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# A Characterization of the Bernoulli and Euler Polynomials.

L. CARLITZ (*)

**1.** The following three multiplication formulas are well-known [5, pp. 18, 24]:

$$B_k(nx) = n^{k-1}\sum_{s=0}^{n-1} B_k\left(x + \frac{s}{n}\right) \tag{1.1}$$

$$E_k(nx) = n^k \sum_{s=0}^{k-1} (-1)^s E_k\left(x + \frac{s}{n}\right) \quad (n \text{ odd}) \tag{1.2}$$

$$E_{k-1}(nx) = -\frac{2n^{k-1}}{k}\sum_{s=0}^{k-1} (-1)^s B_k\left(x + \frac{s}{n}\right) \quad (n \text{ even}), \tag{1.3}$$

where $B_k(x)$, $E_k(x)$ denote the Bernoulli and Euler polynomials in the standard notation,

$$\frac{xe^{xz}}{e^z - 1} = \sum_{k=0}^{\infty} B_k(x)\frac{z^k}{k!}, \tag{1.4}$$

$$\frac{2e^{xz}}{e^z + 1} = \sum_{k=0}^{\infty} E_k(x)\frac{z^k}{k!}. \tag{1.5}$$

Nielsen has observed [4, p. 54] that (1.1) and (1.2) characterize the respective polynomials. More precisely, if a monic polynomial of degree $k$ satisfies (1.1) for a single value $n > 1$, then it is identical with

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$B_k(x)$; similarly if a monic polynomial of degree $k$ satisfies (1.2) for a single odd $n > 1$, then it is identical with $E_k(x)$. The present writer [1] has proved that if $f_k(x), g_{k-1}(x)$ are monic polynomials of degree $k$, $k-1$, respectively, that satisfy

$$g_{k-1}(nx) = -\frac{2n^{k-1}}{k}\sum_{s=0}^{n-1}(-1)^s f_k\left(x+\frac{s}{n}\right) \quad (n \text{ even}) \tag{1.6}$$

for *two* distinct even $k$, then

$$f_k(x) = B_k(x) + c\,, \quad g_{k-1}(x) = E_{k-1}(x)\,, \tag{1.7}$$

where $c$ is a n arbitrary constant.

The writer [2] has generalized (1.1), (1.2), (1.3) in the following way:

$$n^{k-1}\sum_{s=0}^{n-1} B_k\left(\frac{x}{n}+\frac{ms}{n}\right) = m^{k-1}\sum_{r=0}^{m-1} B_k\left(\frac{x}{m}+\frac{nr}{m}\right), \tag{1.8}$$

$$n^k\sum_{s=0}^{n-1}(-1)^s E_k\left(\frac{x}{n}+\frac{ms}{n}\right) = m^k\sum_{r=0}^{m-1}(-1)^r E_k\left(\frac{x}{m}+\frac{nr}{m}\right) \quad (m \equiv n \equiv 1 \pmod 2)\,, \tag{1.9}$$

$$n^k\sum_{s=0}^{n-1}(-1)^s B_{k+1}\left(\frac{x}{n}+\frac{ms}{n}\right) = -\frac{1}{2}(k+1)m^k\sum_{r=0}^{m-1} E_k\left(\frac{x}{m}+\frac{nr}{m}\right) \quad (n \text{ even})\,. \tag{1.10}$$

These results were suggested by the formula for the gamma function

$$\prod_{s=0}^{n-1}\Gamma\left(mx+\frac{ms}{n}\right) = \left(\frac{m}{n}\right)^{mnx+(mn-m-n)/2}(2\pi)^{(n-m)/2}\prod_{r=0}^{m-1}\Gamma\left(nx+\frac{nr}{n}\right) \tag{1.11}$$

due to Schoblock [3, pp. 196-198]. For $m = 1$, (1.11) reduces to the familiar multiplication formula for the gamma function.

The purpose of the present note is to see to what extent the Bernoulli and Euler polynomials are characterized by (1.8), (1.9), (1.10) We show that (1.8) and (1.9) do indeed characterize the Bernoulli and Euler polynomials, respectively, if a monic polynomial of degree $k$ satisfies (1.8) for two unequal values $m, n$, then it is identical with $B_k(n)$; if a monic polynomial of degree $k$ satisfies (1.9) for two unequal *odd* values $m, n$, then it is identical with $E_k(x)$.

The situation for (1.10) is somewhat less simple. We show that if $f_{k+1}(x)$ and $g_k(x)$ are monic polynomials of degree $k+1$, and $k$, respectively, that satisfy

$$(1.12)\qquad n^k\sum_{s=0}^{n-1}(-1)^s f_{k+1}\left(\frac{x}{n}+\frac{ms}{n}\right)= \\ =-\frac{1}{2}(k+1)m^k\sum_{r=0}^{m-1}(-1)g_k\left(\frac{x}{m}+\frac{nr}{m}\right)$$

for two pairs of $m$, $n$ and $m'$, $n'$, where $n$ and $n'$ are even and in addition

$$m'n-mn'\neq 0\ ,$$

then

$$f_{k+1}(x)=B_{k+1}(x)+c\ ,\qquad g_k(x)=E_k(x)\ ,$$

where $c$ is an arbitrary constant. If however (1.12) is assumed only for the single pair $m$, $n$ with $n$ even, then

$$(1.13)\qquad f_{k+1}(x)=a_0+\sum_{j=0}^{k}a_{j+1}m^{k-j}B_{j+1}(x)$$

if and only if

$$(1.14)\qquad g_k(x)=\sum_{j=0}^{k}(j+1)a_{j+1}n^{k+j}E_j(x)\ .$$

Conversely, if $g_k(x)$ is defined by (1.14) then $f_{k+1}(x)$ is determined by (1.13) with $a_0$ arbitrary.

We remark that the results concerning the Euler polynomial can be carried over to the Eulerian polynomials discussed in [1] and [2]; however we shall not do so in the present note.

**2.** We first prove

THEOREM 1. *Let the monic polynomial $f_k(x)$ of degree $k$ satisfy*

$$(2.1)\qquad n^{k-1}\sum_{s=0}^{n-1}f_k\left(\frac{x}{n}+\frac{ms}{n}\right)=m^{k-1}\sum_{r=0}^{m-1}f_k\left(\frac{x}{m}+\frac{nr}{m}\right)$$

*for two distinct (positive) values m, n. Then*

$$f_k(x) = B_k(x) . \tag{2.2}$$

PROOF. Let

$$S_k(x;\, m,\, n) = n^{k-1}\sum_{s=0}^{n-1} B_k\left(\frac{x}{n} + \frac{ms}{n}\right), \tag{2.3}$$

so that, by (1.8),

$$S_k(x;\, m,\, n) = S_k(x;\, n,\, m) \qquad (k = 0, 1, 2, \ldots) . \tag{2.4}$$

It is clear from (2.3) that $S_k(x;\, m,\, n)$ is a monic polynomial of degree $k$. Moreover, from the proof of (1.8), we have

$$\sum_{k=0}^{\infty} S_k(x;\, m,\, n)\frac{z^k}{k!} = \frac{ze^{xz}(e^{mnz}-1)}{(e^{mz}-1)(e^{nz}-1)} . \tag{2.5}$$

Now put

$$f_k(x) = \sum_{j=0}^{k} a_j B_j(x) \qquad (a_k = 1) , \tag{2.6}$$

where the coefficients $a_j$ are independent of $x$ and are uniquely determined by $f_k(x)$. Thus (2.1) becomes

$$n^{k-1}\sum_{j=0}^{k} a_j \sum_{s=0}^{n-1} B_j\left(\frac{x}{n} + \frac{ms}{n}\right) = m^{k-1}\sum_{j=0}^{k} a_j \sum_{r=0}^{m-1} B_j\left(\frac{x}{m} + \frac{nr}{m}\right) .$$

Hence, by (2.3),

$$\sum_{j=0}^{k} a_j n^{k-j} S_j(x;\, m,\, n) = \sum_{j=0}^{k} a_j m^{k-j} S_j(x;\, n,\, m) ,$$

so that, by (2.4),

$$\sum_{j=0}^{k} a_j (n^{k-j} - m^{k-j}) S_j(x;\, m,\, n) = 0 . \tag{2.7}$$

Since $S_j(x; m, n)$ is of precise degree $j$ in $x$, it follows from (2.7) that

$$a_j = 0 \qquad (j = 0, 1, 2, \ldots, k-1)$$

and (2.6) reduces to $f_k(x) = B_k(x)$.

We remark that it follows from (2.5) that

$$(2.8) \qquad mnkS_{k-1}(s;\, m, n) = (mB + nB + x + mn)^k - (mB + nB + x)^k\,,$$

where

$$(2.9) \qquad (mB + nB + x)^k = \sum_{i+j\leqslant k} \frac{k!}{i!\,j!\,(k-i-j)!}\, m^i n^i B_i B_j x^{k-1-j}\,.$$

Alternatively, $(mB + nB + x)^k$ can be exhibited as a Bernoulli polynomial of higher order [5, Ch. 6].

**3.** Turning to (1.9) we shall prove

THEOREM 2. *Let the monic polynomial $g_k(x)$ of degree $k$ satisfy*

$$(3.1) \qquad n^k \sum_{s=0}^{n-1} (-1)^s g_k\left(\frac{x}{n} + \frac{ms}{n}\right) = m^k \sum_{r=0}^{m-1} (-1)^r g_k\left(\frac{x}{m} + \frac{nr}{m}\right)$$

*for two distinct odd values of $m, n$. Then*

$$(3.2) \qquad g_k(x) = E_k(x)\,.$$

PROOF. Let

$$(3.3) \qquad T_k(x;\, m, n) = n^k \sum_{s=0}^{n-1} (-1)^s E_k\left(\frac{x}{n} + \frac{ms}{n}\right),$$

so that, by (1.9),

$$(3.4) \qquad T_k(x;\, m, n) = T_k(x;\, n, m)\,, \qquad (k = 0, 1, 2, \ldots)\,,$$

at least for $m, n$ both odd. It follows from (3.3) that, for $n$ odd, $T_k(x;\, m, n)$ is a monic polynomial of degree $k$. From the proof of (1.9) we have

$$(3.5) \qquad \sum_{k=0}^{\infty} T_k(x;\, m, n) \frac{z^k}{k!} = \frac{2e^{xz}(e^{mnz} + 1)}{(e^{mz} + 1)(e^{nz} + 1)}\,.$$

Now put

$$g_k(x) = \sum_{j=0}^{k} b_j E_j(x) \qquad (b_k = 1)\,, \tag{3.6}$$

where the coefficients $b_j$ are independent of $x$ and are uniquely determined by $g_k(x)$. Thus (3.1) becomes

$$n^k \sum_{j=0}^{k} b_j \sum_{s=0}^{n-1} (-1)^s E_j\left(\frac{x}{n} + \frac{ms}{n}\right) = m^k \sum_{j=0}^{k} b_j \sum_{t=0}^{m-1} (-1)^r E_j\left(\frac{x}{m} + \frac{nr}{m}\right).$$

Thus, by (3.3),

$$\sum_{j=0}^{k} b_j n^{k-1} T_j(x;\, m,\, n) = \sum_{j=0}^{k} b_j m^{k-j} T_j(s;\, n,\, m)\,,$$

so that, by (3.4),

$$\sum_{j=0}^{k} b_j (n^{k-j} - m^{k-j}) T_j(x;\, m,\, n) = 0\,. \tag{3.7}$$

Since $T_j(x;\, m, n)$ is of degree $j$ in $x$, it follows from (3.7) that

$$b_j = 0 \qquad (j = 0, 1, 2, \ldots, k-1)$$

and therefore (3.6) reduces to $g_k(x) = E_k(x)$.

It follows from (3.5) that

$$2mnT_k(x;\, m,\, n) = (\tfrac{1}{2}mC + \tfrac{1}{2}nC + x + mn)^k + \\ + (\tfrac{1}{2}mC + \tfrac{1}{2}nC + x)^k \qquad (m \equiv b \equiv 1 \pmod{\zeta})\,, \tag{3.8}$$

where

$$\left(\frac{1}{2}mC + \frac{1}{2}nC + x\right)^k = \sum_{i+j \leqslant k} \frac{k!}{i!\,j!\,(k-i-j)!}\, 2^{-i-j} m^i n^i C_i C_j x^{k-i-j} \tag{3.9}$$

and [5, p. 28]

$$E_k(x) = (x + \tfrac{1}{2}C)^k\,, \qquad E_k(0) = 2^{-k} C_k\,. \tag{3.10}$$

For $n$ even, it is proved in [2] that

$$\sum_{k=0}^{\infty}\frac{(nz)^k}{k!}\sum_{s=0}^{n-1}(-1)^s E_k\left(\frac{x}{n}+\frac{ms}{n}\right)=\frac{2e^{xz}(1-e^{mnz})}{(e^{mz}+1)(e^{nz}+1)}\,.$$

Since the right hand side is symmetric in $m, n$, it follows that (1.9) holds provided only that $m$ and $n$ have the same parity. The definition (3.3) holds for arbitrary $n$ and therefore

$$\sum_{k=0}^{\infty}\frac{z^k}{k!}T_k(x;\,m,\,n)=\frac{2e^{xz}(1-e^{mnz})}{(e^{mz}+1)(e^{nz}+1)}\quad (n\text{ even})\,. \tag{3.11}$$

We accordingly get

$$\begin{aligned}2mnT_k(x;\,m,\,n) &= \\ &= (\tfrac{1}{2}mC+\tfrac{1}{2}nC+x)^k-(\tfrac{1}{2}mC+\tfrac{1}{2}nC+x+mn)^k\qquad (n\text{ even})\,.\end{aligned} \tag{3.12}$$

Expanding the right member of (3.12) it is clear that, for $n$ even, $T_k(x;\,m,\,n)$ is of degree $k-1$; the coefficient of $x^{k-1}$ is equal to $-mn$.

We now consider the equation (3.1) assuming that both $m$ and $n$ are even. The proof of Theorem 2 applies without change down to and including (3.7). In the present situation $T_j(x;\,m,\,n)$ is of degree $j-1$ for $j\geqslant 1$. Hence we infer that

$$b_j=0\qquad (j=1,2,\dots,k-1)\,.$$

Finally we may state

THEOREM 3. *Let the monic polynomial* $g_k(x)$ *satisfy* (3.1) *for two distinct even values of* $m, n$. *Then*

$$g_k(x)=E_k(x)+c\,, \tag{3.9}$$

*where* $c$ *is an arbitrary constant.*

**4.** Let $f_{k+1}(x)$ be a monic polynomial of degree $k+1$ and let $g_k(x)$ be a monic polynomial of degree $k$. Consider the equation

$$n^k\sum_{s=0}^{n-1}(-1)^s f_{k+1}\left(\frac{x}{n}+\frac{ms}{n}\right)=-\frac{1}{2}(k+1)m^k\sum_{r=0}^{m-1}g_k\left(\frac{x}{m}+\frac{nr}{m}\right) \tag{4.1}$$

for fixed $m$ and fixed even $n$.

Put

$$U_{k+1}(x;\, m,\, n) = n^k \sum_{s=0}^{n-1} (-1)^s B_{k+1}\left(\frac{x}{n} + \frac{ms}{n}\right) \tag{4.2}$$

and

$$V_k(x;\, m,\, n) = m^k \sum_{r=0}^{m-1} E_k\left(\frac{x}{m} + \frac{nr}{m}\right). \tag{4.3}$$

Then by (1.10)

$$U_{k+1}(x;\, m,\, n) = -\tfrac{1}{2}(k+1)\, V_k(x;\, m,\, n)\,. \tag{4.4}$$

By (4.3) it is evident that $V_k(x;\, m,\, n)$ is monic of degree $k$. Hence $U_{k+1}(x;\, m, n)$ is of degree $k$ and with highest coefficient equal to $-\frac{1}{2}(k+1)$.

Let

$$f_{k+1}(x) = \sum_{j=0}^{k+1} a_j B_j(x)\,, \qquad g_k(x) = \sum_{j=0}^{k} b_j E_j(x)\,, \tag{4.5}$$

where the $a_j$, $b_j$ are independent of $x$ and are uniquely determined by $f_{k+1}(x)$ and $g_k(x)$, respectively; in particular, $a_{k+1} = b_k = 1$.

Substituting from (4.5) in (4.1), we get

$$n^k \sum_{j=0}^{k+1} a_j \sum_{s=0}^{n-1} (-1)^s B_j\left(\frac{x}{n} + \frac{ms}{n}\right) = -\frac{1}{2}(k+1) m^k \sum_{j=0}^{k} b_j \sum_{r=0}^{m-1} E_j\left(\frac{x}{m} + \frac{nr}{m}\right),$$

that is

$$\sum_{j=0}^{k+1} a_j n^{k-j+1} U_j(x;\, m,\, n) = -\tfrac{1}{2}(k+1) \sum_{j=0}^{k} b_j m^{k-j} V_j(x;\, m,\, n)\,. \tag{4.6}$$

By (4.4) this reduces to

$$a_0 n^{k+1} U_0(x;\, m,\, n) + \tag{4.7}$$
$$+ \tfrac{1}{2}(k+1) \sum_{j=0}^{k} \{b_j m^{k-j} - (j+1) a_{j+1} n^{k-j}\}\, V_j(x;\, m,\, n) = 0\,.$$

Note that

$$U_0(x;\, m,\, n) = n^{-1} \sum_{s=0}^{n-1} (-1)^s B_0\left(\frac{x}{n} + \frac{ms}{n}\right) = 0\,.$$

Since $V_j(x; m, n)$ is of degree $j$ it follows from (4.7) that

$$b_j m^{k-j} - (j+1) a_{j+1} n^{k-j} = 0 \qquad (j = 0, 1, 2, \dots, k) . \tag{4.8}$$

For $j = k$, (4.8) is automatically satisfied in view of $a_{k+1} = b_k = 1$.

We now assume that (4.1) is satisfied by a second pair of numbers $m', n'$, with $n'$ even. Then by (4.8) we have also

$$b_j m'^{k-j} - (j+1) a_{j+1} n'^{k-j} = 0 \qquad (j = 0, 1, 2, \dots, k) . \tag{4.9}$$

It follows from (4.8) and (4.9) that

$$a_{j+1}\big((m' n)^{k-j} - (mn')^{k-j}\big) = 0 \qquad (j = 0, 1, \dots, k-1) . \tag{4.10}$$

For $j = k-1$, (4.10) reduces to

$$a_k(m' n - mn') = 0 .$$

We therefore assume that

$$m' n - mn' \neq 0 . \tag{4.11}$$

It is then clear that (4.10) implies

$$a_j = 0 \qquad (j = 1, 2, \dots, k) , \tag{4.12}$$

so that

$$b_j = 0 \qquad (j = 0, 1, \dots, k-1) . \tag{4.13}$$

This completes the proof of

THEOREM 4. *Let $f_{k+1}(x)$ and $g_k(x)$ be monic polynomials of degree $k+1$ and $k$; respectively. Assume that*

$$n^k \sum_{s=0}^{n-1} (-1)^s f_{k+1}\left(\frac{x}{n} + \frac{ms}{n}\right) = -\frac{1}{2}(k+1) m^k \sum_{r=0}^{m-1} g_k\left(\frac{x}{m} + \frac{nr}{m}\right) \tag{4.14}$$

*for two pairs of number $m, n$ and $m', n'$, where $n$ and $n'$ are even and in addition*

$$m' n - mn' \neq 0 . \tag{4.15}$$

*Then*

$$f_{k+1}(x) = B_{k+1}(x) + c\,, \qquad g_k(x) = E_k(x)\,, \tag{4.16}$$

*where c is an arbitrary constant.*

If we assume only that (4.14) is satisfied for the pair $m, n$ we get the following

COROLLARY. *Let $f_{k+1}(x)$ and $g_k(x)$ satisfy the hypothesis of Theorem* 4. *Assume that* (4.14) *holds for the pair $m, n$ with $n$ even. Let*

$$f_{k+1}(x) = a_0 + \sum_{j=0}^{k} a_{j+1} m^{k-j} B_{j+1}(x)\,. \tag{4.17}$$

*Then $g_k(x)$ is uniquely determined by*

$$g_k(x) = \sum_{j=0}^{k} (j+1)\, a_{j+1} n^{k-j} E_j(x)\,. \tag{4.18}$$

*Conversely, if $g_k(x)$ is given by* (4.18) *then $f_{k+1}(x)$ is determined by* (4.17) *with $a_0$ arbitrary.*

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