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Identities Involving Two Kinds of q-Euler Polynomials and Numbers

Abdelmejid Bayad
Département de Mathématiques
Université d'Evry Val d'Essonne
Bâtiment I.B.G.B.I., 3ème étage
23 Bd. de France
91037 Evry Cedex
France
abayad@maths.univ-evry.fr

Yoshinori Hamahata
Faculty of Engineering Science
Kansai University
3-3-35 Yamate-cho, Suita-shi
Osaka 564-8680
Japan
hamahata@fc.ritsumei.ac.jp

Abstract

We introduce two kinds of q-Euler polynomials and numbers, and investigate many of their interesting properties. In particular, we establish q-symmetry properties of these q-Euler polynomials, from which we recover the so-called Kaneko-Momiyama identity for the ordinary Euler polynomials, discovered recently by Wu, Sun, and Pan. Indeed, a q-symmetry and q-recurrence formulas among sum of product of these q-analogues Euler numbers and polynomials are obtained. As an application, from these q-symmetry formulas we deduce non-linear recurrence formulas for the product of the ordinary Euler numbers and polynomials.

1 Introduction and preliminaries

1.1 The ordinary Euler numbers and polynomials: An analytic overview

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$. The ordinary Euler polynomials $E_n(x)$ are defined by the generating series

$$\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}.$$

The first few values are

$$E_0(x) = 1, E_1(x) = x - \frac{1}{2}, E_2(x) = x^2 - x, E_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{4}.$$

The ordinary Euler numbers E_n $(n = 0, 1, 2, \ldots)$ are defined by the generating function

$$\frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}. \tag{1}$$

The n^{th}-Euler number and polynomial are connected by the equation: $E_n = E_n(0)$. The first few values are $E_0 = 1$, $E_1 = -1/2$, $E_2 = 0$, $E_3 = 1/4$, and it holds that $E_{2k} = 0$ $(k = 1, 2, 3, \ldots)$.

Remark 1. Note that the Euler numbers E_n which we consider in this paper are different from the Euler numbers defined by M. Abramowitz and I. A. Stegun [1, Ch.23].

From the definition we can easily deduce the following well-known difference formula:

$$(-1)^n E_n(-x) + E_n(x) = 2x^n, (n \in \mathbb{N}). \tag{2}$$

In 2004, K.J. Wu, Z.W. Sun, and H. Pan [13] proved the following important formulae:

$$(-1)^m \sum_{k=0}^{m} \binom{m}{k} E_{n+k}(x) = (-1)^n \sum_{k=0}^{n} \binom{n}{k} E_{m+k}(-x), \tag{3}$$

$$\begin{aligned}(-1)^m \sum_{k=0}^{m+1} \binom{m+1}{k} (n+k+1) E_{n+k}(x)& \\ +(-1)^n \sum_{k=0}^{n+1} \binom{n+1}{k} (m+k+1) E_{m+k}(x) &= 0.\end{aligned} \tag{4}$$

The last identity (4) is an Euler polynomial version of Kaneko-Momiyama relations among Bernoulli numbers. See M. Kaneko [7], H. Momiyama [10], I. M. Gessel [5] and Wu-Sun-Pan [13] for details.

The identity (3) can be viewed as an integral version of the Kaneko-Momiyama type identity for the Euler polynomials. In this present paper, we introduce and investigate two kinds of q-Euler polynomials and numbers. For instance, we find q-analogues for the identities (3) and (4). On the other hand, we also establish a relation between sums of products of our q-Euler polynomials.

1.2 q-shifted factorials

Let $a \in \mathbb{C}$. The *q-shifted factorials* are defined by

$$(a, q)_0 = 1, \quad (a, q)_n = \prod_{k=0}^{n-1} (1 - aq^k) \quad (n = 1, 2, \ldots).$$

If $|q| < 1$, then we define

$$(a, q)_\infty = \lim_{n\to\infty} (a, q)_n = \prod_{k=0}^{\infty} (1 - aq^k).$$

We also denote

$$\begin{aligned}
[x]_q &= \frac{1-q^x}{1-q}, \quad x \in \mathbb{C}, \\
[n]_q! &= \frac{(q, q)_n}{(1-q)^n}, \quad n \in \mathbb{N}, \\
\begin{bmatrix} n \\ k \end{bmatrix}_q &= \frac{[n]_q!}{[k]_q![n-k]_q!}, \quad k, n \in \mathbb{N}, \\
\begin{bmatrix} n \\ i_1, \ldots, i_m \end{bmatrix}_q &= \frac{[n]_q!}{[i_1]_q! \cdots [i_m]_q!}, \quad n, i_1, \ldots, i_m \in \mathbb{N}, \text{ with } i_1 + \cdots + i_m = n.
\end{aligned}$$

1.3 q-Exponential functions

The *q-exponential functions* are given by

$$e_q(z) := \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!}, \text{ and } \quad e_{q^{-1}}(z) := \sum_{n=0}^{\infty} \frac{z^n}{[n]_{q^{-1}}!}. \tag{5}$$

It is easy to see that $[n]_{q^{-1}}! = q^{-\binom{n}{2}} [n]_q!$. Hence

$$e_{q^{-1}}(z) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} z^n}{[n]_q!}.$$

Recently both q-exponential functions are intensively studied in q-calulus and and quatum theory. See I. M. Gessel [4], W. P. Johnson [6] for related topics. As is well-known, these functions are related to the infinite product $(z, q)_\infty$ by

$$e_q(z) = \frac{1}{((1-q)z, q)_\infty}, \qquad e_{q^{-1}}(z) = (-(1-q)z, q)_\infty \,.$$

This yields $e_q(z) e_{q^{-1}}(-z) = 1$.

1.4 q-Euler polynomials and numbers

Definition 2. We define two kinds of q-Euler polynomials $E_n(x,q)$ and $F_n(x,q^{-1})$ $(n = 0,1,2,\ldots)$ by

$$\begin{aligned}\frac{2e_q(xt)}{e_q(t)+1} &= \sum_{n=0}^{\infty} E_n(x,q)\frac{t^n}{[n]_q!},\\ \frac{2e_q(xt)}{e_{q^{-1}}(t)+1} &= \sum_{n=0}^{\infty} F_n(x,q^{-1})\frac{t^n}{[n]_q!}.\end{aligned}$$

We call $E_n(x,q)$ (resp. $F_n(x,q^{-1})$) the *first* (resp. *second*) *q-Euler polynomials*. In particular, we call $E_n(0,q)$ (resp. $F_n(0,q^{-1})$) the *first* (resp. *second*) *q-Euler numbers*.

Example 3.

$$\begin{aligned}E_0(x,q) &= 1, E_1(x,q) = x - \frac{1}{2},\\ E_2(x,q) &= x^2 - \frac{[2]_q}{2}x - \frac{1}{2} + \frac{[2]_q}{4},\\ E_3(x,q) &= x^3 - \frac{[3]_q}{2}x^2 + \left(\frac{[3]_q[2]_q}{4} - \frac{[3]_q}{2}\right)x - \frac{1}{2} - \frac{[3]_q[2]_q}{8} + \frac{[3]_q}{2}.\end{aligned}$$

$$\begin{aligned}F_0(x,q^{-1}) &= 1, F_1(x,q^{-1}) = x - \frac{1}{2},\\ F_2(x,q^{-1}) &= x^2 - \frac{[2]_q}{2}x - \frac{q}{2} + \frac{[2]_q}{4},\\ F_3(x,q^{-1}) &= x^3 - \frac{[3]_q}{2}x^2 + \left(\frac{[3]_q[2]_q}{4} - \frac{[3]_q}{2}q\right)x - \frac{q^3}{2} - \frac{[3]_q[2]_q}{8} + \frac{[3]_q}{2}q.\end{aligned}$$

Remark 4.

1. The reason for introducing both kinds of q-analogue Bernoulli polynomials $E_n(x,q)$ and $F_n(x,q^{-1})$ is that they are needed in the q-analogues of symmetry, difference, recurrence and complementary argument formulas.

2. The case $q = 1$ corresponds to the ordinary Euler polynomials and numbers.

3. In the literature there are many q-analogues of the Euler numbers and polynomials. The q-analogues which we consider here are closely related to q-calculus, q-Jackson integral and hypergeometric series.

4. Various q-analoques of the Euler numbers and polynomials are studied by many mathematicians. For more details for example you can refer to T. Kim [9],C. S. Ryoo [11], Y. Simsek [12] and others.

5. It seems to be difficult to clarify the connections between all the q-analogues of the Euler numbers and polynomials.

2 q-Recurrence, q-Addition, q-Derivative and q-integral formulae

In this section, we establish a series of formulas involving the polynomials $E_n(x,q)$ and $F_n(x,q^{-1})$, like q-addition, q-derivative, q-integral and q-recurrence formulas.

2.1 q-Recurrence formulae

Proposition 5 (q-Recurrence formula)**.** *For any $n \geq 1$, we have*

$$\begin{aligned} E_n(x,q) &= x^n - \frac{1}{2}\sum_{i=0}^{n-1} \begin{bmatrix} n \\ i \end{bmatrix}_q E_i(x,q), \\ F_n(x,q^{-1}) &= x^n - \frac{1}{2}\sum_{i=0}^{n-1} \begin{bmatrix} n \\ i \end{bmatrix}_q F_i(x,q^{-1})q^{\binom{n-i}{2}}. \end{aligned}$$

Proof. As for the first identity, we make use of

$$\left(\sum_{n=0}^{\infty} E_n(x,q)\frac{t^n}{[n]_q!}\right)(e_q(t)+1) = 2e_q(xt).$$

We deduce from this identity

$$\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q E_i(x,q) + E_n(x,q)\right)\frac{t^n}{[n]_q!} = \sum_{n=0} 2x^n\frac{t^n}{[n]_q!},$$

which yields the result. We get the second result in the similar way. □

As $q \to 1$, one has a recurrence formula for the ordinary Euler polynomials:

$$E_n(x) = x^n - \frac{1}{2}\sum_{i=0}^{n-1}\binom{n}{i}E_i(x) \qquad (n \geq 1),$$

then for the Euler numbers E_{2n+1} we recover the well-known recurrence formula

$$E_{2n+1}(x) = -\frac{1}{2}\sum_{k=0}^{2n}\binom{2n+1}{k}E_k \qquad (n \geq 0). \tag{6}$$

2.2 q-Derivative and q-integral

The *q-derivative* of a function f is given by

$$D_q f(x) := \frac{f(x)-f(qx)}{(1-q)x} \qquad (x \neq 0, q \neq 1),$$

where x and qx should be in the domain of f. If f is differentiable on an open set I, then for all $x \in I$,

$$\lim_{q \to 1} D_q f(x) = f'(x).$$

Besides, for all $n \in \mathbb{N}$,

$$\begin{aligned} D_q(x^n) &= [n]_q x^{n-1}, D_q(x,q)_n = -[n]_q(xq,q)_{n-1}, \\ D_{q^{-1}}(x,q)_n &= -[n]_q(x,q)_{n-1}, D_q\left(\frac{x^n}{[n]_q!}\right) = \frac{x^{n-1}}{[n-1]_q!}. \end{aligned}$$

From the last identity, for instance, we have $D_q e_q(x) = e_q(x)$.
Our q-Euler polynomials form "q-Appell sequences":

Proposition 6 (q-Derivative formula)**.** *For any $n \geq 0$, we have*

$$\begin{aligned} D_q E_{n+1}(x,q) &= [n+1]_q E_n(x,q), \\ D_q F_{n+1}(x,q^{-1}) &= [n+1]_q F_n(x,q^{-1}). \end{aligned}$$

Proof. Since

$$\sum_{n=0}^{\infty} D_q E_n(x,q)\frac{t^n}{[n]_q!} = \frac{2te_q(xt)}{e_q(t)+1} = \sum_{n=1}^{\infty}[n]_q E_{n-1}(x,q)\frac{t^n}{[n]_q!},$$

we have the first identity. The second identity can be obtained similarly. □

As $q \to 1$, we have the identities of Appell sequences of the ordinary Euler polynomials:

$$\frac{d}{dx}E_{n+1}(x) = (n+1)E_n(x).$$

For the product of two functions f and g, the following formula holds:

$$\begin{aligned} D_q(f \cdot g)(x) &= g(x)D_q(x) + f(qx)D_q g(x) \\ &= f(x)D_q g(x) + g(qx)D_q f(x). \end{aligned}$$

We next treat the composition of $f(x)$ and $g(x)$. When $g(x) = -x$, the following chain rule for the q-derivative is valid:

$$D_q(f \circ g)(x) = D_q f(g(x))D_q g(x),$$

which will be used in the proofs of Theorems 15 and 20. However, in general, the rule above does not hold. If we modify the definition of the composition of two functions, then a new chain rule for the q-derivative is gained. We refer to I. M. Gessel [4] for this topic.

The *q-Jackson integral* from 0 to a is defined by

$$\int_0^a f(x)d_q x \quad := \quad (1-q)a\sum_{n=0}^{\infty} f(aq^n)q^n$$

provided the infinite sums converge absolutely. The q-Jackson integral in the generic interval $[a,b]$ is given by

$$\int_a^b f(x)d_qx = \int_0^b f(x)d_qx - \int_0^a f(x)d_qx.$$

For any function f we have

$$D_q \int_0^x f(t)d_qt = f(x).$$

Proposition 7 (q-Integral formula)**.** *For any $n \geq 0$,*

$$\begin{aligned}
\int_a^x E_n(t,q)d_qt &= \frac{E_{n+1}(x,q) - E_{n+1}(a,q)}{[n+1]_q}, \\
\int_a^x F_n(t,q^{-1})d_qt &= \frac{F_{n+1}(x,q^{-1}) - F_{n+1}(a,q^{-1})}{[n+1]_q}.
\end{aligned}$$

This result follows from q-derivative formula. As $q \to 1$, we have integral formula for the classical Euler polynomials:

$$\int_a^x E_n(t)dt = \frac{E_{n+1}(x) - E_{n+1}(a)}{n+1}.$$

2.3 q-Binomial formula

Let $q \in \mathbb{C}$, and take two q-commuting variables x and y which satisfy the relation

$$xy = q^{-1}yx.$$

Let $\mathbb{C}_q[x,y]$ be the complex associative algebra with 1 generated by x and y. Then the following identity is valid in the algebra $\mathbb{C}_q[x,y]$:

$$(x+y)^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q x^k y^{n-k}, \qquad n \in \mathbb{N},$$

or alternatively,

$$(x+y)^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} y^k x^{n-k}, \qquad n \in \mathbb{N}.$$

For details, we refer to Andrews-Askey-Roy [2], Gasper-Rahman [3].

2.4 q-Exponential identity

Let x, y be the q-commuting variables satisfying the relation $xy = q^{-1}yx$. Let $\mathbb{C}_q[[x,y]]$ be the complex associative algebra with 1 of formal power series

$$\sum_{m=0}^\infty \sum_{n=0}^\infty a_{m,n} x^m y^n$$

with arbitrary complex coefficients $a_{m,n}$. One knows in Andrews-Askey-Roy [2], Gasper-Rahman [3] that in $\mathbb{C}_q[[x,y]]$, we have the following identity

$$e_q(x+y) = e_q(x)e_q(y).$$

Proposition 8 (q-Addition formula)**.** *Let x, y be the q-commuting variables satisfying the relation $xy = q^{-1}yx$. For any $n \geq 0$, we have*

$$\begin{aligned} E_n(x+y,q) &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q E_k(x,q)y^{n-k}, \\ F_n(x+y,q^{-1}) &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q F_k(x,q^{-1})y^{n-k}. \end{aligned}$$

Particularly, it follows that

$$\begin{aligned} E_n(y,q) &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q E_k(0,q)y^{n-k}, \\ F_n(y,q^{-1}) &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q F_k(0,q^{-1})y^{n-k}. \end{aligned}$$

Proof. The first identity follows from

$$\frac{2e_q((x+y)t)}{e_q(t)+1} = \frac{2e_q(xt)}{e_q(t)+1} \cdot e_q(yt).$$

One can easily prove the remaining identities. □

As $q \to 1$, we have the classical formula:

$$E_{n+1}(x+y) = \sum_{k=0}^{n} \binom{n}{k} E_k(x)y^{n-k}.$$

Particularly, it holds that

$$E_n(y) = \sum_{k=0}^{n} \binom{n}{k} E_k y^{n-k}.$$

At the end of this section, we give a list of limit of q-analogues.

$$\begin{aligned} \lim_{q\to 1} e_q(z) &= \lim_{q\to 1} e_{q^{-1}}(z) = e^z, \\ \lim_{q\to 1} [n]_q &= n, \\ \lim_{q\to 1} [n]_q! &= n!, \\ \lim_{q\to 1} \begin{bmatrix} n \\ k \end{bmatrix}_q &= \binom{n}{k}, \\ \lim_{q\to 1} \begin{bmatrix} n \\ i_1,\ldots,i_m \end{bmatrix}_q &= \binom{n}{i_1,\ldots,i_m} := \frac{n!}{i_1!\cdots i_m!}, \\ \lim_{q\to 1} E_n(x,q) &= \lim_{q\to 1} F_n(x,q^{-1}) = E_n(x). \end{aligned}$$

3 q-symmetry and q-analogues to Kaneko-Momiyama identities

3.1 q-symmetry fo Sums of products

Theorem 9 (Sums of products)**.** *Let m be a given positive integer. Then for any $n \geq 0$, we have*

$$
\begin{aligned}
&(-1)^n \sum_{i_1+\cdots+i_m=n} \begin{bmatrix} n \\ i_1,\ldots,i_m \end{bmatrix}_q F_{i_1}(-x,q^{-1})\cdots F_{i_m}(-x,q^{-1}) \\
&\quad = \sum_{j=0}^{m} (-1)^j 2^{m-j} \binom{m}{j} \sum_{k_1+\cdots k_m=n} \begin{bmatrix} n \\ k_1,\ldots,k_m \end{bmatrix}_q E_{k_1}(x,q)\cdots E_{k_j}(x,q) x^{n-(k_1+\cdots+k_j)}.
\end{aligned}
$$

Remark 10. The above theorem implies the following results.

1. If $m=1$, then

$$
(-1)^n F_n(-x,q^{-1}) + E_n(x,q) = 2x^n. \tag{7}
$$

   This relation can be viewed as a q-difference formula. If $q \to 1$ we recover the usual difference formula for the Euler polynomials (2).

2. If $m=2$, then

$$
\begin{aligned}
&(-1)^n \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q F_i(-x,q^{-1})F_{n-i}(-x,q^{-1}) \\
&\qquad = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q E_i(x,q)E_{n-i}(x,q) - 4\sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q E_i(x,q)x^{n-i} + 4x^n \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q.
\end{aligned}
$$

3. It should be noted that Simsek [12] found formulae for sums of products of another kind of q-Euler polynomials.

Proof. In view of $e_q(t)e_{q^{-1}}(-t)=1$, we have

$$
\frac{1}{e_{q^{-1}}(-t)+1} = 1 - \frac{1}{e_q(t)+1}.
$$

Hence for $m \geq 1$,

$$
\left(\frac{2e_q((-x)(-t))}{e_{q^{-1}}(-t)+1}\right)^m = \left(2e_q(xt) - \frac{2e_q(xt)}{e_q(t)+1}\right)^m.
$$

The left-hand side of the identity is

$$
\sum_{n=0}^{\infty} (-1)^n \sum_{i_1+\cdots+i_m=n} \begin{bmatrix} n \\ i_1,\ldots,i_m \end{bmatrix}_q F_{i_1}(-x,q^{-1})\cdots F_{i_m}(-x,q^{-1}) \frac{t^n}{[n]_q!}.
$$

The right-hand side becomes

$$\sum_{j=0}^{m}(-1)^j\binom{m}{j}\left(\frac{2e_q(xt)}{e_q(t)+1}\right)^j 2^{m-j}e_q(xt)^{m-j}$$
$$=\sum_{j=0}^{m}(-1)^j 2^{m-j}\binom{m}{j}\sum_{n=0}^{\infty}\sum_{k_1+\cdots+k_m=n}\left[\begin{matrix} n \\ k_1,\ldots,k_m\end{matrix}\right]_q E_{k_1}(x,q)\cdots E_{k_j}(x,q)x^{k_{j+1}}\cdots x^{k_m}\frac{t^n}{[n]_q!}.$$

□

As $q \to 1$ in the formula of Theorem 9, we have

Theorem 11. *Let m be a given positive integer. Then for any $n \geq 0$,*

$$(-1)^n\sum_{i_1+\cdots+i_m=n}\left(\begin{matrix} n \\ i_1,\ldots,i_m\end{matrix}\right)E_{i_1}(-x)\cdots E_{i_m}(-x)$$
$$=\sum_{j=0}^{m}(-1)^j 2^{m-j}\binom{m}{j}\sum_{k_1+\cdots k_m=n}\left(\begin{matrix} n \\ k_1,\ldots,k_m\end{matrix}\right)E_{k_1}(x)\cdots E_{k_j}(x)x^{n-(k_1+\cdots+k_j)}.$$

Corollary 12. *Especially in the cases $m = 1, 2$, the following results hold:*

(1) *For any $n \geq 0$, we have $(-1)^n E_n(-x) + E_n(x) = 2x^n$.*

(2) *For any $k \geq 1$, $E_{2k} = 0$.*

(3) *For any $n \geq 0$, we get the* **non-linear recurrence** *formulae*

$$(-1)^n\sum_{i=0}^{n}\binom{n}{i}E_i(-x)E_{n-i}(-x)=\sum_{i=0}^{n}\binom{n}{i}E_i(x)E_{n-i}(x)-4\sum_{i=0}^{n}\binom{n}{i}E_i(x)x^{n-i}+2^{n+2}x^n.$$

(4) *If n is an odd positive integer, then we obtain the well-known* **Euler non-linear recurrence** *formula*

$$\sum_{i=0}^{n}\binom{n}{i}E_iE_{n-i}=2E_n.$$

3.2 q-Symmetry

Theorem 13 (q-Symmetry 1)**.** *For any $m, n \in \mathbb{N}$, we have*

$$(-1)^m\sum_{k=0}^{m}\begin{bmatrix} m \\ k\end{bmatrix}_q E_{n+k}(x,q)q^{-kn+mn}=(-1)^n\sum_{k=0}^{n}\begin{bmatrix} n \\ k\end{bmatrix}_{q^{-1}}F_{m+k}(-x,q^{-1})q^{\binom{n}{2}-\binom{k}{2}}. \tag{8}$$

This identity (8) can be viewed as a q-analogue to the polynomial version of the integral Kaneko-Momiyama's formulae on Euler polynomials (3).

Proof. Let x, y be two q-commuting variables with $xy = q^{-1}yx$. We compute the generating functions

$$\begin{aligned}
L(w,x,y) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^m \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix}_q E_{n+k}(w,q) q^{-kn+mn} \frac{x^m}{[m]_q!}\frac{y^n}{[n]_q!}, \\
R(w,x,y) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^n \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q F_{m+k}(-w,q^{-1}) q^{\binom{n}{2}-\binom{k}{2}} \frac{x^m}{[m]_q!}\frac{y^n}{[n]_q!},
\end{aligned}$$

where w is a commuting variable with x and y.

$$\begin{aligned}
L(w,x,y) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^m \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix}_q E_{n+k}(w,q) q^{-kn} \frac{y^n}{[n]_q!}\frac{x^m}{[m]_q!} \\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^m \sum_{k=0}^{m} E_{n+k}(w,q) q^{-kn} \frac{y^n}{[n]_q!}\frac{x^k}{[k]_q!}\frac{x^{m-k}}{[m-k]_q!} \\
&= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} E_{n+k}(w,q) \frac{(-x)^k}{[k]_q!}\frac{y^n}{[n]_q!}\frac{(-x)^j}{[j]_q!} \\
&= \left(\sum_{i=0}^{\infty} E_i(x,q) \sum_{k=0}^{i} \frac{(-x)^k}{[k]_q!}\frac{y^{i-k}}{[i-k]_q!}\right) e_q(-x) \\
&= \frac{2e_q(w(y-x))}{e_q(y-x)+1} e_q(-x).
\end{aligned}$$

$$\begin{aligned}
R(w,x,y) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^n \sum_{k=0}^{n} F_{m+k}(-w,q^{-1}) \frac{x^m}{[m]_q!}\frac{y^k}{[k]_q!}\frac{q^{\binom{n-k}{2}}y^{n-k}}{[n-k]_q!} \\
&= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{j+k} F_{m+k}(-w,q^{-1}) \frac{x^m}{[m]_q!}\frac{y^k}{[k]_q!}\frac{y^j}{[j]_{q^{-1}}!} \\
&= \left(\sum_{m=0}^{\infty}\sum_{k=0}^{\infty} F_{m+k}(-w,q^{-1}) \frac{x^m}{[m]_q!}\frac{(-y)^k}{[k]_q!}\right) e_q(-y) \\
&= \frac{2e_q(-w(y-x))}{e_{q^{-1}}(x-y)+1} e_{q^{-1}}(-y).
\end{aligned}$$

Hence it follows that

$$\begin{aligned}
R(w,x,y)e_q(y) &= \frac{2e_q(w(y-x))}{e_{q^{-1}}(x-y)+1} \\
&= \frac{2e_q(w(y-x))}{e_q(y-x)+1} e_q(y-x) \\
&= L(w,x,y)e_q(y),
\end{aligned}$$

which provides $R(w,x,y) = L(w,x,y)$. Therefore we can complete the proof. $\square$

As $q \to 1$ in (8) of Theorem 13, we have a symmetric relation for the ordinary Euler polynomials:

Theorem 14. *For any $m, n \in \mathbb{N}$, we have*

$$(-1)^m \sum_{k=0}^{m} \binom{m}{k} E_{n+k}(x) = (-1)^n \sum_{k=0}^{n} \binom{n}{k} E_{m+k}(-x).$$

Theorem 15 (q-Symmetry 2)**.** *For any $m, n \in \mathbb{N}$, we have*

$$(-1)^m \sum_{k=0}^{m+1} \begin{bmatrix} m+1 \\ k \end{bmatrix}_q [n+k+1]_q E_{n+k}(x,q) q^{-k(n+1)-\binom{n}{2}+mn+1}$$
$$+ (-1)^n \sum_{k=0}^{n+1} \begin{bmatrix} n+1 \\ k \end{bmatrix}_{q^{-1}} [m+k+1]_{q^{-1}} F_{m+k}(-x,q^{-1}) q^{k(m+1)+\binom{m}{2}} = 0. \quad (9)$$

Proof. Applying q-derivative formula to the identity (8) in Theorem 13 replaced m, n by $m+1$, $n+1$, respectively, we have the result. □

As $q \to 1$ in (9) of Theorem 15, we have another symmetric formula for the ordinary Euler polynomials:

Theorem 16. *For any $m, n \in \mathbb{N}$, we have*

$$(-1)^m \sum_{k=0}^{m+1} \binom{m+1}{k} (n+k+1) E_{n+k}(x) + (-1)^n \sum_{k=0}^{n+1} \binom{n+1}{k} (m+k+1) E_{m+k}(-x) = 0. \quad (10)$$

This can be regarded as an Euler polynomial version of Kaneko-Momiyama formulae for Bernoulli numbers. To be precise, put $m = n$ and $x = 0$ in (10). Then we have an analogue of Kaneko's formula:

Theorem 17. *For any $n \in \mathbb{N}$,*

$$\sum_{k=0}^{n+1} \binom{n+1}{k} (n+k+1) E_{n+k} = 0.$$

Then we obtain the nice formula

$$E_{2n+1} = -\frac{1}{n+1} \sum_{k=0}^{n} \binom{n+1}{k} (n+k+1) E_{n+k}. \quad (11)$$

Remark 18. The formula (11) has a strong resemblance to the usual recurrence (6), ($0 \leq k \leq 2n$). Using the formula (6) and according to the fact that $E_k = 0$ for k even positive integers, we need the n first terms with odd indexes k to compute E_{2n+1}. But the recurrence formula (11) needs only half the number of those terms (E_k with $n \leq k \leq 2n$ with k odd) to calculate E_{2n+1}.

Put $x = 0$ in (10). Then we have an analogue of Momiyama's formula:

Theorem 19. *For any $m, n \in \mathbb{N}$, we have*

$$(-1)^m \sum_{k=0}^{m+1} \binom{m+1}{k} (n+k+1)E_{n+k} + (-1)^n \sum_{k=0}^{n+1} \binom{n+1}{k} (m+k+1)E_{m+k} = 0.$$

Using q-integral formula to (8) in Theorem 13, we have

Theorem 20 (q-Symmetry 3)**.** *For any $m, n \in \mathbb{N}$ and $a, b \in \mathbb{R}$,*

$$(-1)^m \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix}_q \frac{E_{n+k+1}(a,q) - E_{n+k+1}(b,q)}{[n+k+1]_q} q^{-kn+mn} + (-1)^n \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} \frac{F_{m+k+1}(-a,q^{-1}) - F_{m+k+1}(-b,q^{-1})}{[m+k+1]_q} q^{\binom{n}{2}-\binom{k}{2}} = 0.$$

As $q \to 1$, we get

Theorem 21. *For any $m, n \in \mathbb{N}$ and $a, b \in \mathbb{R}$,*

$$(-1)^m \sum_{k=0}^{m} \binom{m}{k} \frac{E_{n+k+1}(a) - E_{n+k+1}(b)}{n+k+1} + (-1)^n \sum_{k=0}^{n} \binom{n}{k} \frac{E_{m+k+1}(-a) - E_{m+k+1}(-b)}{m+k+1} = 0.$$

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