{k=1}^{\infty} \frac{h_k}{k^2} (1-x^{2k}) = \frac{1}{2} \int_x^1 \frac{1}{t} \ln^2 \left( \frac{1+t}{1-t} \right) dt,$$

we have

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{h_k^2}{k^2} &= \sum_{k=1}^{\infty} \frac{h_k}{k^2} \int_0^1 \frac{1-x^{2k}}{1-x^2} dx \\ &= \frac{1}{2} \int_0^1 \left( \frac{1}{1-x^2} \int_x^1 \frac{1}{t} \ln^2 \left( \frac{1+t}{1-t} \right) dt \right) dx. \end{aligned}$$

Exchanging the order of the integration, we get

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{h_k^2}{k^2} &= \frac{1}{2} \int_0^1 \left( \frac{1}{t} \ln^2 \left( \frac{1+t}{1-t} \right) \int_0^t \frac{1}{1-x^2} dx \right) dt. \\ &= \frac{1}{4} \int_0^1 \frac{1}{t} \ln^3 \left( \frac{1+t}{1-t} \right) dt. \end{aligned}$$

Using the substitution  $x = (1-t)/(1+t)$  and the well-known fact that

$$\int_0^1 x^k \ln^3 x dx = -\frac{6}{(k+1)^3},$$

we find

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{h_k^2}{k^2} &= -\frac{1}{2} \int_0^1 \frac{\ln^3 x}{1-x^2} dx \\ &= -\frac{1}{2} \sum_{k=0}^{\infty} \int_0^1 x^{2k} \ln^3 x dx = 3 \sum_{k=0}^{\infty} \frac{1}{(2k+1)^4} = \frac{45}{16} \zeta(4) \end{aligned} \tag{20}$$

Another path out of (3) is to bring in complex variables. Since

$$\frac{1}{i} \tan^{-1}(iz) = \tanh^{-1} z = \frac{1}{2} \ln \left( \frac{1+z}{1-z} \right)$$

Replacing  $x$  by  $ix$  in (3), we obtain

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} h_k}{k} x^{2k} = (\tan^{-1} x)^2. \quad (21)$$

This series may be evaluated at values such as  $x = 2 - \sqrt{3}, \sqrt{3}/3, 1$  explicitly:

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} h_k (2 - \sqrt{3})^{2k}}{k} = \frac{\pi^2}{144} = \frac{3}{72} \zeta(2), \quad (22)$$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} h_k}{3^k k} = \frac{\pi^2}{36} = \frac{1}{6} \zeta(2), \quad (23)$$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} h_k}{k} = \frac{\pi^2}{16} = \frac{3}{8} \zeta(2). \quad (24)$$

Similarly, applying differentiation and integration to (21), we deduce the corresponding formulas

$$\sum_{k=1}^{\infty} (-1)^{k-1} h_k x^{2k} = \frac{x}{1+x^2} \tan^{-1} x, \quad (25)$$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} h_k}{k^2} x^{2k} = 2 \int_0^x \frac{(\tan^{-1} t)^2}{t} dt. \quad (26)$$

In particular, we find

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} h_k}{3^k} = \frac{\sqrt{3}}{24} \pi, \quad (27)$$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} h_k}{k^2} = G \pi - \frac{7}{4} \zeta(3), \quad (28)$$

where  $G$  is the Catalan's constant which is defined by

$$G = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}.$$

Finally, following the excellent suggestion of an anonymous referee, recalling that

$$h_k = H_{2k} - \frac{1}{2} H_k, \quad (29)$$

we find from (19)

$$\sum_{k=1}^{\infty} \frac{1}{k^2} H_{2k} = \sum_{k=1}^{\infty} \frac{1}{k^2} h_k + \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k^2} H_k = \frac{11}{4} \zeta(3). \quad (30)$$

Furthermore, in terms of the multiple series [7]

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{ij(i+j)} = 2\zeta(3), \quad \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{(-1)^{i+j}}{ij(i+j)} = \frac{1}{4} \zeta(3),$$

the difference gives

$$\sum_{i,j>1, i+j=\text{odd}} \frac{1}{ij(i+j)} = \frac{7}{8} \zeta(3).$$

Setting  $i + j = 2k + 1$  and using partail fractions, we have

$$\begin{aligned} \sum_{i,j>1, i+j=\text{odd}} \frac{1}{ij(i+j)} &= \sum_{k=1}^{\infty} \sum_{j=1}^{2k} \frac{1}{j(2k+1-j)(2k+1)} \\ &= \sum_{k=1}^{\infty} \frac{1}{(2k+1)^2} \sum_{j=1}^{2k} \left( \frac{1}{j} + \frac{1}{2k+1-j} \right) \\ &= \sum_{k=1}^{\infty} \frac{1}{(2k+1)^2} 2H_{2k}. \end{aligned}$$

Thus,

$$\sum_{k=1}^{\infty} \frac{1}{(2k+1)^2} H_{2k} = \frac{7}{16} \zeta(3). \quad (31)$$

Subsequently, we have

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} H_{2k} &= \sum_{k=1}^{\infty} \frac{1}{(2k+1)^2} H_{2k} + \\ \sum_{k=1}^{\infty} \frac{1}{(2k-1)^3} + \sum_{k=1}^{\infty} \frac{1}{2k(2k-1)^2} &= \frac{21}{16} \zeta(3) + \frac{1}{8} (\pi^2 - 8 \ln 2). \end{aligned} \quad (32)$$

From this and the known result [1]

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} H_k = \frac{1}{4} (\pi^2 - \pi^2 \ln 2 - 8 \ln 2 + 7\zeta(3)),$$

we finally get

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} h_k = \frac{7}{16} \zeta(3) + \frac{3}{4} \zeta(2) \ln 2. \quad (33)$$

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(Concerned with sequences [A000045](#), [A000984](#), [A001008](#), and [A005408](#).)

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