

Euler-Seidel matrices over \mathbb{F}_p

Nesrin TUTAŞ*

Department of Mathematics, Akdeniz University, Antalya, Turkey

Received: 19.09.2012 • Accepted: 29.03.2013 • Published Online: 09.12.2013 • Printed: 20.01.2014

Abstract: A Euler–Seidel matrix is determined by an infinite sequence whose elements are given by recursion. The recurrence relations are investigated for numbers and polynomials such as hyperharmonics, Lucas numbers, and Euler and Genocchi polynomials. Linear recurring sequences in finite fields are employed, for instance, in coding theory and in several branches of electrical engineering. In this work, we define the period of a Euler–Seidel matrix over a field \mathbb{F}_p with p elements, where p is a prime number. We give some results for the matrix whose initial sequence is $\{s_r(n)\}_{n=0}^{\infty}$, where $s_r(n) = \sum_{k=0}^n \binom{n}{k}^r$, $n \geq 0$, and r is a fixed positive number. The numbers $s_r(n)$ play an important role in combinatorics and number theory. These numbers are known as Franel numbers for $r = 3$.

Key words: Euler–Seidel matrix

1. Introduction

The Euler–Seidel matrix corresponding to the infinite sequence $\{a_n\}_{n=0}^{\infty}$ was introduced in [3]. In [3] an algorithm that is useful for investigating recurrence relations and identities for numbers and polynomials was described. Other authors applied this algorithm to hyperharmonic and ordinary and incomplete Fibonacci and Lucas numbers in [2], and they gave some closed formulas for Bernoulli, Euler, and Genocchi polynomials. In this paper we investigate the behavior of the Euler–Seidel matrix in characteristic p .

The Euler–Seidel matrix corresponding to the initial sequence $\{a_n\}_{n=0}^{\infty}$ is determined by the sequences $a_{[k,n]}$ whose elements are recursively given by

$$\begin{aligned} a_{[0,n]} &= a_n, \quad n \geq 0, \\ a_{[k,n]} &= a_{[k-1,n]} + a_{[k-1,n+1]}, \quad n \geq 0, \quad k \geq 1, \end{aligned}$$

where $a_{[0,n]}$ is the initial row's n th entry, and $a_{[k,n]}$ is the (k, n) th entry of the matrix. It can be seen easily that $a_{[k,n]} = \sum_{r=0}^k \binom{k}{r} a_{[0,n+r]}$ from the recurrence relation. If the zeroth row is given, then one can find the zeroth column, and vice versa. More precisely, the zeroth column and row of the matrix depend on each other in the following way:

$$a_{[n,0]} = \sum_{r=0}^n \binom{n}{r} a_{[0,r]} \quad \text{and} \quad a_{[0,n]} = \sum_{r=0}^n \binom{n}{r} (-1)^{n-r} a_{[r,0]}.$$

*Correspondence: ntutas@akdeniz.edu.tr

2000 AMS Mathematics Subject Classification: 11B50, 11B83.

If $a(t) = \sum_{n=0}^{\infty} a_{[0,n]} t^n$ and $A(t) = \sum_{n=0}^{\infty} a_{[0,n]} \frac{t^n}{n!}$ are the ordinary and the exponential generating functions of the sequence $a_{[0,n]}$, then the ordinary and the exponential generating functions of the sequence $a_{[n,0]}$ are given by

$$\bar{a}(t) = \sum_{n=0}^{\infty} a_{[n,0]} t^n = \frac{1}{1-t} a\left(\frac{t}{1-t}\right), \quad \bar{A}(t) = \sum_{n=0}^{\infty} a_{[n,0]} \frac{t^n}{n!} = e^t A(t).$$

These identities were proven by Euler and Seidel, respectively [4, 7].

Throughout this work we will denote the Euler–Seidel matrix of an initial sequence $\{a_n\}_{n=0}^{\infty}$ by $S(a_n)$, while $[x]$ will denote the integer part of x . In the next section we define the period of the Euler–Seidel matrix in the case that $a_n \in \mathbb{F}_p$ for all $n \geq 0$. Here \mathbb{F}_p denotes a finite field with p elements, where p is a prime number. We will also study the special case in which the initial sequence $\{a_n\}_{n=0}^{\infty}$ is given by $\{s_r(n)\}_{n=0}^{\infty}$. For any $r \in \mathbb{N}$ we consider the sums $s_r(n) = \sum_{k=0}^n \binom{n}{k}^r$, $n \geq 0$. These sums have been studied by many authors

[5, 6, 8]. Apart from the trivial recurrences for $s_1(n) = 2^n$ and $s_2(n) = \binom{2n}{n}$, Calkin claimed that for $3 \leq r \leq 9$, there is no closed form for $s_r(n)$ in [1]. Franel was the first to obtain recurrences for $s_3(n)$ and $s_4(n)$ [5, 6]. For nonnegative integers a, b and a prime number p , Lucas’s theorem expresses the following congruence relation:

$\binom{a}{b} \equiv \prod_{i=0}^t \binom{a_i}{b_i} \pmod{p}$, where $a = a_0 + a_1p + \dots + a_t p^t$ and $b = b_0 + b_1p + \dots + b_t p^t$ are the base p expansions of a and b , respectively. In this work, if $n = \alpha_0 + \alpha_1p + \dots + \alpha_t p^t$ and $0 \leq \alpha_i < p$, then we can calculate $s_r(n)$

as the product $\prod_{i=0}^t s_r(\alpha_i) \pmod{p}$ by using Lucas’s theorem. Moreover, if k is the order of 2 in \mathbb{F}_p^* , which is

the largest multiplicative subgroup of the field \mathbb{F}_p with p elements, where p is a prime number, then the period

of the Euler–Seidel matrix of an initial sequence $\{s_r(n)\}_{n=0}^{\infty}$ is kp^{p-1} , and we have $\sum_{\gamma=1}^k \binom{k}{\gamma} s_r(p^{p-1}\gamma + j) = 0$

for any odd integer $r > 0$ and $j \in \mathbb{N}$.

2. Euler–Seidel matrices over \mathbb{F}_p

Example 1 If $\{a_n\}_{n=0}^{\infty} = \{c\}_{n=0}^{\infty}$ is an initial sequence and c is a fixed positive integer, then the k th row of $S(a_n)$ is the sequence $\{2^k c\}_{n=0}^{\infty}$. If $2^k \equiv 1 \pmod{p}$, then the k th row of $S(a_n)$ and $\{a_n\}_{n=0}^{\infty}$ are the same. If $p = 2$, $\{a_n\}_{n=0}^{\infty} = \{0\}_{n=0}^{\infty}$ or $\{a_n\}_{n=0}^{\infty} = \{1\}_{n=0}^{\infty}$, then the k th row of $S(a_n)$ is zero, $k \geq 1$. In case $\{a_n\}_{n=0}^{\infty}$ is a sequence of period 2, then the k th row of $S(a_n)$ is the sequence $\{2^{k-1}(a_0 + a_1)\}_{n=0}^{\infty}$. If $2^{k-1} \equiv 1 \pmod{p}$, then the k th row and the first row are the same.

Unless otherwise stated we will make the following main assumptions: we assume that p is an *odd prime number*, \mathbb{F}_p is a finite field with p elements and \mathbb{F}_p^* is its largest multiplicative subgroup, and $\{a_n\}_{n=0}^{\infty}$ is a periodic sequence over \mathbb{F}_p and its period is an *odd integer*, $a_i + a_{i+1} \not\equiv 0 \pmod{p}$, $i \geq 1$.

The period of a sequence over a nonempty set A is a well-known concept. After the following lemma, we will define the period of the Euler–Seidel matrix in Definition 3.

Lemma 2 Let p be an odd prime number and \mathbb{F}_p be a finite field with p elements. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of finite period over \mathbb{F}_p . Then there exists a positive integer r such that the r th row of the Euler–Seidel matrix $S(a_n)$ is the same as $\{a_n\}_{n=0}^{\infty}$.

Proof Let I denote the zeroth row of the Euler-Seidel matrix. The row X can be obtained by performing left shift on the zeroth row. Then the Euler-Seidel matrix $S(a_n)$ can be constructed as follows:

$$\begin{array}{lcl} I & : & a_0 \quad a_1 \quad a_2 \quad a_3 \quad \cdots \\ X & : & a_1 \quad a_2 \quad a_3 \quad a_4 \quad \cdots \\ I + X & : & a_0 + a_1 \quad a_1 + a_2 \quad a_2 + a_3 \quad a_3 + a_4 \quad \cdots \end{array}$$

The r th row of the matrix is denoted by $(1 + X)^r$, $r \geq 1$. If the period of $\{a_n\}_{n=0}^\infty$ is k , then actually it can be represented as $(1 + X)^r \pmod{X^k - 1}$. One studies the order of $1 + X$ in the $\mathbb{F}_p[X]/\langle X^k - 1 \rangle$. If $h(X)$ is an irreducible factor of $X^k - 1$ of degree m , then the finite field $\mathbb{F}_p[X]/\langle h(X) \rangle$ has a cyclic multiplicative group of order $p^m - 1$. The order of $1 + X$ divides $p^m - 1$. Hence, we have the lemma. \square

Definition 3 Let p be an odd prime number and let $\{a_n\}_{n=0}^\infty$ be a sequence with finite period over \mathbb{F}_p . Let S_j denote the j th row of the Euler-Seidel matrix $S(a_n)$. The Euler-Seidel matrix $S(a_n)$ is called periodic over \mathbb{F}_p , if there exists a nonnegative integer k such that $S_r = S_k$, $k < r$, for some r . The smallest positive integer r is called the period of the matrix $S(a_n)$ over \mathbb{F}_p , denoted by $\text{per}(S(a_n))$. If $\{a_n\}_{n=0}^\infty = \{0\}_{n=0}^\infty$, we define the period of $S(a_n)$ to be 1.

Example 4 Take $\{a_n\}_{n=0}^\infty = \{n\}_{n=0}^\infty$, $n \geq 0$ as the initial sequence. Then one can observe that $a_{[t,j]} = a_{[2,0]}a_{[t-2,j+1]}$, $t \geq 2$. The (t, j) entry of $S(n)$ can be expressed in the row number t and the column number j as

$$a_{[t,j]} = \begin{cases} 4^k(2j + t), & \text{if } t \text{ is odd,} \\ 4^k(j + k), & \text{if } t \text{ is even,} \end{cases}$$

where $k = \lfloor \frac{t}{2} \rfloor$. The period of the matrix $S(n)$ can be determined by viewing the entries modulo p , and the period is $p(p - 1)$.

Lemma 5 Let p be an odd prime number and let $\{a_n\}_{n=0}^\infty$ be a sequence over \mathbb{F}_p with period p . For any positive integers t, k and $0 \leq j < p$, the following statements hold for the Euler-Seidel matrix $S(a_n)$:

- (a) $a_{[t, kp+j]} = a_{[t,j]}$.
- (b) $a_{[kp+t,j]} = 2^k a_{[t,j]}$, $0 \leq t < p - 1$.
- (c) $a_{[p-1,j]} = a_{[0,j]} + \sum_{\lambda=1}^{\frac{p-1}{2}} \{a_{[0,j+2\lambda]} - a_{[0,j+(2\lambda-1)]}\}$.
- (d) $a_{[0,j]} = a_{[p^n,j]} - a_{[0,p^n+j]}$, for any $n \geq 1$.

Proof The lemma follows directly from the definition of $S(a_n)$. \square

Here we note that these equalities in Lemma 5 are independent from the ordering of the terms of the initial sequence.

Proposition 6 Let p be an odd prime number and let $\{a_n\}_{n=0}^\infty$ be a sequence over \mathbb{F}_p . Let $S(a_n)$ be the Euler-Seidel matrix corresponding to $\{a_n\}_{n=0}^\infty$. Then we have:

- (a) If p is the period of $\{a_n\}_{n=0}^\infty$, then $\text{per}(S(a_n)) = pk$, where k is the order of 2 in \mathbb{F}_p^* .

(b) Let p be the period of $\{a_n\}_{n=0}^\infty$ and $\{b_n\}_{n=0}^\infty = (a_0, \dots, a_0, \dots, a_{p-1}, \dots, a_{p-1}, a_0, \dots)$, where a_t is repeated p times for each t , $0 \leq t \leq p-1$. Then $\text{per}(S(b_n)) = p^2k$, where k is the order of 2 in \mathbb{F}_p^* .

(c) Let $\{a_n^{(1)}\}_{n=0}^\infty, \{a_n^{(2)}\}_{n=0}^\infty, \dots, \{a_n^{(m)}\}_{n=0}^\infty$ be m distinct sequences of period p over \mathbb{F}_p . For $j \in \mathbb{N}$, we write $j = \alpha_{m-1}p^{m-1} + \dots + \alpha_1p + \alpha_0$, where $0 \leq \alpha_i \leq p-1$, $0 \leq i \leq m-1$. Let $\{c_n\}_{n=0}^\infty$ be a sequence with j th entry $c_j = (a_{\alpha_{m-1}}^{(1)}, \dots, a_{\alpha_1}^{(m-1)}, a_{\alpha_0}^{(m)})$. Then $\text{per}(S(c_n)) = p^mk$, where k is the order of 2 in \mathbb{F}_p^* .

(d) The Euler–Seidel matrix $S(a_n)$ is the product of a Pascal and a Hankel matrix. If p is the period of the sequence $\{a_n\}_{n=0}^\infty$, then $S(a_n)$ can be considered as a square matrix over \mathbb{F}_p .

Proof (a) By Lemma 5 (b), we have $a_{[kp,j]} = 2^k a_{[0,j]}$. On the other hand, if k is the order of 2 in \mathbb{F}_p^* , then $2^k \equiv 1 \pmod{p}$. For proving (b), we again use Lemma 5 (b), $a_{[kp^2,j]} = 2^k a_{[0,j]}$. The assertion in (c) can be proven using similar ideas and induction. (d) Using the notation of the proof of Lemma 2, we form the Euler–Seidel matrix $\mathbf{S} = S(a_n)$ as $\mathbf{S} = \mathbf{L}\mathbf{X}$, where

$$\mathbf{L} = \begin{pmatrix} \binom{0}{0} & 0 & 0 & 0 & \cdots \\ \binom{1}{0} & \binom{1}{1} & 0 & 0 & \cdots \\ \binom{2}{0} & \binom{2}{1} & \binom{2}{2} & 0 & \cdots \\ \binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \binom{3}{3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}, \mathbf{X} = \begin{pmatrix} 1 \\ X \\ X^2 \\ X^3 \\ X^4 \\ \vdots \end{pmatrix}, \mathbf{S} = \begin{pmatrix} 1 \\ 1+X \\ (1+X)^2 \\ (1+X)^3 \\ (1+X)^4 \\ \vdots \end{pmatrix}$$

and $(1+X)^r$ is the r th row of \mathbf{S} . Here we note that \mathbf{L} is a Pascal and \mathbf{X} is a Hankel matrix. Since a Euler–Seidel matrix has finite period over \mathbb{F}_p , we can identify \mathbf{S} with a finite-type matrix. In our case, its type will be $pk \times p$, where k is of the order of 2. On the other hand, since the initial sequence is periodic, we can take \mathbf{S} as a square matrix of type $pk \times pk$ matrix. \square

Lemma 7 With the notation from Proposition 6, $\mathbf{L}^\mu \mathbf{X}$ and $\mathbf{L}^{-\mu} \mathbf{X}$ are matrices whose r th rows are $(\mu+X)^r$ and $(-\mu+X)^r$, for $r \geq 0$, $\mu \geq 1$, respectively. The (n, j) th entry of $\mathbf{L}^2 \mathbf{X}$ is $\sum_{\lambda=0}^n \sum_{r=0}^n \binom{n}{\lambda} \binom{\lambda}{r} 2^{n-\lambda} a_{r+j}$. If $\mu = p$ equals the period of $\{a_n\}_{n=0}^\infty$, then $\mathbf{L}^{-p} \mathbf{X} = \mathbf{L}^p \mathbf{X} = \mathbf{X}$.

Proof This lemma can be proven by direct calculation. \square

Lemma 8 Let p and q be distinct odd prime numbers, and let $\{a_n\}_{n=0}^\infty$ be a periodic sequence over \mathbb{F}_p with period q . Let $p = qs + u$, $0 < u < q$, and $u, s, j \in \mathbb{N}$. Then the following statements hold:

(a) For $k \in \mathbb{N}$, we have

$$a_{[k(p+1), j]} = \sum_{\lambda=0}^k \sum_{\gamma=0}^k \binom{k}{\lambda} \binom{k}{\gamma} a_{[0, \gamma u + \lambda + j]} = \sum_{\gamma=0}^k \binom{k}{\gamma} a_{[k, \gamma u + j]}.$$

(b) If $u = 1$, then $\text{per}(S(a_n)) = p-1$.

(c) If $u = q-1$, then $\text{per}(S(a_n)) = (p-1)q$.

(d) Especially if $q = 2$, $a_0 \not\equiv -a_1 \pmod{p}$ and k is the order of 2 in \mathbb{F}_p^* , then $\text{per}(S(a_n)) = k$.

Proof (a) Let $p = qs + u$, $0 < u < q$, and $u, s, j \in \mathbb{N}$. Using induction, we have

$$a_{[kp,j]} = \sum_{\gamma=0}^k \binom{k}{\gamma} a_{[0,\gamma u+j]}$$

and we obtain

$$\begin{aligned} a_{[kp+k,j]} &= \sum_{\gamma=0}^k \binom{k}{\gamma} a_{[kp,j]} \\ &= \sum_{\gamma=0}^k \binom{k}{\gamma} \sum_{\lambda=0}^k \binom{k}{\lambda} a_{[0,\gamma u+\lambda+j]} = \sum_{\gamma=0}^k \binom{k}{\gamma} a_{[k,\gamma u+j]}. \end{aligned}$$

(b) By direct calculation, we have

$$a_{[p-1,j]} = a_{[0,j]} + \sum_{\lambda=1}^s (-1)^{\lambda q} a_{[0,\lambda q+j]} + \sum_{\lambda=1}^{q-1} (-1)^{\lambda} [(-1)^q + \dots + (-1)^{sq}] a_{[0,\lambda+j]}$$

and since s is an even number, we get $a_{[p-1,j]} = a_{[0,j]} = a_j$. (c) Since s is an odd number and $0 \leq j < q$, we have

$$\begin{aligned} a_{[p-1,j]} &= \sum_{\lambda=0}^{p-1} (-1)^{\lambda} a_{[0,\lambda+j]} = \sum_{\lambda=0}^{q-1} (-1)^{\lambda} a_{[0,\lambda+j]} + \sum_{\lambda=0}^{u-1} (-1)^{\lambda+sq} a_{[0,\lambda+sq+j]} \\ &= (-1)^{q-1} a_{[0,q-1+j]} = a_{[0,q-1+j]}. \end{aligned}$$

Hence, $\text{per}(S(a_n)) = (p-1)q$. For (d), it is enough to see that any entry of the k th row is $2^k(a_0 + a_1)$. \square

Theorem 9 Let p and q be distinct odd primes, where $p = qs + u$, $0 < u < q$, $u, s \in \mathbb{N}$. Let $\{a_n\}_{n=0}^{\infty}$ be a nonconstant sequence over \mathbb{F}_p with period q . Then the following statements hold:

(a) If d is of the order of u in \mathbb{F}_q^* and r is a positive integer such that

$$r = \min \left\{ \frac{p^{d/2} - 1}{l} \in \mathbb{N} : (1+x)^{\frac{p^{d/2}-1}{l}} = x^i, \text{ for some } i \leq q-1 \right\},$$

then we have

$$\text{per}(S(a_n)) = \begin{cases} rq, & \text{if } d \text{ is even} \\ p^d - 1, & \text{if } d \text{ is odd.} \end{cases}$$

(b) If β is a sequence formed by means of $\{a_n\}_{n=0}^{\infty}$ such that each a_i is repeated p times in β , $0 \leq i \leq q-1$, then the period of the Euler–Seidel matrix corresponding to the new initial sequence β is $\text{per}(S(a_n))p$.

Proof (a) Let $p = qs + u$, $0 < u < q$, $u, s \in \mathbb{N}$. It is well known that $X^q - 1 = (X-1)\Phi_q(X)$, where $\Phi_q(X)$ is the q th cyclotomic polynomial. Since $(p, q) = 1$, the polynomial $\Phi_q(X)$ factors into $(q-1)/d$ monic

irreducible polynomials in $\mathbb{F}_p[X]$ of the same degree d . Let K be the splitting field of any such irreducible factor over \mathbb{F}_p . We have $[K : \mathbb{F}_p] = d$, where d is the order of u in \mathbb{F}_q^* . Let $\theta \in K$ be a root of such a factor of the polynomial $\Phi_q(X)$. Then K^* has $p^d - 1$ elements, $\theta^q = 1$ and $q|(p^d - 1)$. If d is an even natural number, then $q|(p^{d/2} + 1)$ and

$$(1 + \theta)^{p^{d/2}-1} = \frac{1 + \theta^{p^{d/2}}}{1 + \theta} = \frac{1 + \theta^{-1}}{1 + \theta} = \theta^{q-1}.$$

Therefore, the order of $1 + \theta$ divides $(p^{d/2} - 1)q$. If d is an odd natural number, we have $(1 + \theta)^{p^d-1} = \frac{1 + \theta^{p^d}}{1 + \theta} = 1$ and d is minimal since d is of the order of u . For proving (b), one should see that the period of the initial sequence will be pq , since $(X^{pq} - 1) = (X^q - 1)^p$. □

Example 10 (a) Firstly, we take $p = 5$, $q = 3$, and $\{a_n\}_{n=0}^\infty = \{1, 0, 2, 1, \dots\}$, and then $\text{per}(S(a_n)) = 12$.

(b) Now we take $p = 11$, $q = 5$, and $\{a_n\}_{n=0}^\infty = \{4, 2, 3, 1, 0, \dots\}$. In this case, $\text{per}(S(a_n)) = 10$. If we choose $\{a_n\}_{n=0}^\infty = \{7, 3, 5, 2, 10, 8, 4, \dots\}$ and $q = 7$, then the period of the matrix will be 1330.

For an illustration of the results above, we present the following Euler–Seidel matrices:

$$\left(\begin{array}{cccccccc} 1 & 0 & 2 & 1 & 0 & 2 & 1 & \dots \\ 1 & 2 & 3 & 1 & 2 & 3 & 1 & \dots \\ 3 & 0 & 4 & 3 & 0 & 4 & 3 & \dots \\ 3 & 4 & 2 & 3 & 4 & 2 & 3 & \dots \\ 2 & 1 & 0 & 2 & 1 & 0 & 2 & \dots \\ 3 & 1 & 2 & 3 & 1 & 2 & 3 & \dots \\ 4 & 3 & 0 & 4 & 3 & 0 & 4 & \dots \\ 2 & 3 & 4 & 2 & 3 & 4 & 2 & \dots \\ 0 & 2 & 1 & 0 & 2 & 1 & 0 & \dots \\ 2 & 3 & 1 & 2 & 3 & 1 & 2 & \dots \\ 0 & 4 & 3 & 0 & 4 & 3 & 0 & \dots \\ 4 & 2 & 3 & 4 & 2 & 3 & 4 & \dots \\ 1 & 0 & 2 & 1 & 0 & 2 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \right) \quad \left(\begin{array}{cccccccc} 4 & 2 & 3 & 1 & 0 & \dots \\ 6 & 5 & 4 & 1 & 4 & \dots \\ 0 & 9 & 5 & 5 & 10 & \dots \\ 9 & 3 & 10 & 4 & 10 & \dots \\ 1 & 2 & 3 & 3 & 8 & \dots \\ 3 & 5 & 6 & 0 & 9 & \dots \\ 8 & 0 & 6 & 9 & 1 & \dots \\ 8 & 6 & 4 & 10 & 9 & \dots \\ 3 & 10 & 3 & 8 & 6 & \dots \\ 2 & 2 & 0 & 3 & 9 & \dots \\ 4 & 2 & 3 & 1 & 0 & \dots \\ 6 & 5 & 4 & 1 & 4 & \dots \\ 0 & 9 & 5 & 5 & 10 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \right)$$

a) the case $p=5$, $q=3$, and $\{a_n\}_{n=0}^\infty = \{1, 0, 2, 1, 0, \dots\}$.

b) the case $p=11$, $q=5$, and $\{a_n\}_{n=0}^\infty = \{4, 2, 3, 1, 0, \dots\}$.

3. Euler–Seidel matrices with initial sequence $\{s_r(n)\}_{n=0}^\infty$

In this section, we will explain the Euler–Seidel matrix with initial sequence $\{s_r(n)\}_{n=0}^\infty$, where $s_r(n) = \sum_{k=0}^n \binom{n}{k}^r$, $n \geq 0$, for any $r \in \mathbb{N}$. Using the period of the Euler–Seidel matrix, we can express $s_r(n)$ in subsequent terms of the sequence.

Firstly, we choose a fixed number $n \in \mathbb{N}$, for any odd prime number p .

Lemma 11 For a fixed natural number n , let $\{s_r(n)\}_{r=0}^\infty$ be the initial sequence of the Euler–Seidel matrix S over \mathbb{F}_p . Then the following statements hold:

- (a) $s_{kp+j}(n) = \sum_{\lambda=0}^n \binom{n}{\lambda}^k \binom{n}{\lambda}^j = s_{k+j}(n)$, for $k \geq 1$, $j, k \in \mathbb{N}$.
- (b) $s_{r+m(p-1)}(n) = s_r(n)$, $m \in \mathbb{N}$.
- (c) $\text{per}(S(s_r(n))) = p - 1$ and $s_r(n) = \sum_{\lambda=0}^{p-1} \sum_{\gamma=0}^n \binom{p-1}{\lambda} \binom{n}{\gamma}^{\lambda+r}$.

Proof (a) and (b) can be proved by induction. (c) We have $s_{p-1}(n) = s_0(n)$ and $s_p(n) = s_1(n)$. Denoting the first row of the Euler–Seidel matrix by $1+X$, we have, since $X^{p-1} = 1$, that $(1+X)^{p-1} = (1+X^p)/(1+X) = 1$. This means that the zeroth row of the matrix and row $(p - 1)$ are the same. Therefore, we have

$$s_r(n) = \sum_{\lambda=0}^{p-1} \binom{p-1}{\lambda} s_{\lambda+r}(n) = \sum_{\lambda=0}^{p-1} \sum_{\gamma=0}^n \binom{p-1}{\lambda} \binom{n}{\gamma}^{\lambda+r}.$$

□

On the other hand, if we choose a fixed number $r \in \mathbb{N}$, and $k \in \mathbb{N}$, then the following congruence properties of $s_r(n)$ over \mathbb{F}_p can be proven:

- (a) $s_r(p) = s_r(1) = 2$, and $s_r(kp) = s_r(k)$.
- (b) $s_r(p - 1) = \begin{cases} 1, & \text{if } r \text{ is odd} \\ 0, & \text{if } r \text{ is even.} \end{cases}$
- (c) $s_r(kp + i) = s_r(k)s_r(i)$, for $1 \leq i \leq p - 1$.

Theorem 12 *Let r be a fixed nonnegative integer and let p be an odd prime number, $n \in \mathbb{N}$. Let $\{s_r(n)\}_{n=0}^\infty$ be an initial sequence of the Euler–Seidel matrix over \mathbb{F}_p and $a_{[i,j]}$ be the (i, j) th entry of the matrix. Then the following statements hold:*

- (a) *We have $a_{[kp,\mu p+\xi]} = a_{[k,\mu]}a_{[0,\xi]}$, where μ, ξ, k are nonnegative integers and $0 \leq \xi < p - 1$.*
- (b) *For any positive integer m , we have $a_{[p^m,0]} = a_{[1,0]}$. If $j = \alpha_0 + \alpha_1 p + \dots + \alpha_t p^t$, and $0 \leq \alpha_i \leq p - 1$, $0 \leq t < m$, we have $a_{[p^m,j]} = a_{[1,0]}a_{[0,j]}$, and if $t \geq m$, then $a_{[p^m,j]} = a_{[0,j]}$.*
- (c) *If $\beta = kp + t$, $\xi + r = p$ and $0 \leq \xi < p - 1$, then we have*

$$a_{[\beta,\mu p+\xi]} = \begin{cases} a_{[k,\mu]} \sum_{\lambda=0}^{r-1} \binom{t}{\lambda} a_{[0,\xi+\lambda]} + a_{[k,\mu+1]} \sum_{\gamma=r}^t \binom{t}{\gamma} a_{[0,\gamma-r]}, & \xi \neq 0, r \leq t, \\ a_{[k,\mu]} a_{[t,\xi]}, & \xi \neq 0, r > t \text{ or} \\ & \xi = 0. \end{cases}$$

- (d) *If k is of the order of 2 in \mathbb{F}_p^* , then $\text{per}(S(s_r(n))) = kp^{p-1}$. For any odd integer $r > 0$ and $j \in \mathbb{N}$, we have $\sum_{\gamma=1}^k \binom{k}{\gamma} s_r(p^{p-1}\gamma + j) = 0$.*

Proof In order to prove our theorem, we must use Lucas’s theorem several times. Using the congruence properties of $s_r(n)$, we get

$$a_{[kp,\mu p+\xi]} = \sum_{\lambda=0}^k \binom{k}{\lambda} s_r((\lambda + \mu)p + \xi) = a_{[k,\mu]} a_{[0,\xi]}$$

and $a_{[p^m,0]} = a_{[0,0]}^m a_{[1,0]}$. If $j = \alpha_0 + \alpha_1 p + \dots + \alpha_t p^t$ and $0 \leq \alpha_i \leq p-1$, $t < m$, we obtain

$$a_{[p^m,j]} = \{a_{[0,p^{m-t}+\alpha_t]} + a_{[0,\alpha_t]}\} \prod_{i=0}^{t-1} a_{[0,\alpha_i]} = a_{[1,0]} a_{[0,j]}.$$

Similarly, if $t \geq m$, we have

$$a_{[p^m,j]} = a_{[0,\alpha_m+\dots+\alpha_t p^{t-m+1}]} \cdot \prod_{i=0}^{m-1} a_{[0,\alpha_i]} = \prod_{i=0}^t a_{[0,\alpha_i]} = a_{[0,j]}.$$

This proves (a) and (b). For proving (c), let $j = \mu p + \xi$, $\beta = kp + t$, where $0 \leq \xi$, $t < p-1$. Now we call $a_{[kp,j]}$ by $\tilde{b}_{[0,j]}$. In case $\xi = 0$, we get

$$a_{[kp+t,j]} = \tilde{b}_{[t,j]} = \sum_{\lambda=0}^t \binom{t}{\lambda} \tilde{b}_{[0,j+\lambda]}$$

and

$$a_{[kp,\mu p+\lambda]} = \sum_{\gamma=0}^k \binom{k}{\gamma} s_r((\gamma + \mu)) s_r(\lambda) = s_r(\lambda) a_{[k,\mu]}.$$

Then $\tilde{b}_{[t,j]} = a_{[k,\mu]} a_{[t,0]}$. If $\xi \neq 0$, $p = \xi + r$ and $r > t$, then $\xi + t < p$ and

$$\tilde{b}_{[t,j]} = \sum_{\lambda=0}^t \binom{t}{\lambda} \tilde{b}_{[0,\mu p+\xi+\lambda]} = \sum_{\lambda=0}^t \binom{t}{\lambda} a_{[k,\mu]} a_{[0,\xi+\lambda]} = a_{[k,\mu]} a_{[t,\xi]}.$$

If $r < t$, then

$$\begin{aligned} \tilde{b}_{[t,j]} &= a_{[k,\mu]} \sum_{\lambda=0}^{r-1} \binom{t}{\lambda} a_{[0,\xi+\lambda]} + \sum_{\gamma=r}^t \binom{t}{\gamma} \tilde{b}_{[0,(\mu+1)p+\gamma-r]} \\ &= a_{[k,\mu]} \sum_{\lambda=0}^{r-1} \binom{t}{\lambda} a_{[0,\xi+\lambda]} + a_{[k,\mu+1]} \sum_{\gamma=r}^t \binom{t}{\gamma} a_{[0,\gamma-r]}. \end{aligned}$$

Now we will prove (d). If $j = \alpha_0 + \alpha_1 p + \dots + \alpha_{p-2} p^{p-2}$, $\alpha_i = p-1$, $0 \leq i \leq p-2$, and r is an odd integer, then $s_r(j) = s_r(p-1)^{p-1} = s_r(0)$ and $s_r(j+1) = s_r(1)$. Then the period of the initial sequence is p^{p-1} . Therefore, for a positive integer k we have $(1+X)^{kp^{p-1}} = 2^k$ and if k is of the order of 2 in \mathbb{F}_p^* , then $\text{per}(S(s_r(n))) = kp^{p-1}$. Hence for any $j \in \mathbb{N}$, we have

$$a_{[kp^{p-1},j]} = \sum_{\gamma=0}^k \binom{k}{\gamma} s_r(p^{p-1}\gamma + j) = a_{[0,j]} \text{ and } \sum_{\gamma=1}^k \binom{k}{\gamma} s_r(p^{p-1}\gamma + j) = 0.$$

□

Acknowledgment

The author would like to thank Peter Beelen and the anonymous referees for many helpful suggestions and comments.

References

- [1] Calkin, N. Factors of sums of powers of binomial coefficients. *Acta Arith.* 86, 17–26 (1998).
- [2] Dil, A., Mező, I. Symmetric algorithm for hyperharmonic and Fibonacci Numbers. *Appl. Math. Comput.* 206, 942–951 (2008).
- [3] Dumont, D. Matrices d'Euler-Seidel. *Seminaire Lotharingien de Combinatoire.* B05c 1981.
- [4] Euler, L. *De transformatione serierum.* Opera Omnia 1: Teubner 1913.
- [5] Franel, J. On a question of Laisant. *L'intermédiaire des Mathématiciens* 1, 45–47 (1894).
- [6] Franel, J. On a question of J. Franel. *L'intermédiaire des Mathématiciens* 2, 33–35 (1895).
- [7] Seidel, P. Über eine einfache Entstehungsweise der Bernoullischen Zahlen und einiger verwandten. *Sitzungsberichte der Münh. Akad. Math. Phys.* 7, 157–187 (1877).
- [8] Yuan, J., Lu, Z., Schmidt, A. On recurrences for sums of powers of binomial coefficients. *J. Number Theory* 128, 2784–2794 (2008).