EULER–FROBENIUS NUMBERS AND ROUNDING

SVANTE JANSON

Abstract. We study the Euler–Frobenius numbers, a generalization of the Eulerian numbers, and the probability distribution obtained by normalizing them. This distribution can be obtained by rounding a sum of independent uniform random variables; this is more or less implicit in various results and we try to explain this and various connections to other areas of mathematics, such as spline theory.

The mean, variance and (some) higher cumulants of the distribution are calculated. Asymptotic results are given. We include a couple of applications to rounding errors and election methods.

1. Introduction

The *Euler–Frobenius polynomial* $P_{n,\rho}(x)$ can be defined by

$$\frac{P_{n,\rho}(x)}{(1-x)^{n+1}} = \Bigl(\rho + x\frac{\mathrm{d}}{\mathrm{d}x}\Bigr)^n \frac{1}{1-x} = \sum_{j=0}^{\infty}(j+\rho)^n x^j, \tag{1.1}$$

or, equivalently, by the recursion formula

$$P_{n,\rho}(x) = \bigl(nx + \rho(1-x)\bigr)P_{n-1,\rho}(x) + x(1-x)P'_{n-1,\rho}(x), \qquad n \geqslant 1, \tag{1.2}$$

with $P_{0,\rho}(x) = 1$; see Appendix A for details, some further results and references. Here $n = 0,1,2,\dots$, and ρ is a parameter that can be any complex number, although we shall be interested mainly in the case $0 \leqslant \rho \leqslant 1$. (The special cases $\rho = 0,1$ yield the *Eulerian polynomials*, see below.)

It is immediate from (1.2) that $P_{n,\rho}(x)$ is a polynomial in x of degree at most n. We write

$$P_{n,\rho}(x) = \sum_{k=0}^{n} A_{n,k,\rho}x^k. \tag{1.3}$$

The recursion (1.2) can be translated to the recursion

$$A_{n,k,\rho} = (k+\rho)A_{n-1,k,\rho} + (n-k+1-\rho)A_{n-1,k-1,\rho}, \qquad n \geqslant 1, \tag{1.4}$$

where we let $A_{n,k,\rho} = 0$ if $k \notin \{0,\dots,n\}$. Following [37], we call these numbers *Euler–Frobenius numbers*. See Table 1 for the first values. (The special cases $\rho = 0,1$ yield the *Eulerian numbers*, see below.)

Date: 15 May, 2013.

2010 *Mathematics Subject Classification.* 60C05; 05A15, 11B68, 41A15, 60E05, 60F05.

Partly supported by the Knut and Alice Wallenberg Foundation.

$n\backslash k$	0	1	2	3
0	1			
1	ρ	$1-\rho$		
2	ρ^2	$1+2\rho-2\rho^2$	$1-2\rho+\rho^2$	
3	ρ^3	$1+3\rho+3\rho^2-3\rho^3$	$4-6\rho^2+3\rho^3$	$1-3\rho+3\rho^2-\rho^3$

TABLE 1. The Euler–Frobenius numbers $A_{n,k,\rho}$ for small n.

We usually regard ρ as a fixed parameter, but we note that $P_{n,\rho}(x)$ also is a polynomial in ρ, see (A.8); thus the Euler–Frobenius numbers $A_{n,k,\rho}$ are polynomials in ρ, as also follows from (1.4). (Some papers conversely consider $P_{n,\rho}(x)$ as a polynomial in ρ, with x as a parameter; see e.g. [8; 78] and Appendix B.)

It follows from (1.4) that if $0 \leqslant \rho \leqslant 1$, then $A_{n,k,\rho} \geqslant 0$, so if we normalize by dividing by $\sum_{k=0}^{n} A_{n,k,\rho} = P_{n,\rho}(1) = n!$, see (A.4), we obtain a probability distribution on $\{0,\dots,n\}$; we call this distribution the *Euler–Frobenius distribution* and let $\mathfrak{E}_{n,\rho}$ denote a random variable with this distribution, i.e.

$$\mathbb{P}(\mathfrak{E}_{n,\rho} = k) = A_{n,k,\rho}/P_{n,\rho}(1) = A_{n,k,\rho}/n!. \tag{1.5}$$

Equivalently, $\mathfrak{E}_{n,\rho}$ has the probability generating function

$$\mathbb{E}\, x^{\mathfrak{E}_{n,\rho}} = \sum_{k=0}^{n} \mathbb{P}(\mathfrak{E}_{n,\rho} = k)x^k = \frac{P_{n,\rho}(x)}{P_{n,\rho}(1)} = \frac{P_{n,\rho}(x)}{n!}. \tag{1.6}$$

With a minor abuse of notation, we also denote this distribution by $\mathfrak{E}_{n,\rho}$.

Remark 1.1. Since $A_{1,0,\rho} = \rho$ and $A_{1,1,\rho} = 1-\rho$, see Table 1, the condition $0 \leqslant \rho \leqslant 1$ is also necessary for (1.5) to define a probability distribution for all $n \geqslant 1$. (We will extend the definition of $\mathfrak{E}_{n,\rho}$ to arbitrary ρ later, see (3.12), but (1.5) holds only for $\rho \in [0,1]$.)

The main purpose of the present paper is to show that this distribution occurs when rounding sums of uniform random variables, and to give various consequences and connections to other problems. Our main result can be stated as follows. (A proof is given in Section 3.)

Theorem 1.2. *Let $U_1,\dots,U_n$ be independent random variables uniformly distributed on $[0,1]$, and let $S_n := \sum_{i=1}^{n} U_i$. Then, for every $n \geqslant 1$ and $\rho \in [0,1]$, the random variable $\lfloor S_n+\rho \rfloor$ has the Euler–Frobenius distribution $\mathfrak{E}_{n,1-\rho}$, i.e.,*

$$\mathbb{P}\big(\lfloor S_n+\rho\rfloor = k\big) = \frac{A_{n,k,1-\rho}}{n!}, \qquad k \in \mathbb{Z}. \tag{1.7}$$

Theorem 1.2 can also be stated geometrically, see Section 2.

The distribution of S_n was calculated already by Laplace [54, pp. 257–260] (who used it for a statistical test showing that the orbits of the planets are not randomly distributed, while the orbits of the comets seem to be

$n\backslash k$	0	1	2	3	4	5	6
0	1						
1	1						
2	1	1					
3	1	4	1				
4	1	11	11	1			
5	1	26	66	26	1		
6	1	57	302	302	57	1	
7	1	120	1191	2416	1191	120	1

TABLE 2. The Eulerian numbers $A_{n,k,1} = A_{n,k+1,0}$ for small n. The row sums are $n!$.

random [54, pp. 261–265]), see also e.g. [29, Theorem I.9.1]. The case $\rho = 0$ (or $\rho = 1$) of Theorem 1.2, which is a connection between the distribution of S_n and Eulerian numbers (see below), is well-known, see e.g. [91], [31], [87], [75]. The case $\rho = 1/2$, which means standard rounding of S_n, is given in [13]. Moreover, the theorem is implicit in e.g. [78, Lecture 3], but I have not seen it stated explicitly in this form. This paper is therefore partly expository, trying to explain some of the many connections to other results in various areas. (However, we do not attempt to give a complete history. Furthermore, there are many papers on algebraic and other aspects of Euler–Frobenius polynomials and numbers that are not mentioned here.)

Before discussing rounding and Theorem 1.2 further, we return to the Euler–Frobenius polynomials and numbers to give some background.

The cases $\rho = 0$ and $\rho = 1$ are equivalent; we have $P_{n,0}(x) = xP_{n,1}(x)$ and thus $A_{n,k+1,0} = A_{n,k,1}$ and $\mathfrak{E}_{n,0} \overset{\mathrm{d}}{=} \mathfrak{E}_{n,1} + 1$ for all $n \geqslant 1$, as follows directly from (1.1) or by induction from (1.2) or (1.4). (Cf. (1.7), where the left-hand side obviously has the corresponding property.) This is the most important case and appears in many contexts. (Carlitz [8] remarks that these polynomials and numbers have been frequently rediscovered.) The numbers $A_{n,k,1}$ and the polynomials $P_{n,1}$ were studied already by Euler [24; 25; 26], and the numbers $A_{n,k,1}$ are therefore called *Eulerian numbers*, see [65, A173018] and Table 2; the usual modern notation is $\left\langle {n \atop k} \right\rangle$ [39], [64, §26.14]. Similarly, the polynomials $P_{n,1}(x)$ are usually called *Eulerian polynomials.* (Notation varies, and these names are also used for the shifted versions that we denote by $A_{n,k,0}$ and $P_{n,0}(x)$, see e.g. [65, A008292, A173018 and A123125]. Already Euler used both versions: $P_{n,0}$ in [24] and $P_{n,1}$ in [25; 26].)

Euler [24; 25; 26] used these numbers to calculate the sum of series; see also [46] and [32]. (In particular, Euler [26] calculated the sum of the divergent series $\sum_{k=1}^{\infty}(-1)^{k-1}k^n$ for integers $n \geqslant 0$; in modern terminology he found the Abel sum as $2^{-n-1}P_{n,1}(-1)$ by letting $x \to -1$ in (1.1).) They have since appeared in many other contexts. For example, the Eulerian

$n\backslash k$	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	6	1				
3	1	23	23	1			
4	1	76	230	76	1		
5	1	237	1682	1682	237	1	
6	1	722	10543	23548	10543	722	1

TABLE 3. The Eulerian numbers of type B, $B_{n,k} = 2^n A_{n,k,\frac{1}{2}}$, for small n. The row sums are $2^n n!$.

number $A_{n,k,1} = \left\langle{n \atop k}\right\rangle$ equals the number of permutations of length n with k descents (or ascents), see e.g. [74, Chapter 8.6], [18, Chapter 10] or [88, Section 1.3]; this well-known combinatorial interpretation is often taken as the definition of Eulerian numbers. (In the terminology introduced above, the number of descents in a random permutation thus has the Euler–Frobenius distribution $\mathfrak{E}_{n,1}$.) Furthermore, the Eulerian numbers also enumerate permutations with k exceedances, see again [88, Section 1.3], where also further related combinatorial interpretations are given. See also [15] for an enumeration with staircase tableaux, and [33] for further related results. Some other examples where Eulerian numbers and polynomials appear are number theory [35], [58, p. 328], summability [67, p. 99], statistics [52], control theory [98], and splines [78; 79], [80, Table 2, p. 137] (see also Appendix B).

The case $\rho = 1/2$ also occurs in several contexts. In this case, it is often more convenient to consider the numbers $B_{n,k} := 2^n A_{n,k,1/2}$ which are integers and satisfy the recursion

$$B_{n,k} = (2k+1)B_{n-1,k} + (2n-2k+1)B_{n-1,k-1}, \qquad n \geqslant 1, \tag{1.8}$$

with $B_{0,0} = 1$ (and $B_{n,k} = 0$ if $k \notin \{0,\dots,n\}$), see [65, A060187] and Table 3. These numbers are sometimes called *Eulerian numbers of type B*. They seem to have been introduced by MacMahon [58, p. 331] in number theory. (It seems likely that they were used already by Euler, who in [26] also says, without giving the calculation, that he can prove similar results for $\sum_{k=1}^{\infty}(-1)^k(2k-1)^m$; see [46] for a calculation using $P_{n,1/2}$ and methods of [26].) The numbers $B_{n,k}$ also have combinatorial interpretations, for example as the numbers of descents in signed permutations, i.e., in the hyperoctahedral group [5; 14; 76]. Furthermore, the numbers $B_{n,k}$ and the distribution $\mathfrak{E}_{n,1/2}$ appear in the study of random staircase tableaux [17]. $P_{n,1/2}(x)$ and $B_{n,k}$ appear in spline theory [78, Lecture 3.4] (see Appendix B). They also appear (as do $P_{n,1}(x)$ and $A_{n,k,1}$) in [34], as special cases of more general polynomials.

The general polynomials $P_{n,\rho}$ were perhaps first introduced by Carlitz [8] (in the form $P_{n,\rho}(x)/(x-1)^n$, cf. (1.9) below). They are important in spline theory, see e.g. [78, Lecture 3], [79], [92], [73] and Appendix B. They appear also (as a special case) in the study of random staircase tableaux [47]. Note also that the function (1.1) is the special case $s \in \{0, -1, -2, \dots\}$ of Lerch's transcendental function $\Phi(z, s, \rho) = \sum_{j=0}^{\infty} (j + \rho)^{-s} z^j$, see [55], [64, §25.14] and e.g. [94] with further references. The general Eulerian Numbers $A_{n,k}(a, d)$ defined by [99] equal our $d^n A_{n,k+1,1-a/d}$.

The special case $\rho = 1/N$ where $N \geqslant 1$ is an integer appears in combinatorics. The integers $N^n A_{n,k,1/N}$ (cf. $B_{n,k}$ above, which is the case $N = 2$) enumerate indexed permutations with k descents (or with k exceedances), generalizing the cases $N = 1$ (permutations) and $N = 2$ (signed permutations) above, see [89].

Remark 1.3. Frobenius [35] studied the Eulerian polynomials $P_{n,1}$ in detail (with applications to number theory); he also gave them the name Eulerian (in German). The Eulerian polynomials have sometimes been called *Euler–Frobenius polynomials* (see e.g. [78, p. 22] and [98]), and the generalization (1.1) considered here has been called *generalized Euler–Frobenius polynomials* by various authors (e.g. [61; 71; 82; 73]), but this has also been simplified by dropping "generalized" and calling them too just *Euler–Frobenius polynomials* (e.g. [60; 72; 38; 37]). We follow the latter usage, for convenience rather than for historical accuracy. (As far as I know, neither Euler nor Frobenius considered this generalization.) The names *Frobenius–Euler polynomials* and *numbers* are also used in the literature (e.g. [84]). The reader should note that also other generalizations of Eulerian polynomials have been called Euler–Frobenius polynomials, and that, conversely, other names have been used for our Euler–Frobenius polynomials (1.1). Note futher that *Euler numbers* and *Euler polynomials* (usually) mean something different, see Remark A.3.

Remark 1.4. As said above, the notation varies. Examples of other notations for our $P_{n,\rho}(x)$ are $H_n(\rho, x)$ (e.g. [60; 73]) and $P_n(x, 1-\rho)$ (e.g. [93]). A different notation used by e.g. Frobenius [35] and Carlitz [8] (in the classical case $\rho = 1$) is $R_n = P_{n,1}$ and H^n or $H_n(x) = P_{n,1}(x)/(x - 1)^n$. Carlitz [8] uses for the general case

$$H_n(u \mid \lambda) = \frac{P_{n,1-u}(\lambda)}{(\lambda - 1)^n} = \sum_{j=0}^{n} \binom{n}{j} u^{n-j} H_j(\lambda), \tag{1.9}$$

where the last equality follows from (A.9). Similarly, e.g. Schoenberg [78] uses $A_n(x; t)$ for our $(1 - t^{-1})^{-n} P_{n,x}(t^{-1}) = (t - 1)^{-n} P_{n,1-x}(t)$ (which thus equals $H_n(x \mid t)$ in (1.9)), cf. (A.14); he further uses $\Pi_n(t)$ for $P_{n,1}(t)$ and $\rho_n(t)$ for $2^n P_{n,1/2}(t)$.

Remark 1.5. Many other combinatorial numbers satisfy recursion formulas similar to (1.4); see [97] for a general version. There are also many other generalizations of Eulerian numbers and polynomials that have been defined by various authors; for a few examples, see [7; 10], [9], [12], [21], [95], [84],

[83], [99]. In particular, note the generalized Eulerian numbers $A(r, s \mid \alpha, \beta)$ defined by Carlitz and Scoville [12]; the Euler–Frobenius numbers are the special case $A_{n,k,\rho} = A(n-k, k \mid \rho, 1-\rho)$.

As said above, the case $\rho = 0$ (or $\rho = 1$) of Theorem 1.2 is well-known. We end this section by recalling the simple proof by Stanley [87] giving an explicit connection between $\lfloor S_n \rfloor$ and the number of descents in a random permutation, which, as said above, has the distribution $\mathfrak{E}_{n,1}$; we give it here in probabilistic formulation rather than the original geometric, cf. Theorem 2.1:

In the notation of Theorem 1.2, let V_j be the fractional part $\{S_j\} := S_j - \lfloor S_j \rfloor$; then $V_1, \dots, V_n$ is another sequence of independent uniformly distributed random variables. Thus the number of descents in a random permutation of length n has the same distribution as $\sum_{i=2}^{n} \mathbf{1}\{V_{i-1} > V_i\}$. On the other hand, $V_{i-1} > V_i$ exactly when the sequence $S_1, \dots, S_n$ passes an integer; thus $\mathbf{1}\{V_{i-1} > V_i\} = \lfloor S_i \rfloor - \lfloor S_{i-1} \rfloor$ and this sum equals $\lfloor S_n \rfloor$.

An extension of this proof to the case $\rho = 1/N$ and indexed permutations is given in [89, Theorem 50]; a modification for the case $\rho = 1/2$ and (one version of) descents in signed permutations is given in [76].

Section 2 gives a geometric formulation of Theorem 1.2 and some related results. Section 3 gives a proof of Theorem 1.2 together with further connections between the distribution of S_n and Euler–Frobenius numbers. Section 4 introduces ρ-rounding, and states Theorem 1.2 using it. Section 5 uses this to derive results on the characteristic function and moments of the Euler–Frobenius distribution. Section 6 shows asymptotic normality and gives further asymptotic results. Section 7 gives applications to a well known problem on rounding. Section 8 gives applications to an election method. Finally, the appendices give further background and connections to other results.

We let throughout U and $U_1, U_2, \dots$ denote independent uniform random variables in $[0,1]$, and $S_n := \sum_{i=1}^{n} U_i$.

2. Volumes of slices

Theorem 1.2 can also be stated geometrically as follows. A proof is given in Section 3.

Theorem 2.1. *Let $Q^n := [0,1]^n$ be the n-dimensional unit cube and let, for $s \in \mathbb{R}$, Q^n_s be the slice*

$$Q^n_s := \Bigl\{ (x_i)_1^n \in Q^n : s - 1 \leqslant \sum_{i=1}^{n} x_i \leqslant s \Bigr\}. \tag{2.1}$$

Then the volume of $Q^n_{k+\rho}$ is $A_{n,k,\rho}/n!$, for all $n \geqslant 1$, $k \in \mathbb{Z}$ and $\rho \in [0,1]$.

As said above, the case $\rho = 0$ (or $\rho = 1$), when the volumes are given by the Eulerian numbers $A_{n,k,1}$, is well-known [31; 87; 45; 13; 76].

The case $\rho = 1/2$, which corresponds to standard rounding of S_n in Theorem 1.2, and where the result can be stated using the Eulerian numers of type B $B_{n,k}$ in (1.8), is treated in [13] (with reference to an unpublished technical memorandum [86]) and in [76].

[13] gives also the $(n-1)$-dimensional area of the slice $\{(x_i)_1^n \in Q^n : \sum_{i=1}^n x_i = s\}$. This equals, by simple geometry, $\sqrt{n}$ times the density function of S_n at s, which by (3.5) below equals (except when $n = 1$ and $s = 1$)

$$\frac{\sqrt{n}}{(n-1)!} A_{n-1,\lfloor s \rfloor,\{s\}}. \tag{2.2}$$

See also [69] and [13], and the further references in the latter, for related results on more general slices of cubes. Furthermore, [76] give related results, involving Eulerian numbers, on some slices of a simplex.

Mixed volumes of two consequtive slices Q_k^n and Q_{k+1}^n (with integer k) are studied by [22], and further by [96] where relations to our $A_{n,k,\rho}$ (and $f_{n+1}(x)$ in Theorem 3.2 below) are given based on the fact [22] that the Minkowski sum $\lambda Q_k^n + Q_{k+1}^n = (\lambda+1)Q_{k+1/(\lambda+1)}^n$.

3. The distribution of S_n

Let $F_n(x)$ be the distribution function and $f_n(x) = F_n'(x)$ the density function of $S_n := \sum_{i=1}^n U_i$. Then $f_1(x)$, the density function of $S_1 = U_1$, is the indicator function $\mathbf{1}_{[0,1]}$ of the interval $[0,1]$, and f_n is the n-fold convolution $\mathbf{1}_{[0,1]} * \cdots * \mathbf{1}_{[0,1]}$. Hence, for $n \geqslant 1$,

$$f_{n+1}(x) = f_n * \mathbf{1}_{[0,1]}(x) = \int_0^1 f_n(x-y)\,\mathrm{d}y = F_n(x) - F_n(x-1). \tag{3.1}$$

Note that the density $f_n(x)$ is continuous for $n \geqslant 2$, as a convolution of bounded, integrable functions (or by (3.1), since $F_n(x)$ is continuous for $n \geqslant 1$), while $f_1(x)$ is discontinuous at $x = 0$ and $x = 1$. We regard $f_1(x)$ as undetermined at these two points, and we will tacitly assume that $x \neq 0, 1$ in equations involving $f_1(x)$ (such as (3.3) when $n = 1$).

The distribution of S_n was, as said above, calculated already by Laplace [54, pp. 257–260] (by taking the limit of a discrete version), see also e.g. Feller [28, XI.7.20] (where the result is attributed to Lagrange) and [29, Theorem I.9.1] (with a simple proof using (3.1) and induction); the result is the following. (A more general formula for the sum of independent uniform random variables on different intervals is given by Pólya [69].) We use the notation $(x)_+^n := (\max(x,0))^n$, interpreted as 0 when $x \leqslant 0$ and $n \geqslant 0$.

Theorem 3.1 (E.g. [54], [29]). *For $n \geqslant 1$, S_n has the distribution function*

$$F_n(x) := \mathbb{P}(S_n \leqslant x) = \frac{1}{n!}\sum_{j=0}^n (-1)^j \binom{n}{j} (x-j)_+^n \tag{3.2}$$

and density function

$$f_n(x) := F_n'(x) = \frac{1}{(n-1)!}\sum_{j=0}^{n}(-1)^j\binom{n}{j}(x-j)_+^{n-1}. \tag{3.3}$$

□

It is easy to see that Theorems 1.2 and 2.1 are equivalent to the following relation between the densities f_n and the Euler–Frobenius numbers. (The case $\rho = 0$ is noted in [23]. Moreover, the relation is well-known in the spline setting, see e.g. [78, Theorem 3.2] (for $\rho = 1$) and [92; 93; 81; 82]; see also (for the case $\rho = 0$ or 1) [96; 44].)

Theorem 3.2. *For integers $n \geqslant 0$ and $k \in \mathbb{Z}$, and $\rho \in [0,1]$,*

$$f_{n+1}(k+\rho) = \frac{A_{n,k,\rho}}{n!}. \tag{3.4}$$

Equivalently, for every real x,

$$f_{n+1}(x) = A_{n,\lfloor x\rfloor,\{x\}}/n!. \tag{3.5}$$

We first verify that, as claimed above, Theorems 1.2, 2.1 and 3.2 are equivalent; we then prove the three theorems.

Proof of Theorem 1.2 $\Longleftrightarrow$ Theorem 2.1 $\Longleftrightarrow$ Theorem 3.2. By replacing ρ by $1-\rho$, (1.7) can be written

$$\mathbb{P}\big(\lfloor S_n + 1 - \rho\rfloor = k\big) = \frac{A_{n,k,\rho}}{n!}, \qquad \rho \in [0,1],\ k \in \mathbb{Z}. \tag{3.6}$$

In Theorem 2.1, the volume of Q_s^n equals $\mathbb{P}\big(s-1 \leqslant S_n < s\big)$. Taking $s = k+\rho$, we have

$$\begin{aligned} k+\rho-1 \leqslant S_n < k+\rho &\iff k \leqslant S_n + 1 - \rho < k+1 \\ &\iff \lfloor S_n + 1 - \rho\rfloor = k, \end{aligned}$$

and thus Theorem 2.1 is equivalent to (3.6).

Similarly, by (3.1), at least when $n \geqslant 1$,

$$f_{n+1}(k+\rho) = \mathbb{P}(k+\rho-1 < S_n \leqslant k+\rho) = \mathbb{P}(\lfloor S_n + 1 - \rho\rfloor = k), \tag{3.7}$$

so (3.4) is equivalent to (3.6). Hence all three theorem are equivalent to (3.6). (The trivial and partly exceptional case $n = 0$ of Theorem 3.2 can be verified directly.) □

Proof of Theorems 1.2, 2.1, 3.2. It suffices to prove one of the theorems; we chose the version (3.5). We do this by calculating the Laplace transform of both sides, showing that they are equal. (Note that both sides vanish for $x < 0$.) This implies that the two sides are equal a.e., and since both sides are continuous on each interval $[k, k+1)$, they are equal for every x. (In the trivial case $n = 0$, we exclude $x = 0, 1$ as said above.) Alternatively, for $n \geqslant 1$, we can see directly that both sides of (3.5) are continuous on $\mathbb{R}$, using (A.7) for the right-hand side.

The Laplace transform of $f_{n+1}(x)$ is

$$\int_0^\infty f_{n+1}(x)e^{-sx}\,\mathrm{d}x = \mathbb{E}\,e^{-sS_{n+1}} = \bigl(\mathbb{E}\,e^{-sU_1}\bigr)^{n+1} = \Bigl(\int_0^1 e^{-sx}\,\mathrm{d}x\Bigr)^{n+1} = \Bigl(\frac{1-e^{-s}}{s}\Bigr)^{n+1}. \tag{3.8}$$

For $A_{n,\lfloor x\rfloor,\{x\}}$ we obtain, using (1.3) and (1.1),

$$\begin{aligned}\int_0^\infty A_{n,\lfloor x\rfloor,\{x\}}e^{-sx}\,\mathrm{d}x &= \sum_{k=0}^\infty \int_0^1 A_{n,k,\rho}e^{-s(k+\rho)}\,\mathrm{d}\rho \\ &= \int_0^1 e^{-s\rho}\sum_{k=0}^\infty A_{n,k,\rho}e^{-sk}\,\mathrm{d}\rho = \int_0^1 e^{-s\rho}P_{n,\rho}(e^{-s})\,\mathrm{d}\rho \\ &= \int_0^1 e^{-s\rho}\bigl(1-e^{-s}\bigr)^{n+1}\sum_{j=0}^\infty (j+\rho)^n e^{-js}\,\mathrm{d}\rho \\ &= \bigl(1-e^{-s}\bigr)^{n+1}\int_0^\infty x^n e^{-sx}\,\mathrm{d}x = n!\Bigl(\frac{1-e^{-s}}{s}\Bigr)^{n+1}. \end{aligned} \tag{3.9}$$

hence the two Laplace transforms are equal, which completes the proof. $\square$

An alternative proof is by induction in n, using the derivative of (3.1) and (A.21). We leave this to the reader.

Remark 3.3. By (3.4), the basic recursion (1.4) is equivalent to the recursion formula

$$f_{n+1}(x) = \frac{1}{n}\bigl(xf_n(x) + (n+1-x)f_n(x-1)\bigr) \tag{3.10}$$

for the density functions f_n. This formula is well-known in spline theory, see [80, (4.52)–(4.53)]. Conversely, (3.10) implies (3.4), and thus also Theorems 1.2 and 2.1, by induction.

Remark 3.4. By (3.4) and (3.3), for $n \geqslant 1$ and $\rho \in [0,1]$,

$$A_{n,k,\rho} = \sum_{j=0}^{n+1}(-1)^j\binom{n+1}{j}(k+\rho-j)_+^n = \sum_{j=0}^{k}(-1)^j\binom{n+1}{j}(k+\rho-j)^n. \tag{3.11}$$

This is another well-known formula, at least for the Eulerian case $\rho = 1$. It has been used to extend the definition of the Euler–Frobenius numbers to arbitrary real n by [6] (Eulerian numbers, $\rho = 1$) and [56] (note that $A(x,n)$ in [56] equals our $A_{n,\lfloor x\rfloor,\{x\}}$).

We extend the definition (1.5) of $\mathfrak{E}_{n,\rho}$ for $\rho \in [0,1]$ to arbitrary real ρ by defining, for any $\rho \in \mathbb{R}$,

$$\mathfrak{E}_{n,\rho} := \mathfrak{E}_{n,\{\rho\}} - \lfloor\rho\rfloor. \tag{3.12}$$

As above, we use $\mathfrak{E}_{n,\rho}$ also to denote the distribution of this random variable. Since $\mathfrak{E}_{n,0} \overset{\mathrm{d}}{=} \mathfrak{E}_{n,1}+1$, and we only are interested in the distribution of $\mathfrak{E}_{n,\rho}$, (3.12) is consistent with our previous definition (1.5) for all $\rho \in [0,1]$. Note, however, that (1.5) holds only for $\rho \in [0,1]$. (See also Remark 1.1.)

This rather trivial extension is sometimes convenient. Theorems 1.2 and 3.2 extend immediately:

Theorem 3.5. *For any real ρ and $n \geqslant 1$,*

$$\mathfrak{E}_{n,\rho} \overset{\mathrm{d}}{=} \lfloor S_n + 1 - \rho \rfloor \tag{3.13}$$

and, for any $k \in \mathbb{Z}$,

$$\mathbb{P}(\mathfrak{E}_{n,\rho} = k) = \mathbb{P}(\lfloor S_n + 1 - \rho \rfloor = k) = A_{n,k+\lfloor \rho \rfloor,\{\rho\}}/n! = f_{n+1}(k+\rho) \tag{3.14}$$

and

$$\mathbb{P}(\mathfrak{E}_{n,\rho} \leqslant k) = \mathbb{P}(\lfloor S_n + 1 - \rho \rfloor \leqslant k) = F_n(k+\rho). \tag{3.15}$$

Proof. By (3.12) and Theorem 1.2,

$$\mathfrak{E}_{n,\rho} := \mathfrak{E}_{n,\{\rho\}} - \lfloor \rho \rfloor \overset{\mathrm{d}}{=} \lfloor S_n + 1 - \{\rho\} \rfloor - \lfloor \rho \rfloor = \lfloor S_n + 1 - \rho \rfloor,$$

which also implies (3.15). Moreover, by (3.12), (1.5) and (3.5),

$$\mathbb{P}(\mathfrak{E}_{n,\rho} = k) = \mathbb{P}\big(\mathfrak{E}_{n,\{\rho\}} = k + \lfloor \rho \rfloor\big) = A_{n,k+\lfloor \rho \rfloor,\{\rho\}}/n! = f_{n+1}(k+\rho). \qquad \square$$

4. Rounding

Let $\rho \in [0,1]$ and define *ρ-rounding* of real numbers by rounding a number x down (to the nearest integer) if its fractional part $\{x\}$ is less that ρ, and up (to the nearest integer) otherwise. We denote ρ-rounding by $\lfloor x \rceil_\rho$, and can state the definition as

$$\lfloor x \rceil_\rho = n \iff n - 1 + \rho \leqslant x < n + \rho, \tag{4.1}$$

or, equivalently,

$$\lfloor x \rceil_\rho = \lfloor x + 1 - \rho \rfloor. \tag{4.2}$$

In particular, $\lfloor x \rceil_1 = \lfloor x \rfloor$ (rounding down), $\lfloor x \rceil_0 = \lceil x \rceil$ (rounding up), except when x is an integer, and $\lfloor x \rceil_{1/2}$ is standard rounding (except perhaps when $\{x\} = 1/2$).

Remark 4.1. As seen from these examples, in the case $\{x\} = \rho$, the definition made above (for definiteness) is not obviously the best choice. Often it is better to leave this case ambiguous, allowing rounding both up and down. However, we will be interested in roundings of continuous random variables, and then this exceptional case has probability 0 and may be ignored.

We define $\lfloor x \rceil_\rho$ by (4.2) for arbitrary $\rho \in \mathbb{R}$. This will be convenient later, although it is strictly speaking not a "rounding" when $\rho \notin [0,1]$.

Example 4.2. One use of ρ-rounding is in the study of election methods; more precisely methods for proportional elections using party lists. (In the United States, such methods are used, under different names, for apportionment of the seats in the House of Representatives among the states.) Several important such methods are *divisor methods*, and most of them can be described as giving a party with v votes $\lfloor v/D \rceil_\rho$ seats, where ρ is a given number and the divisor D is chosen such that the total number of seats is a predetermined number (the house size). The main examples are $\rho = 0$ (*d'Hondt's method = Jefferson's method*) and $\rho = 1/2$ (*Sainte-Laguë's method = Webster's method*). Some other important proportional election methods are *quota methods*, which again can be described as giving a party with v votes $\lfloor v/D \rceil_\rho$ seats, where now D (in this setting called the *quota*) is given by some formula and ρ is chosen such that the total number of seats is the house size. The most important example is to take D as the *simple quota* (also called *Hare quota*), i.e., the average number of votes per seat (*the method of greatest remainder = Hare's method = Hamilton's method*). We return to election methods in Sections 7 and 8. See further [50, Appendices A and B] and [2], [53], [70]. (In the study of election methods, usually $\rho \in [0, 1]$, but occasionally other values of ρ are used, see [50].)

By (4.2), yet another formulation of Theorem 1.2 is the following.

Theorem 4.3. *For every $\rho \in \mathbb{R}$ and $n \geqslant 1$, the random variable $\lfloor S_n \rceil_\rho$ has the Euler–Frobenius distribution $\mathfrak{E}_{n,\rho}$. In particular, if $\rho \in [0, 1]$, then*

$$\mathbb{P}\big(\lfloor S_n \rceil_\rho = k\big) = \mathbb{P}(\mathfrak{E}_{n,\rho} = k) = \frac{A_{n,k,\rho}}{n!}, \qquad k \in \mathbb{Z}, \tag{4.3}$$

and more generally, for any real ρ,

$$\mathbb{P}\big(\lfloor S_n \rceil_\rho = k\big) = \mathbb{P}(\mathfrak{E}_{n,\rho} = k) = \frac{A_{n,k+\lfloor \rho \rfloor,\{\rho\}}}{n!}, \qquad k \in \mathbb{Z}. \tag{4.4}$$

Proof. The first claim is immediate from (4.2) and Theorem 3.5. This yields (4.3) by (1.5) and then (4.4) by (3.12). ∎

In other words, defining $Z_{n,\rho} := \lfloor S_n \rceil_\rho$, we have

$$Z_{n,\rho} := \lfloor S_n \rceil_\rho = \lfloor S_n + 1 - \rho \rfloor \sim \mathfrak{E}_{n,\rho}. \tag{4.5}$$

In particular, when $\rho \in [0, 1]$, $Z_{n,\rho}$ has the probability generating function (1.6).

Janson [49] studied roundings using the notation, for $\alpha \in \mathbb{R}$,

$$X_\alpha := \lfloor X + \alpha \rfloor - \alpha + 1. \tag{4.6}$$

Comparing with (4.5), we see that in this notation,

$$Z_{n,\rho} = (S_n)_{1-\rho} - \rho. \tag{4.7}$$

We state a corollary of Theorem 4.3 for standard rounding (i.e., $\lfloor x \rceil_{1/2}$), which again shows the special importance of the cases $\rho = 0, \frac{1}{2}, 1$ of the Euler–Frobenius numbers.

Corollary 4.4. *Let $\widetilde{U}_1,\dots,\widetilde{U}_n$ be independent random variables uniformly distributed on $[-\frac12,\frac12]$, and let $\widetilde{S}_n := \sum_{i=1}^n \widetilde{U}_i$, with $n \geqslant 1$. Then, $\lfloor \widetilde{S}_n \rfloor_{1/2}$ has the distribution $\mathfrak{E}_{n,(n+1)/2}$, and thus, for $k \in \mathbb{Z}$,*

$$\mathbb{P}\bigl(\lfloor \widetilde{S}_n \rfloor_{1/2} = k\bigr) = \begin{cases} A_{n,k+n/2,1/2}/n!, & n \text{ even},\\ A_{n,k+(n+1)/2,0}/n! = A_{n,k+(n-1)/2,1}/n!, & n \text{ odd}. \end{cases}$$

Proof. We can take $\widetilde{U}_i := U_i - \frac12$, and then, using (4.5),

$$\lfloor \widetilde{S}_n \rfloor_{1/2} = \Bigl\lfloor \widetilde{S}_n + \frac12 \Bigr\rfloor = \Bigl\lfloor S_n - \frac{n-1}{2} \Bigr\rfloor = \lfloor S_n \rfloor_{(n+1)/2} \overset{\mathrm{d}}{=} \mathfrak{E}_{n,(n+1)/2}.$$

The result follows by (4.4). ☐

5. Characteristic function and moments

We use the results in Section 4 to derive further results for the Euler–Frobenius distribution $\mathfrak{E}_{n,\rho}$. As said in the introduction, we also use $\mathfrak{E}_{n,\rho}$ to denote a random variable with this distribution. Since $\mathfrak{E}_{n,\rho} \overset{\mathrm{d}}{=} Z_{n,\rho}$ by (4.5), we can just as well consider $Z_{n,\rho} := \lfloor S_n \rfloor_\rho$.

We begin with an expression for the characteristic function and moment generating function of the Euler–Frobenius distribution $\mathfrak{E}_{n,\rho}$. (Cf. [37, Lemma 2.4] where an equivalent formula is given.) We denote the characteristic function of a random variable X by φ_X. Note that if $\rho \in [0,1]$ (and the general case can be reduced to this by (3.12)), we have by (1.6)

$$\varphi_{\mathfrak{E}_{n,\rho}}(t) := \mathbb{E}\, e^{\mathrm{i}t\mathfrak{E}_{n,\rho}} = \frac{P_{n,\rho}(e^{\mathrm{i}t})}{n!} \tag{5.1}$$

and, more generally, for all $t \in \mathbb{C}$, the moment generating function

$$\mathbb{E}\, e^{t\mathfrak{E}_{n,\rho}} = \frac{P_{n,\rho}(e^{t})}{n!}. \tag{5.2}$$

Theorem 5.1. *Let $n \geqslant 1$ and $\rho \in \mathbb{R}$. The characteristic function of $\mathfrak{E}_{n,\rho}$ is given by*

$$\varphi_{\mathfrak{E}_{n,\rho}}(t) = \mathrm{i}^{-n-1} e^{-\mathrm{i}\rho t}\bigl(e^{\mathrm{i}t}-1\bigr)^{n+1} \sum_{k=-\infty}^{\infty} \frac{e^{-2\pi \mathrm{i} k\rho}}{(t+2\pi k)^{n+1}}. \tag{5.3}$$

Equivalently, the moment generating function is, for all $t \in \mathbb{C}$,

$$\mathbb{E}\, e^{t\mathfrak{E}_{n,\rho}} = e^{-\rho t}\bigl(e^{t}-1\bigr)^{n+1} \sum_{k=-\infty}^{\infty} \frac{e^{-2\pi \mathrm{i} k\rho}}{(t+2\pi k\mathrm{i})^{n+1}}. \tag{5.4}$$

(For $t \in 2\pi\mathbb{Z}$ or $t \in 2\pi\mathrm{i}\mathbb{Z}$, respectively, the expressions are interpreted by continuity.)

Proof. This can be reduced to the case $\rho \in [0,1]$, and then (5.3) is (5.1) together with a special case of an expansion found by Lerch [55, (4) and (5)] (take $s = -n$ there); nevertheless, we give a probabilistic proof (valid for all ρ) using a general formula for rounded stochastic variables in [49].

Since U_i has the characteristic function

$$\varphi_U(t) := \mathbb{E}\, e^{\mathrm{i}tU} = \frac{e^{\mathrm{i}t}-1}{\mathrm{i}t}, \tag{5.5}$$

the sum S_n has the characteristic function

$$\varphi_{S_n}(t) = \varphi_U(t)^n = \Bigl(\frac{e^{\mathrm{i}t}-1}{\mathrm{i}t}\Bigr)^n. \tag{5.6}$$

The formula in [49, Theorem 2.1] now yields

$$\begin{aligned}\mathbb{E}\, e^{\mathrm{i}t(S_n)_{1-\rho}} &= \sum_{k=-\infty}^{\infty} e^{2\pi\mathrm{i}k(1-\rho)}\varphi_U(t+2\pi k)\varphi_{S_n}(t+2\pi k)\\ &= \sum_{k=-\infty}^{\infty} e^{2\pi\mathrm{i}k(1-\rho)}\left(\frac{e^{\mathrm{i}t}-1}{\mathrm{i}(t+2\pi k)}\right)^{n+1},\end{aligned}$$

which yields (5.3) by (4.5) and (4.7).

This derivation tacitly assumes that t is real, but the sum in (5.3) converges for every complex $t \in \mathbb{C}\setminus 2\pi\mathbb{Z}$, and defines a meromorphic function with poles in $2\pi\mathbb{Z}$; thus the right-hand side of (5.3) is an entire function of $t \in \mathbb{C}$. So is also the left-hand side, since it is a trigonometric polynomial. Hence (5.3) is valid for all complex t, and replacing t by t/i, we obtain (5.4). □

Moments of arbitrary order can be obtained from the moment generating function by differentiation. For moments of order at most n, this leads to simple results.

Lemma 5.2. *Let $n \geqslant 1$ and $\rho \in \mathbb{R}$. The random variables $\mathfrak{E}_{n,\rho}+\rho$ and S_{n+1} have the same moments up to order n:*

$$\mathbb{E}\,(\mathfrak{E}_{n,\rho}+\rho)^m = \mathbb{E}\, S_{n+1}^m, \qquad 1 \leqslant m \leqslant n. \tag{5.7}$$

Proof. Since $\mathfrak{E}_{n,\rho}+\rho \overset{\mathrm{d}}{=} Z_{n,\rho}+\rho = (S_n)_{1-\rho}$ by (4.7), this follows by [49, Theorem 2.3], noting that

$$\tilde{\varphi}(t) := \frac{e^{\mathrm{i}t}-1}{\mathrm{i}t}\varphi_{S_n}(t) = \left(\frac{e^{\mathrm{i}t}-1}{\mathrm{i}t}\right)^{n+1} \tag{5.8}$$

has its n first derivatives 0 at $t = 2\pi n$, $n \in \mathbb{Z}\setminus\{0\}$. (Alternatively and equivalently, this follows from (5.4), noting that the terms with $k \neq 0$ give no contribution to the m:th derivative at $t = 0$ for $m \leqslant n$, since the factor $(e^t-1)^{n+1}$ vanishes to order $n+1$ there.) □

This leads to the following results, shown by Gawronski and Neuschel [37, Lemmas 4.1 and 4.2] by related but more complicated calculations. (In the classical case $\rho = 1$, the cumulants were given already by David and Barton [18, p. 153].)

Theorem 5.3. *For any $\rho \in \mathbb{R}$,*

$$\mathbb{E}\,\mathfrak{E}_{n,\rho} = \frac{n+1}{2} - \rho, \qquad n \geqslant 1, \tag{5.9}$$

and

$$\operatorname{Var} \mathfrak{E}_{n,\rho} = \frac{n+1}{12}, \qquad n \geqslant 2. \tag{5.10}$$

More generally, for $2 \leqslant m \leqslant n$, the m:th cumulant $\varkappa_m(\mathfrak{E}_{n,\rho})$ is independent of ρ, and is given by

$$\varkappa_m(\mathfrak{E}_{n,\rho}) = \varkappa_m(S_{n+1}) = (n+1)\varkappa_m(U) = (n+1)\frac{B_m}{m}, \qquad 2 \leqslant m \leqslant n, \tag{5.11}$$

where B_m is the m:th Bernoulli number. In particular, $\varkappa_m(\mathfrak{E}_{n,\rho}) = 0$ if m is odd with $3 \leqslant m \leqslant n$.

For Bernoulli numbers, see e.g. [39, Section 6.5] and [64, §24.2(i)].

Proof. For the mean we have by Lemma 5.2, for $n \geqslant 1$,

$$\mathbb{E}\,\mathfrak{E}_{n,\rho} + \rho = \mathbb{E}\,S_{n+1} = (n+1)\,\mathbb{E}\,U = \frac{n+1}{2}, \tag{5.12}$$

which gives (5.9).

For higher moments, we note that the m:th cumulant $\varkappa_m$ can be expressed as a polynomial in moments of order at most m; hence Lemma 5.2 implies that $\varkappa_m(\mathfrak{E}_{n,\rho} + \rho) = \varkappa_m(S_{n+1})$ for $m \leqslant n$. Moreover, for any random variable (with $\mathbb{E}\,|X|^m < \infty$) and any real number a, $\varkappa_m(X+a) = \varkappa_m(X)$, since $\varkappa_m(X+a)$ is the m:th derivative at 0 of $\log \mathbb{E}\,e^{t(X+a)} = at + \log \mathbb{E}\,e^{tX}$. Hence,

$$\varkappa_m(\mathfrak{E}_{n,\rho}) = \varkappa_m(\mathfrak{E}_{n,\rho} + \rho) = \varkappa_m(S_{n+1}), \qquad 2 \leqslant m \leqslant n. \tag{5.13}$$

Furthermore, since S_{n+1} is the sum of the $n+1$ i.i.d. random variables U_i, $i \leqslant n+1$, $\varkappa_m(S_{n+1}) = (n+1)\varkappa_m(U)$. Finally, the cumulants of the uniform distribution are given by $\varkappa_m(U) = B_m/m$ for $m \geqslant 2$; this, as is well-known, is shown by a simple calculation: (for $|t| < 2\pi$; see [64, §24.2.1] for the last step)

$$\begin{aligned}\sum_{m=0}^{\infty} m\varkappa_m(U)\frac{t^m}{m!} &= t\frac{\mathrm{d}}{\mathrm{d}t}\sum_{m=0}^{\infty} \varkappa_m(U)\frac{t^m}{m!} = t\frac{\mathrm{d}}{\mathrm{d}t}\log \mathbb{E}\,e^{tU} = t\frac{\mathrm{d}}{\mathrm{d}t}\log\frac{e^t-1}{t} \\ &= t\frac{e^t}{e^t-1} - 1 = \frac{t}{e^t-1} + t - 1 = \sum_{m=0}^{\infty} B_m\frac{t^m}{m!} + t - 1.\end{aligned}$$

Combining these facts gives (5.11). The special case $m = 2$ yields (5.10), since $\varkappa_2$ is the variance and $B_2 = 1/6$. □

We note also that Theorem 3.2 and Fourier inversion for the distribution of S_{n+1} yield the following integral formula. (This is well-known in the settings of S_n, and also for splines, see e.g. [77, Theorem 3].) The case $\rho = 0$ is given by [63], see also [96]. (This integral formula is used in [63] to define an extension $A(n,x)$ of Eulerian numbers to real x, which by (5.14) equals the Euler–Frobenius number $A_{n,\lfloor x\rfloor,\{x\}}$. The formula is further extended to real n in [56]; see also Remark 3.4.) The special cases (5.15)–(5.16) are given by [37].

Theorem 5.4. *If $n \geqslant 1$, $k \in \mathbb{Z}$ and $\rho \in [0,1]$, then*

$$\begin{aligned} A_{n,k,\rho} &= \frac{n!}{\pi} \int_{-\infty}^{\infty} e^{\mathrm{i}t(2k+2\rho-n-1)} \Bigl(\frac{\sin t}{t}\Bigr)^{n+1} \,\mathrm{d}t \\ &= n!\,\frac{2}{\pi} \int_{0}^{\infty} \cos\bigl(t(2k+2\rho-n-1)\bigr) \Bigl(\frac{\sin t}{t}\Bigr)^{n+1} \,\mathrm{d}t. \end{aligned} \tag{5.14}$$

In particular, for $k \geqslant 1$,

$$A_{2k-1,k,0} = (2k-1)!\,\frac{2}{\pi} \int_{0}^{\infty} \Bigl(\frac{\sin t}{t}\Bigr)^{2k} \,\mathrm{d}t, \tag{5.15}$$

$$A_{2k,k,1/2} = (2k)!\,\frac{2}{\pi} \int_{0}^{\infty} \Bigl(\frac{\sin t}{t}\Bigr)^{2k+1} \,\mathrm{d}t. \tag{5.16}$$

Proof. By Theorem 3.2, Fourier inversion using (5.6), and finally replacing t by $2t$,

$$\begin{aligned} \frac{A_{n,k,\rho}}{n!} &= f_{n+1}(k+\rho) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\mathrm{i}t(k+\rho)} \varphi_{S_{n+1}}(t) \,\mathrm{d}t \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\mathrm{i}t(k+\rho)} \Bigl(\frac{e^{\mathrm{i}t}-1}{\mathrm{i}t}\Bigr)^{n+1} \,\mathrm{d}t \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} e^{\mathrm{i}t(n+1-2k-2\rho)} \Bigl(\frac{\sin t}{t}\Bigr)^{n+1} \,\mathrm{d}t \end{aligned}$$

and the result follows by replacing t by $-t$, noting that $\sin t/t$ is an even function. ◻

6. Asymptotic normality and large deviations

It is well-known that the Eulerian distribution $\mathfrak{E}_{n,0}$ or $\mathfrak{E}_{n,1}$ is asymptotically normal, and that furthermore a local limit theorem holds, i.e., the Eulerian numbers $A_{n,k,1} = A_{n,k+1,0}$ can be approximated by a Gaussian function for large n. This has been proved by various authors using several different metods, see below; most of the methods generalize to $\mathfrak{E}_{n,\rho}$ and $A_{n,k,\rho}$ for arbitrary $\rho \in [0,1]$. The basic central limit theorem for $\mathfrak{E}_{n,\rho}$ can be stated as follows. (Recall that the mean and variance are given by Theorem 5.3.)

Theorem 6.1. *$\mathfrak{E}_{n,\rho}$ is asymptotically normal as $n \to \infty$, for any real ρ, i.e.,*

$$\frac{\mathfrak{E}_{n,\rho} - \mathbb{E}\,\mathfrak{E}_{n,\rho}}{(\operatorname{Var}\mathfrak{E}_{n,\rho})^{1/2}} \xrightarrow{\mathrm{d}} N(0,1) \tag{6.1}$$

or, more explicitly and simplified,

$$\frac{\mathfrak{E}_{n,\rho} - n/2}{n^{1/2}} \xrightarrow{\mathrm{d}} N\Bigl(0, \frac{1}{12}\Bigr). \tag{6.2}$$

Proof. Immediate by (4.5) and the central limit theorem for $S_n = \sum_{i=1}^n U_i$. Alternatively, the theorem follows by the method of moments from the formula (5.11) for the cumulants in Theorem 5.3, which implies that the normalized cumulants $\varkappa_m(\mathfrak{E}_{n,\rho}/(\operatorname{Var}\mathfrak{E}_{n,\rho})^{1/2})$ converge to 0 for any $m \geqslant 3$. Several other proofs are described below. □

One refined verison is the following local limit theorem with an asymptotic expansion proved by Gawronski and Neuschel [37]; we let as in (5.11) B_m denote the Bernoulli numbers and let $H_m(x)$ denote the Hermite polynomials, defined e.g. by

$$H_m(x) := (-1)^m e^{x^2/2} \frac{\mathrm{d}^m}{\mathrm{d}x^m} e^{-x^2/2}, \tag{6.3}$$

see [66, p. 137]. (These are the orthogonal polynomials for the standard normal distribution, see e.g. [48].)

Theorem 6.2 (Gawronski and Neuschel [37])**.** *There exist polynomials q_ν, $\nu \geqslant 1$, such that, for any $\ell \geqslant 0$, as $n \to \infty$, uniformly for all $k \in \mathbb{Z}$ and $\rho \in [0,1]$,*

$$\frac{A_{n,k,\rho}}{n!} = \sqrt{\frac{6}{\pi(n+1)}}\, e^{-x^2/2} \left(1 + \sum_{\nu=1}^{\ell} \frac{q_\nu(x)}{(n+1)^\nu}\right) + O\left(n^{-\ell-3/2}\right), \tag{6.4}$$

where

$$x = \left(k + \rho - \frac{n+1}{2}\right)\sqrt{\frac{12}{n+1}}. \tag{6.5}$$

Explicitly,

$$q_\nu(x) = 12^\nu \sum H_{2\nu+2s}(x) 6^s \prod_{m=1}^{\nu} \frac{1}{k_m!} \left(\frac{B_{2m+2}}{(m+1)(2m+2)!}\right)^{k_m}, \tag{6.6}$$

summing over all non-negative integers $(k_1, \dots, k_\nu)$ with $k_1 + 2k_2 + \dots + \nu k_\nu = \nu$ and letting $s = k_1 + \dots + k_\nu$.

The polynomial $q_\nu(x)$ has degree 4ν. The first two are:

$$q_1 = -\frac{1}{20} H_4(x) = -\frac{x^4 - 6x^2 + 3}{20}, \tag{6.7}$$

$$q_2 = \frac{1}{800} H_8(x) + \frac{1}{105} H_6(x) = \frac{21x^8 - 428x^6 + 2010x^4 - 1620x^2 - 195}{16800}. \tag{6.8}$$

Remark 6.3. We state Theorem 6.2 using an expansion in negative powers of $n+1$. Of course, it is possible to use powers of n instead, but then the polynomials $q_i(x)$ will be modified.

Before giving a proof of Theorem 6.2, we give some history and discuss various methods. The perhaps first proof of asymptotic normality for Eulerian numbers (i.e., Theorem 6.1 in the classical case $\rho = 1$) was given by

David and Barton [18], using the generating function (A.11) below to calculate cumulants. Bender [3, Example 3.5] used (for the equivalent case $\rho = 0$) instead a singularity analysis of the generating function (A.11) to obtain this and further results; see also Flajolet and Sedgewick [30, Example IX.12, p. 658].

The representation (A.23) of $\mathfrak{E}_{n,\rho}$ as a sum of independent (but not identically distributed) Bernoulli variables was used by Carlitz et al [11] to show asymptotic normality (global and local central limit theorems and a Berry–Esseen estimate) for Eulerian numbers (i.e. for $\rho = 0$). Siraždinov [85] gave (also for $\rho = 0$) a local limit theorem including the second order term ($\nu = 1$) in Theorem 6.2. (We have not been able to obtain the original reference, but we believe he used this representation.) More recently, Gawronski and Neuschel [37] have used this method for a general $\rho \in [0, 1]$ to show a global central limit theorem and the refined local limit theorem Theorem 6.2 above.

Furthermore, (A.23) was also used (for $\rho = 0$) by Bender [3] to obtain a local limit theorem from the global central limit theorem (proved by him by other methods, as said above).

Tanny [91] showed global and local central limit theorems (for $\rho = 0$) using the representation $\mathfrak{E}_{n,0} \overset{\mathrm{d}}{=} \lfloor S_n \rfloor + 1$, see our Theorems 1.2 and 3.5, together with the standard central limit theorem for S_n; see also Sachkov [75, Section 1.3.2]. This too extends to arbitrary ρ.

Esseen [23] used instead (still for $\rho = 0$) the relation given here as (3.4) with the density function of S_{n+1}, together with the standard local limit theorem for S_n. This too extends to arbitrary ρ, and is perhaps the simplest method to obtain local limit theorems. Moreover, as noted by [23], it easily yields an asymptotic expansion with arbitrary many terms as in Theorem 6.2. (For $\rho = 0$, [23] gave the second term explicitly; as mentioned above, this term was also given by Siraždinov [85].) Furthermore, the first three terms ($\nu \leqslant 2$) in Theorem 6.2 were given (for arbitrary ρ) by Nicolas [63], using essentially the same method, but stated in analytic formulation rather than probabilistic. (See also, for the case $\rho = 0$, [100].)

Esseen [23] also pointed out that $\mathfrak{E}_{n,1} = \mathfrak{E}_{n,0} - 1$, regarded as the number of descents in a random permutation, can be represented as

$$\mathfrak{E}_{n,1} \overset{\mathrm{d}}{=} \sum_{i=1}^{n-1} \mathbf{1}\{U_i > U_{i+1}\}, \tag{6.9}$$

where $U_1, U_2, \dots$ are i.i.d. with the distribution $U(0, 1)$; the global central limit theorem thus follows immediately from the standard central limit theorem for m-dependent sequences. (However, we do not know any similar representation for the case $\rho \in (0, 1)$.)

Proof of Theorem 6.2. We use the method of [23], and note that the theorem follows immediately from (3.4) and the local limit theorem for $f_n(x)$, see e.g. [66, Theorem VII.15 and (VI.1.14)], using $\varkappa_m(U_i) = B_m/m$ for $m \geqslant 2$ and noting that $B_m = 0$ for odd $m \geqslant 3$. (As said above, the proof in [37] is somewhat different, although it also uses [66].) $\square$

Remark 6.4. By using [66, Theorem VII.17] in the proof of Theorem 6.2 (and taking as many terms as needed), the error term in (6.4) is improved to $O\bigl((1+|x|^K)^{-1}n^{-\ell-3/2}\bigr)$, for any fixed $K>0$. This is, however, a superficial improvement, since this is trivial for $k+\rho\notin[0,n+1]$ (when $A_{n,k,\rho}=0$), and otherwise easily follows from (6.4) by increasing ℓ.

Although Theorem 6.2 holds uniformly for all k, it is of interest mainly when $k=n/2+O(\sqrt{n\log n})$, when $x=O(\sqrt{\log n})$, since for $|k-n/2|$ larger, the main term in (6.4) is smaller than the error term. (This holds also for the improved version in Remark 6.4.) However, we can also easily obtain asymptotic estimates for other k by the same method, now using (3.4) and large deviation estimates for the density $f_{n+1}(x)$, which are obtained by standard arguments. (The saddle point method, which in this context is essentially the same as using Cramér's [16] method of conjugate distributions. See e.g. [19, Chapter 2] or [51, Chapter 27] for more general large deviation theory, and [30] for the saddle point method.) In the classical case $\rho=0$, this was done by Esseen [23] (for $1\leqslant k\leqslant n$), improving an earlier result by Bender [3] ($\varepsilon n\leqslant k\leqslant(1-\varepsilon)n$ for any $\varepsilon>0$) who used a related argument using the generating function (A.11). The result extends immediately to any ρ as follows.

Let $\psi(t)$ be the moment generating function of $U\sim U(0,1)$, i.e.,

$$\psi(t)=\frac{e^t-1}{t}, \tag{6.10}$$

and let for $a\in(0,1)$

$$m(a):=\min_{-\infty<t<\infty}e^{-at}\psi(t). \tag{6.11}$$

Since $\log\psi(t)$ is convex (a general property of moment generating functions, and easily verified directly in this case), and the derivative $(\log\psi)'$ increases from 0 to 1, the minimum is attained at a unique $t(a)\in(-\infty,\infty)$ for each $a\in(0,1)$, which is the solution to the equation

$$a=(\log\psi)'(t)=\frac{e^t}{e^t-1}-\frac{1}{t}=\frac{1}{1-e^{-t}}-\frac{1}{t}. \tag{6.12}$$

Set

$$\sigma^2(a):=(\log\psi)''(t(a))=\frac{1}{t(a)^2}-\frac{e^{t(a)}}{(e^{t(a)}-1)^2}=\frac{1}{t(a)^2}-\frac{1}{\sinh^2 t(a)}, \tag{6.13}$$

interpreted (by continuity) as $(\log\psi)''(0)=1/12$ when $a=1/2$ and thus $t(a)=0$.

Theorem 6.5. *Assume $\rho\in[0,1]$ and $0<k+\rho<n+1$. Then*

$$\frac{A_{n,k,\rho}}{n!}=\frac{(m(a))^{n+1}}{\sqrt{2\pi(n+1)\sigma^2(a)}}\bigl(1+O(n^{-1})\bigr), \tag{6.14}$$

where $a := (k+\rho)/(n+1)$, *uniformly in all* k *and* ρ *such that* $0 < k+\rho < n+1$.

Proof. We follow Esseen [23] with some additions. By (3.4) (and replacing n by $n-1$), the statement is equivalent to the estimate

$$f_n(x) = \frac{(m(a))^n}{\sqrt{2\pi n \sigma^2(a)}}\bigl(1 + O(n^{-1})\bigr), \tag{6.15}$$

where $a = x/n$, uniformly for $x \in (0, n)$. To see this, we use Fourier inversion (assuming $n \geqslant 2$), noting that the characteristic function of S_n is $\psi(\mathrm{i}t)^n$, and shift the line of integration to the saddle point $t(a)$:

$$\begin{aligned} f_n(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\mathrm{i}xt}\psi(\mathrm{i}t)^n \,\mathrm{d}t = \frac{1}{2\pi\mathrm{i}}\int_{-\infty\mathrm{i}}^{\infty\mathrm{i}} \bigl(e^{-az}\psi(z)\bigr)^n \,\mathrm{d}z \\ &= \frac{1}{2\pi\mathrm{i}}\int_{t(a)-\infty\mathrm{i}}^{t(a)+\infty\mathrm{i}} \bigl(e^{-az}\psi(z)\bigr)^n \,\mathrm{d}z = \frac{1}{2\pi}\int_{-\infty}^{\infty} g\bigl(t(a)+iu\bigr)^n \,\mathrm{d}u, \end{aligned} \tag{6.16}$$

where $g(z) := e^{-az}\psi(z)$. (Thus $g(t(a)) = m(a)$.)

If we assume $x \in [1, n-1]$ and thus $a \in [1/n, 1-1/n]$, then $t(a) = O(n)$ by (6.12). Routine estimates (and the change of variable $u = s/\sqrt{n}$) show that then the integral over $|u| \leqslant n^{-.49}$ yields the right-hand side of (6.15), uniformly in x, while the remaining integral is smaller by a factor $O(n^{-100})$ (for example). We omit the details.

If $0 < x < 1$, so $0 < a < 1/n$, then $f_n(x) = x^{n-1}/(n-1)!$, while (6.12) and (6.11) yield $t(a) = -a^{-1} + O(a^{-2}e^{-1/a}) = -n/x + O(e^{-n/2x})$ and

$$m(a) = e^{-at(a)}\psi(t(a)) = ae^{-1}\bigl(1 + O(e^{-n/2})\bigr); \tag{6.17}$$

then (6.14) is easily verified directly, using Stirling's formula. The case $n-1 < x < n$ is symmetric. We again omit the details. ☐

As remarked by Esseen [23], it is possible to obtain an expansion with further terms in (6.14) by the same method.

The saddle point method is standard in problems of this type. However, it is perhaps surprising that it can be used with a uniform relative error bound for all $t(a) \in (-\infty, \infty)$.

7. More on rounding

Let $p_1, \dots, p_n$ be given probabilities (or proportions) with $\sum_{i=1}^n p_i = 1$ and let N be a (large) integer. Suppose that we want to round Np_i to integers, by standard rounding or, more generally, by ρ-rounding for some given $\rho \in \mathbb{R}$. It is often desirable that the sum of the results is exactly N, but this is, of course, not always the case. We thus consider the discrepancy

$$\Delta_\rho := \sum_{i=1}^n \lfloor Np_i \rfloor_\rho - N. \tag{7.1}$$

More generally, we may round $(N+\gamma)p_i$ for some fixed $\gamma \in \mathbb{R}$ and we define the discrepancy

$$\Delta_{\rho,\gamma} := \sum_{i=1}^{n} \lfloor (N+\gamma)p_i \rceil_\rho - N. \tag{7.2}$$

Example 7.1. If a statistical table is presented as percentages, with all percentages rounded to integers, we have this situation with $N = 100$; the percentages do not necessarily add up to 100, and the error is given by $\Delta_{1/2}$. Rounding to other accuracies correspond to other values of N. This has been studied by several authors, see Mosteller, Youtz and Zahn [62] and Diaconis and Freedman [20].

Example 7.2. The general idea of a proportional election method is that a given number N of seats are to be distributed among n parties which have obtained $v_1, \dots, v_n$ votes. With $p_i := v_i / \sum_{j=1}^n v_j$, the proportion of votes for party i, the party should ideally get Np_i seats, but the number of seats has to be an integer so some kind of rounding procedure is needed. (See Example 4.2 and the reference given there for some important methods used in practice.)

A simple attempt would be to round Np_i to the nearest integer and give $\lfloor Np_i \rceil_{1/2}$ seats to party i. More generally, we might fix some $\rho \in [0,1]$ and give the party $\lfloor Np_i \rceil_\rho$ seats. Of course, this is not a workable election method since the sum in general is not exactly equal to N, and the error is given by Δ_ρ. (In principle the method could be used for elections if one accepts a varying size of the elected house, but I don't know any examples of it being used.) Nevertheless, this can be seen as the first step in an algorithm implementing divisor methods, see Happacher and Pukelsheim [41] and Pukelsheim [70]. In this context it is also useful to consider the more general $\lfloor (N+\gamma)p_i \rceil_\rho$ for some given $\gamma \in \mathbb{R}$, see again [41] and [70]; then the error is given by (7.2).

Example 7.3. Roundings of $(N+\gamma)p_i$ and thus (7.2) occur also in the study of quota methods of elections, for example *Droop's method* where we take $\gamma = 1$ and adjust ρ to obtain the sum N, see [50, Appendix B].

If we assume that the proportions p_i are random, it is thus of interest to find the distribution of the discrepancy $\Delta_{\rho,\gamma}$, and in particular of $\Delta_{\rho,0} = \Delta_\rho$. We consider the asymptotic distribution of $\Delta_{\rho,\gamma}$ as $N \to \infty$.

The simplest assumption is that $(p_1, \dots, p_n)$ is uniformly distributed over the $n-1$-dimensional unit simplex $\{(p_i)_1^n \in \mathbb{R}_+^n : \sum_i p_i = 1\}$, but it turns out (by Weyl's lemma, see e.g. [50, Lemma 4.1 and Lemma C.1]) that the asymptotic distribution is the same for any absolutely continuous distribution of $(p_i)_1^{n-1}$, and we state our results for this setting.

Remark 7.4. Alternatively, it is also possible to consider a fixed $(p_i)_1^n$ (for almost all choices) and let N be random as in [50, Section 1] (with $N \xrightarrow{\mathrm{p}} \infty$).

The same asymptotic results are obtained in this case too, but we leave the details to the reader.

In the standard case $\rho = 1/2$ and $\gamma = 0$, the asymptotic distribution of the discrepancy $\Delta_{1/2}$ was found by Diaconis and Freedman [20], assuming (as we do here) that $(p_i)_1^n$ have an absolutely continuous distribution on the unit simplex. (The cases $n = 3, 4$, with uniformly distributed probabilities $(p_i)_1^n$, were earlier treated by Mosteller, Youtz and Zahn [62].) This was extended to Δ_ρ with arbitrary ρ (still with $\gamma = 0$) by Balinski and Rachev [1]. Happacher and Pukelsheim [41; 42] considered general ρ and γ (assuming uniformly distributed $(p_i)_1^n$) and found asymptotics of the mean and variance of $\Delta_{\rho,\gamma}$ [41] and (at least in the case $\gamma = n(\rho-\frac{1}{2})$) the asymptotic distribution [42]. The exact distribution of $\Delta_{\rho,\gamma}$ for a finite N (assuming uniform $(p_i)_1^n$) was given by Happacher [40, Sections 2 and 3]. Furthermore, asymptotic results for the probability $\mathbb{P}(\Delta_{\rho,\gamma} = 0)$ of no discrepancy have also been given (assuming uniform $(p_i)_1^n$) by Kopfermann [53, p. 185] ($\rho = 1/2$, $\gamma = 0$), and Gawronski and Neuschel [37] ($\rho = 1/2$, $\gamma = 0$); the latter paper moreover gives the connection to Euler–Frobenius numbers. (The other papers referred to here state the results using S_{n-1} or the density function f_n of S_n, or the explicit sum in (3.3) for the latter.)

We state and extend these asymptotic results for $\Delta_{\rho,\gamma}$ as follows. Equivalent reformulations of (7.3)–(7.4) (including the versions in the references above) can be given using (3.14), see also (7.9) below.

Theorem 7.5. *Suppose that $(p_1, \dots, p_n)$ are random with an absolutely continuous distribution on the $n-1$-dimensional unit simplex. Then, as $N \to \infty$, for any fixed real ρ and γ, $n \geqslant 2$ and $k \in \mathbb{Z}$,*

$$\Delta_{\rho,\gamma} \xrightarrow{\mathrm{d}} \mathfrak{E}_{n-1,n\rho-\gamma}. \tag{7.3}$$

In other words,

$$\mathbb{P}(\Delta_{\rho,\gamma} = k) \to \frac{A_{n-1,k+\lfloor n\rho-\gamma \rfloor,\{n\rho+\gamma\}}}{(n-1)!}. \tag{7.4}$$

Furthermore,

$$\mathbb{E}\,\Delta_{\rho,\gamma} \to \mathbb{E}\,\mathfrak{E}_{n-1,n\rho-\gamma} = n\Big(\frac{1}{2} - \rho\Big) + \gamma \tag{7.5}$$

and if $n \geqslant 3$,

$$\operatorname{Var}\Delta_{\rho,\gamma} \to \operatorname{Var}\mathfrak{E}_{n-1,n\rho-\gamma} = \frac{n}{12}. \tag{7.6}$$

Proof. Let $X_i := (N+\gamma)p_i + 1 - \rho$ and $Y_i := \{X_i\}$. Thus, by the definition (4.2), $\lfloor (N+\gamma)p_i \rfloor_\rho = \lfloor X_i \rfloor$ and hence by (7.2),

$$\begin{aligned}\Delta_{\rho,\gamma} &= \sum_{i=1}^n \lfloor X_i \rfloor - N = \sum_{i=1}^n (X_i - Y_i) - N \\ &= N + \gamma + n(1-\rho) - \sum_{i=1}^n Y_i - N \\ &= \gamma + n(1-\rho) - \sum_{i=1}^n Y_i. \end{aligned} \tag{7.7}$$

Since this is an integer, and $Y_n \in [0,1)$, it follows that

$$\Delta_{\rho,\gamma} = \left\lfloor \gamma + n(1-\rho) - \sum_{i=1}^{n-1} Y_i \right\rfloor. \tag{7.8}$$

As $N \to \infty$, the fractional parts $(\{Np_i\})_1^{n-1}$ converge jointly in distribution to the independent uniform random variables $(U_i)_1^{n-1}$, see [50, Lemma C.1], and since $Y_i = \{\{Np_i\} + \gamma p_i + 1 - \rho\}$, the same holds for $(Y_i)_1^{n-1}$. Thus (7.8) implies, using $1 - U_i \overset{\mathrm{d}}{=} U_i$,

$$\begin{aligned}\Delta_{\rho,\gamma} &\xrightarrow{\mathrm{d}} \left\lfloor \gamma + n(1-\rho) - \sum_{i=1}^{n-1} U_i \right\rfloor \overset{\mathrm{d}}{=} \left\lfloor \gamma + 1 - n\rho + \sum_{i=1}^{n-1} U_i \right\rfloor \\ &= \lfloor \gamma + 1 - n\rho + S_{n-1} \rfloor = \lfloor S_{n-1} \rfloor_{n\rho-\gamma}. \end{aligned} \tag{7.9}$$

Since $\lfloor S_{n-1} \rfloor_{n\rho-\gamma} \overset{\mathrm{d}}{=} \mathfrak{E}_{n-1,n\rho-\gamma}$ by Theorem 4.3, this proves (7.3); furthermore, Theorem 4.3 yields also (7.4). Since $\Delta_{\rho,\gamma}$ is uniformly bounded by (7.7), (7.3) implies moment convergence and thus (7.5)–(7.6) follow by Theorem 5.3. □

Remark 7.6. In the case of uniformly distributed probabilities $(p_i)_1^n$, Happacher and Pukelsheim [41] have given a more precise form of the moment asymptotics (7.5)–(7.6) with explicit higher order terms.

Example 7.7. We see from (7.5) that the choice $\gamma = n(\rho - \frac{1}{2})$ yields $\mathbb{E}\,\Delta_{\rho,\gamma} \to 0$, so the discrepancy is asymptotically unbiased. This was shown by Happacher and Pukelsheim [41], who therefore recommend this choice of γ when the aim is to try to avoid a discrepancy; see also [42].

For standard rounding, $\rho = 1/2$ and $\gamma = 0$, Theorem 7.5 yields the result of Diaconis and Freedman [20], which we state as a corollary.

Corollary 7.8. *With assumptions as in Theorem 7.5,*

$$\Delta_{1/2} \xrightarrow{\mathrm{d}} \mathfrak{E}_{n-1,n/2}. \tag{7.10}$$

In other words,

$$\mathbb{P}(\Delta_{1/2} = k) \to \frac{A_{n-1,k+\lfloor n/2 \rfloor,\{n/2\}}}{(n-1)!}. \tag{7.11}$$

Using the notation of Corollary 4.4, the result can also be written

$$\Delta_{1/2} \xrightarrow{\mathrm{d}} \lfloor \widetilde{S}_{n-1} \rfloor_{1/2}. \tag{7.12}$$

Proof. Theorem 7.5 yields (7.10)–(7.11), and (7.12) then follows by Corollary 4.4. □

The asymptotic distribution in Corollary 7.8 can also be described as $\mathfrak{E}_{n-1,0}$ when n is even and $\mathfrak{E}_{n-1,1/2}$ when n is odd, in both cases centred by subtracting the mean $\lfloor n/2 \rfloor$. Note that in this case the asymptotic distribution is symmetric, e.g. by (A.16).

Example 7.9. Taking $k = 0$ in Corollary 7.8 we obtain the asymptotic probability that standard rounding yields the correct sum:

$$\mathbb{P}(\Delta_{1/2} = 0) \to A_{n-1,\lfloor n/2 \rfloor,\{n/2\}}/(n-1)!. \tag{7.13}$$

The limit is precisely the value for $\mathbb{P}\big(\lfloor \widetilde{S}_{n-1} \rfloor_{1/2} = 0\big)$ given by Corollary 4.4.

Remark 7.10. Even if $(p_1, \dots, p_n)$ is uniformly distributed, the result in Theorem 7.5 is in general only asymptotic and not exact for finite N because of edge effects. For a simple example, if $\rho = 1/2$ and $N = 1$, then $\mathbb{P}(\Delta_{1/2} = 0) = \mathbb{P}\big(\sum_1^n \lfloor p_i \rfloor_{1/2} = 1\big) = n\,\mathbb{P}\big(p_1 > 1/2\big) = n2^{1-n}$, which differs from the asymptotical value in (7.13) for $n \geqslant 4$. (It is much smaller for large n, since it decreases exponentially in n.) See [40] for an exact formula for finite N.

8. Example: The method of greatest remainder

By (3.15), (3.12) and (1.5), the distribution function F_n of S_n can be expressed using the Euler–Frobenius distribution or using Euler–Frobenius numbers. As another example involving election methods, consider again an election as in Example 7.2, with N seats distributed among $n \geqslant 3$ parties having proportions $p_1, \dots, p_n$ of the votes. Let $s_1, \dots, s_n$ be the number of seats assigned to the parties by the method of greatest remainder (Hare's method; Hamilton's method), see Example 4.2, and consider the bias $\Delta'_1 := s_1 - Np_1$ for party 1. Let again $(p_1, \dots, p_n)$ be random as in Theorem 7.5, and let $N \to \infty$; or let $(p_1, \dots, p_n)$ be fixed and N random, with conditions as in [50]. It is shown in [50, Theorems 3.13 and 6.1] that then

$$\Delta'_1 \xrightarrow{\mathrm{d}} \hat{\Delta} := \widetilde{U}_0 + \frac{1}{n} \sum_{i=1}^{n-2} \widetilde{U}_i = \widetilde{U}_0 + \frac{1}{n} \widetilde{S}_{n-2}, \tag{8.1}$$

where $\widetilde{U}_i := U_i - \frac{1}{2}$ are independent and uniform on $[-\frac{1}{2}, \frac{1}{2}]$.

The limit distribution in (8.1) has density function, using (3.15) and (3.12),

$$\begin{aligned} f_{\hat\Delta}(x) &= \mathbb{P}\Bigl(\frac{1}{n}\widetilde{S}_{n-2} \in \bigl(x-\tfrac12, x+\tfrac12\bigr)\Bigr) \\ &= \mathbb{P}\bigl(\widetilde{S}_{n-2} \in (nx-n/2, nx+n/2)\bigr) \\ &= \mathbb{P}\bigl(S_{n-2} \in (nx-1, nx+n-1)\bigr) \\ &= F_{n-2}(nx+n-1) - F_{n-2}(nx-1) \\ &= \mathbb{P}\bigl(0 \leqslant \mathfrak{E}_{n-2,nx} \leqslant n-1\bigr) \\ &= \mathbb{P}\bigl(\lfloor nx\rfloor \leqslant \mathfrak{E}_{n-2,\{nx\}} < \lfloor nx\rfloor + n\bigr). \end{aligned} \tag{8.2}$$

This can also by (1.5) be expressed in the Euler–Frobenius numbers:

$$f_{\hat\Delta}(x) = \frac{1}{(n-2)!}\sum_{j=0}^{n-1} A_{n-2,\lfloor nx\rfloor+j,\{nx\}}. \tag{8.3}$$

By (8.1), $|\hat\Delta| \leqslant 1-1/n < 1$; in particular, $f_{\hat\Delta}(x) = 0$ for $x \notin (-1,1)$, as also can be seen from (8.2) or (8.3). Note also that, for every x,

$$\sum_{i=-\infty}^{\infty} f_{\hat\Delta}(x+i) = \sum_{i=-\infty}^{\infty} \mathbb{P}\bigl(\lfloor nx\rfloor+ni \leqslant \mathfrak{E}_{n-2,\{nx\}} < \lfloor nx\rfloor+ni+n\bigr) = 1. \tag{8.4}$$

(This means that $\{\hat\Delta\}$ is uniformly distributed on $[0,1]$, which also easily is seen directly from (8.1).) It follows that, for all $x \in [0,1]$ and $k \in \mathbb{Z}$,

$$\mathbb{P}\bigl(\lfloor\hat\Delta\rfloor = k \mid \{\hat\Delta\} = x\bigr) = \frac{f_{\hat\Delta}(k+x)}{\sum_{i=-\infty}^{\infty} f_{\hat\Delta}(x+i)} = f_{\hat\Delta}(k+x). \tag{8.5}$$

Of course, the fractional part

$$\{\Delta_1'\} = \{-Np_1\} = 1 - \{Np_1\} \tag{8.6}$$

(unless Np_1 is an integer). Thus, for $x \in [0,1)$ and a small $dx > 0$, with $y := 1-x$,

$$\{\Delta_1'\} \in (x, x+dx) \iff \{Np_1\} \in (y-dx, y).$$

Conditioned on this event, for $k \in \mathbb{Z}$,

$$\begin{aligned} \mathbb{P}\bigl(\lfloor\Delta_1'\rfloor = k \mid \{\Delta_1'\} \in (x,x+dx)\bigr) &= \frac{\mathbb{P}(\Delta_1' \in (k+x, k+x+dx))}{\mathbb{P}(\{\Delta_1'\} \in (x,x+dx))} \\ &\to \frac{\mathbb{P}(\hat\Delta \in (k+x, k+x+dx))}{\mathbb{P}(\{\hat\Delta\} \in (x,x+dx))} \end{aligned} \tag{8.7}$$

where the right-hand side as $dx \to 0$ converges to, see (8.5),

$$\mathbb{P}\bigl(\hat\Delta = k+x \mid \{\hat\Delta\} = x\bigr) = f_{\hat\Delta}(k+x). \tag{8.8}$$

This leads to the following result.

Theorem 8.1. *Let $p_1 \in (0,1)$ be fixed and suppose that $(p_2,\dots,p_n)$ have an absolutely continuous distribution in the simplex $\{(p_i)_2^n \in \mathbb{R}_+^{n-1} : \sum_2^n p_i = 1-p_1\}$, where $n \geqslant 3$. Then, for the method of greatest remainder, as $N \to \infty$,*

$$\mathbb{P}\bigl(\Delta_1' = k - \{Np_1\}\bigr) = f_{\hat{\Delta}}\bigl(k - \{Np_1\}\bigr) + o(1) \tag{8.9}$$

with $f_{\hat{\Delta}}$ given by (8.2)–(8.3), for every $k \in \mathbb{Z}$. Thus,

$$\mathbb{P}\bigl(\Delta_1' = -\{Np_1\}\bigr) = \mathbb{P}\bigl(\mathfrak{E}_{n-2,\{-nNp_1\}} \leqslant n - \lceil n\{Np_1\}\rceil\bigr) + o(1),$$
$$\mathbb{P}\bigl(\Delta_1' = 1-\{Np_1\}\bigr) = \mathbb{P}\bigl(\mathfrak{E}_{n-2,\{-nNp_1\}} > n - \lceil n\{Np_1\}\rceil\bigr) + o(1).$$

The result (8.9) is trivial unless $k \in \{0,1\}$, since the probability is 0 otherwise.

Proof. Let $X_i := Np_i$. Now X_1 is deterministic, but the fractional parts $(\{X_i\})_{i=2}^{n-1}$ converge to independent uniform random variables $(U_i)_{i=2}^{n-1}$ as in Section 7. The seat bias Δ_1' depends only on the fractional parts $\{X_i\}$, and it is an a.e. continuous function $h(\{X_1\},\dots,\{X_n\})$ of them, and it follows that for any subsequence of N with $\{X_1\} = \{Np_1\} \to y \in [0,1]$,

$$\Delta_1' \xrightarrow{\mathrm{d}} Y(y) := h(y, U_2, \dots, U_{n-1}, \{-y - U_2 - \dots - U_{n-1}\}). \tag{8.10}$$

Moreover, the distribution of $Y(y)$ depends continuously on $y \in [0,1]$, with $Y(0) = Y(1)$.

If we now would let p_1 be random and uniform in some small interval, and scale $(p_2,\dots,p_n)$ correspondingly, and then condition on $\{Np_1\} \in (y - dx, y)$, for $y \in (0,1]$, then (8.7) would apply. The left hand side of (8.7) is asymptotically the average of $\mathbb{P}(\lfloor Y(z)\rfloor = k)$ for $z \in (x, x+dx)$ with $x = 1-y$, and by the continuity of the distribution of $Y(z)$, we can let $dx \to 0$ and conclude from (8.7) and (8.8) that $Y(y)$ has the distribution in (8.8), i.e.,

$$\mathbb{P}(Y(y) = k+1-y) = f_{\hat{\Delta}}(k+1-y). \tag{8.11}$$

If (8.9) would not hold, for some k, then there would be a subsequence such that $\mathbb{P}\bigl(\Delta_1' = k - \{Np_1\}\bigr)$ converges to a limit different from $f_{\hat{\Delta}}\bigl(k - \{Np_1\}\bigr)$. We may moreover assume that $\{Np_1\}$ converges to some $y \in [0,1]$, but then (8.10) and (8.11) would yield a contradiction. This shows (8.9).

The final formulas follow by taking $k = 0,1$ in (8.9) and using (8.2). □

Remark 8.2. For other quota methods, [50, Theorems 3.13 and 6.1] provide similar results. In particular, for Droop's method, (8.1) is replaced by

$$\Delta_1' \xrightarrow{\mathrm{d}} p_1 - \frac{1}{n} + \hat{\Delta}, \tag{8.12}$$

with $\hat{\Delta}$ as in (8.1), and thus the argument above shows that (8.9) is replaced by

$$\mathbb{P}\bigl(\Delta_1' = k - \{Np_1\}\bigr) = f_{\hat{\Delta}}\bigl(k - \{Np_1\} - p_1 + 1/n\bigr) + o(1). \tag{8.13}$$

There are also similar results for divisor methods, see [50, Theorems 3.7 and 6.1]. In particular, for Sainte-Laguë's method,

$$\Delta_1' \xrightarrow{\mathrm{d}} \widetilde{U}_0 + p_1 \widetilde{S}_{n-2}; \tag{8.14}$$

this too can be expressed using Euler–Frobenius distributions or Euler–Frobenius numbers as above, but the result is more complicated and omitted.

Appendix A. Euler–Frobenius polynomials

We collect in this appendix some known facts for easy reference. (Some are used above; others are included because we find them interesting and perhaps illuminating.) We note first that the sum in (1.1) is absolutely convergent for $|x| < 1$, and that the second equality holds there (or as an equality of formal power series). The first equality in (1.1) (which is valid for all complex $x \neq 1$) can be written as a "Rodrigues formula"

$$P_{n,\rho}(x) = (1-x)^{n+1}\Bigl(\rho + x\frac{\mathrm{d}}{\mathrm{d}x}\Bigr)^n \frac{1}{1-x}, \tag{A.1}$$

which yields the recursion

$$P_{n,\rho}(x) = (1-x)^{n+1}\Bigl(\rho + x\frac{\mathrm{d}}{\mathrm{d}x}\Bigr)\Bigl(P_{n-1,\rho}(x)(1-x)^{-n}\Bigr), \qquad n \geqslant 1; \tag{A.2}$$

after expansion, this yields (1.2).

We note that by the recursion (1.2) and induction

$$A_{n,0,\rho} = P_{n,\rho}(0) = \rho^n, \tag{A.3}$$

and

$$\sum_{k=0}^{n} A_{n,k,\rho} = P_{n,\rho}(1) = n!; \tag{A.4}$$

moreover, by (1.4) and induction,

$$A_{n,n,\rho} = (1-\rho)^n. \tag{A.5}$$

In particular, if $\rho \neq 1$, then $A_{n,n,\rho} \neq 0$ and $P_{n,\rho}$ has degree exactly n for every n.

The case $\rho = 1$ is special; in this case $A_{n,n,\rho} = 0$ for $n \geqslant 1$, so $P_{n,\rho}$ has degree $n-1$ for $n \geqslant 1$. In fact, if follows directly from (1.1) that

$$P_{n,0}(x) = xP_{n,1}(x), \qquad n \geqslant 1; \tag{A.6}$$

hence, as said in the introduction, the Eulerian numbers appear twice as

$$A_{n,k+1,0} = A_{n,k,1}, \qquad n \geqslant 1. \tag{A.7}$$

A binomial expansion in (1.1) shows that $P_{n,\rho}$ can be expressed using the classical special case $\rho = 0$ as

$$P_{n,\rho}(x) = \sum_{i=0}^{n} \binom{n}{i} \rho^i (1-x)^i P_{n-i,0}(x), \tag{A.8}$$

which shows that $P_{n,\rho}(x)$, and thus also each $A_{n,k,\rho}$, is a polynomial in ρ of degree at most n. By (A.8), the n:th degree term is $(1-x)^n\rho^n$; hence the degree in ρ is exactly n for $P_{n,\rho}(x)$ for any $x \neq 1$ (recall that $P_{n,\rho}(1) = n!$ does not depend on ρ) and for $A_{n,k,\rho}$ for any $k = 0, \dots, n$. (The leading term of $A_{n,k,\rho}$ is $(-1)^k\binom{n}{k}\rho^n$.)

Similarly, another binomial expansion in (1.1) yields

$$\begin{aligned} P_{n,\rho}(x) &= \sum_{i=0}^{n} \binom{n}{i} (\rho-1)^i (1-x)^i P_{n-i,1}(x) \\ &= \sum_{i=0}^{n} \binom{n}{i} (1-\rho)^i (x-1)^i P_{n-i,1}(x). \end{aligned} \tag{A.9}$$

For $\rho = 0$, this yields, using (A.6), the recursion formula used by Frobenius [35, (6.) p. 826] to define the Eulerian polynomials.

The definition (1.1) yields (and is equivalent to) the generating function

$$\sum_{n=0}^{\infty} \frac{P_{n,\rho}(x)}{(1-x)^{n+1}} \frac{z^n}{n!} = \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} (j+\rho)^n x^j \frac{z^n}{n!} = \sum_{j=0}^{\infty} e^{jz+\rho z} x^j = \frac{e^{\rho z}}{1-xe^z} \tag{A.10}$$

or, equivalently,

$$\sum_{n=0}^{\infty} P_{n,\rho}(x) \frac{z^n}{n!} = \frac{(1-x)e^{\rho z(1-x)}}{1-xe^{z(1-x)}}. \tag{A.11}$$

(See also David and Barton [18, pp. 150–152] and Flajolet and Sedgewick [30, Example III.25, p. 209] for the case $\rho = 1$.) In particular, for the classical case $\rho = 1$, (A.10) can be written

$$\frac{1-x}{e^z-x} = \frac{(1-x)e^{-z}}{1-xe^{-z}} = \sum_{n=0}^{\infty} \frac{P_{n,1}(x)}{(1-x)^n} \frac{(-z)^n}{n!} = \sum_{n=0}^{\infty} \frac{P_{n,1}(x)}{(x-1)^n} \frac{z^n}{n!}, \tag{A.12}$$

discovered by Euler [25, §174, p. 391]; this is sometimes taken as a definition, see e.g. Riordan [74, p. 39] and Carlitz [8]. More generally, we similarly obtain, cf. (1.9) and [8],

$$\sum_{n=0}^{\infty} \frac{P_{n,1-u}(x)}{(x-1)^n} \frac{z^n}{n!} = e^{zu} \frac{1-x}{e^z-x}. \tag{A.13}$$

From (A.10) one easily obtains the symmetry relation

$$x^n P_{n,\rho}(x^{-1}) = P_{n,1-\rho}(x), \tag{A.14}$$

or equivalently, by (1.3),

$$A_{n,n-k,\rho} = A_{n,k,1-\rho}, \tag{A.15}$$

which also easily is proved by induction. In terms of the random variables $\mathfrak{E}_{n,\rho}$ defined in (1.5), this can be written

$$\mathfrak{E}_{n,1-\rho} \overset{\mathrm{d}}{=} n - \mathfrak{E}_{n,\rho}. \tag{A.16}$$

Remark A.1. If we define the homogeneous two-variable polynomials

$$\hat{P}_{n,\rho}(x,y) := \sum_{k=0}^{n} A_{n,k,\rho} x^k y^{n-k}, \tag{A.17}$$

so that $\hat{P}_{n,\rho}(x,y) = y^n P_{n,\rho}(x/y)$ and $P_{n,\rho}(x) = \hat{P}_{n,\rho}(x,1)$, the symmetry (A.14)–(A.15) takes the form $\hat{P}_{n,\rho}(x,y) = \hat{P}_{n,1-\rho}(y,x)$. (In particular, in the classical case $\rho = 1$, this together with (A.6) shows that $y^{-1}\hat{P}_{n,1}(x,y)$ is a symmetric homogeneous polynomial of degree $n-1$ [35].) The recursion (1.2) becomes

$$\hat{P}_{n,\rho}(x,y) = \Bigl((1-\rho)x+\rho y+xy\frac{\partial}{\partial x}+xy\frac{\partial}{\partial y}\Bigr)\hat{P}_{n-1,\rho}(x,y), \qquad n \geqslant 1. \tag{A.18}$$

For $x < 1$, the sum in (1.1) can be differentiated in ρ termwise (for all $\rho \in \mathbb{C}$), which yields, for $n \geqslant 1$,

$$\frac{\partial}{\partial\rho}\frac{P_{n,\rho}(x)}{(1-x)^{n+1}} = \sum_{j=0}^{\infty} n(j+\rho)^{n-1}x^j = \frac{nP_{n-1,\rho}(x)}{(1-x)^n} \tag{A.19}$$

and thus (for all x, since we deal with polynomials)

$$\frac{\partial}{\partial\rho}P_{n,\rho}(x) = n(1-x)P_{n-1,\rho}(x), \qquad n \geqslant 1. \tag{A.20}$$

Equivalently, by (1.3),

$$\frac{\partial}{\partial\rho}A_{n,k,\rho} = n\bigl(A_{n-1,k,\rho} - A_{n-1,k-1,\rho}\bigr), \qquad n \geqslant 1. \tag{A.21}$$

Remark A.2. As remarked by Frobenius [35] in the case $\rho = 1$, it follows from the recursion (1.2) that if $0 < \rho \leqslant 1$, the roots of $P_{n,\rho}$ are real, negative and simple, see [92]. (To see this, use induction and consider the values of $P_{n,\rho}$ at the roots of $P_{n-1,\rho}$, and at 0 and $-\infty$; it follows from (1.2) that these values will be of alternating signs, and thus there must be roots of $P_{n,\rho}$ between them, and this accounts for all roots of $P_{n,\rho}$. We omit the details. The argument also shows that the roots of $P_{n,\rho}$ and $P_{n-1,\rho}$ are interlaced. For more general results of this kind, see e.g. [97] and [57, Proposition 3.5].) Recall that if $0 < \rho < 1$ there are n roots, and if $\rho = 1$ only $n-1$ (for $n \geqslant 1$); in this case we can regard $-\infty$ as an additional root. Furthermore, by (A.6) this extends to $\rho = 0$: $P_{n,0}$ has n roots which are simple, real and non-positive, with 0 being a root in this case (for $n \geqslant 1$).

Hence, for $0 \leqslant \rho \leqslant 1$ and $n \geqslant 1$, $P_{n,\rho}$ has n roots $-\infty \leqslant -\lambda_{n,n} < \cdots < -\lambda_{n,1} \leqslant 0$. It follows that the probability generating function (1.6) of $\mathfrak{E}_{n,\rho}$ can be written as

$$\mathbb{E}\, x^{\mathfrak{E}_{n,\rho}} = \frac{P_{n,\rho}(x)}{P_{n,\rho}(1)} = \prod_{j=1}^{n}\frac{x+\lambda_{n,j}}{1+\lambda_{n,j}}, \tag{A.22}$$

which shows that

$$\mathfrak{E}_{n,\rho} \overset{\mathrm{d}}{=} \sum_{j=1}^{n} I_j \tag{A.23}$$

where $I_j \sim \mathrm{Be}(1/(1+\lambda_{n,j}))$ are independent indicator variables. This stochastic representation can be used to show asymptotic properties of the Euler–Frobenius numbers from standard results for sums of independent random variables, see [11] ($\rho = 1$), [37] and Section 6.

The fact that all roots of $P_{n,\rho}$ are real implies further that the sequence $A_{n,k,\rho}$, $k = 0, \dots, n$, is log-concave ([43, Theorem 53, p. 52]; see also Newton's inequality [43, Theorem 51, p. 52]). In particular, the sequence is unimodal. In other words, the distribution of $\mathfrak{E}_{n,\rho}$ is log-concave and unimodal.

Further results on the asymptotics and distribution of the roots of $P_{n,\rho}$ are given in e.g. [27], [36], [72], [38].

Remark A.3. The Eulerian polynomials and numbers should not be confused with the *Euler polynomials* and *Euler numbers*, but there are well-known connections. (See e.g. [35, §§8, 17]; see also [64, §24.1] and the references there for some historical remarks on names and notations.) First, the *Euler polynomials* $E_n(x)$ are defined by their generating function [64, §24.2]

$$\sum_{n=0}^{\infty} E_n(x)\frac{z^n}{n!} = \frac{2e^{xz}}{e^z+1}. \tag{A.24}$$

Taking $x = -1$ in (A.10) we see that

$$E_n(\rho) = 2^{-n}P_{n,\rho}(-1) = 2^{-n}\sum_{k=0}^{n} A_{n,k,\rho}(-1)^k. \tag{A.25}$$

Similarly, the *Euler numbers* E_n [65, A000364], [64, §24.2], which are defined as the coefficients in the Taylor (Maclaurin) series

$$\frac{1}{\cosh t} = \frac{2e^t}{e^{2t}+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!} \tag{A.26}$$

are by (A.24)–(A.25) given by

$$E_n = 2^n E_n(1/2) = P_{n,1/2}(-1). \tag{A.27}$$

The Euler numbers E_n vanish for odd n, and the numbers E_{2n} alternate in sign, $E_{2n} = (-1)^n|E_{2n}|$; the positive numbers $|E_{2n}|$ are also known as *secant numbers* since they are the coefficients in [25, p. 432]

$$\sec t := \frac{1}{\cos t} = \sum_{n=0}^{\infty} |E_{2n}| \frac{t^{2n}}{(2n)!}. \tag{A.28}$$

Furthermore, returning to the classical case (Euler's case) $\rho = 1$ and taking $x = \mathrm{i}$ in (A.10) we find

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{P_{n,1}(\mathrm{i})}{(1-\mathrm{i})^{n+1}} \frac{t^n}{n!} &= \frac{e^t}{1-\mathrm{i}e^t} = \frac{e^t(1+\mathrm{i}e^t)}{1+e^{2t}} = \frac{\mathrm{i}}{2} + \frac{\mathrm{i}}{2}\frac{e^{2t}-1}{e^{2t}+1} + \frac{e^t}{e^{2t}+1} \\ &= \frac{\mathrm{i}}{2} + \frac{\mathrm{i}}{2}\tanh t + \frac{1}{2\cosh t} \\ &= \frac{\mathrm{i}}{2} + \frac{\mathrm{i}}{2}\sum_{m=0}^{\infty}(-1)^m T_{2m+1}\frac{t^{2m+1}}{(2m+1)!} + \frac{1}{2}\sum_{m=0}^{\infty} E_{2m}\frac{t^{2m}}{(2m)!},\end{aligned}$$

where T_n are the *tangent numbers* [65, A000182], [64, §24.15] defined as the coefficients in the Taylor (Maclaurin) series

$$\tan t = \sum_{n=0}^{\infty} T_n \frac{t^n}{n!} = \sum_{m=0}^{\infty} T_{2m+1}\frac{t^{2m+1}}{(2m+1)!}; \tag{A.29}$$

note that $T_n = 0$ when n is even. Hence, for $n \geqslant 1$,

$$P_{n,1}(\mathrm{i}) = \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle \mathrm{i}^k = \begin{cases} 2^m \mathrm{i}^m T_n, & n = 2m+1, \\ (1-\mathrm{i})2^{m-1}(-\mathrm{i})^m E_n, & n = 2m. \end{cases} \tag{A.30}$$

For even n we can also write this as

$$P_{2m,1}(\mathrm{i}) = (\mathrm{i}^m - \mathrm{i}^{m+1})2^{m-1}|E_{2m}|. \tag{A.31}$$

For odd n we can use the relation [64, 24.15.4]

$$T_{2m-1} = (-1)^{m-1}\frac{2^{2m}(2^{2m}-1)}{2m}B_{2m} \tag{A.32}$$

with the *Bernoulli numbers* B_{2m} [64, §24.2] and write (A.30) as

$$P_{n,1}(\mathrm{i}) = (-2i)^m \frac{2^{n+1}(2^{n+1}-1)}{n+1}B_{n+1}, \qquad n = 2m+1. \tag{A.33}$$

Similarly, taking $\rho = 1$ and $x = -1$ in (A.10) we find

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{P_{n,1}(-1)}{2^{n+1}} \frac{t^n}{n!} &= \frac{e^t}{e^t+1} = \frac{1}{2} + \frac{1}{2}\frac{e^t-1}{e^t+1} = \frac{1}{2} + \frac{1}{2}\tanh\frac{t}{2} \\ &= \frac{1}{2} + \frac{1}{2}\sum_{m=0}^{\infty}(-1)^m T_{2m+1}\frac{(t/2)^{2m+1}}{(2m+1)!}.\end{aligned}$$

Hence, for $n \geqslant 1$,

$$P_{n,1}(-1) = \sum_{k=0}^{n-1}(-1)^k \left\langle {n \atop k} \right\rangle = (-1)^{(n-1)/2} T_n = \begin{cases} (-1)^m T_n, & n = 2m+1, \\ 0, & n = 2m, \end{cases} \tag{A.34}$$

which for odd n can be expressed in B_{n+1} using (A.32).

Frobenius [35] studied also $P_{n,1}(\zeta)$ for other roots of unity ζ, using the name *Euler numbers of the mth order* for $P_{n,1}(\zeta)/(\zeta-1)^m$ when ζ is a primitive mth root of unity (see also [90, p. 163]); the standard Euler numbers above are the case $m=4$ ($\zeta=\mathrm{i}$) apart from a factor $1+\mathrm{i}$, see (A.30)–(A.31).

Remark A.4. For $p\in(0,1)$, (1.1) can be rewritten as

$$\frac{P_{n,\rho}(p)}{(1-p)^n}=\sum_{j=0}^{\infty}(j+\rho)^n(1-p)p^j=\mathbb{E}(X_p+\rho)^n, \tag{A.35}$$

where $X_p\sim\mathrm{Ge}(p)$ has a geometric distribution. In particular, the moments of a geometric distribution are, using (A.6), for $n\geqslant 1$,

$$\mathbb{E}\,X_p^n=(1-p)^{-n}P_{n,0}(p)=p(1-p)^{-n}P_{n,1}(p), \tag{A.36}$$

and the central moments are, using (A.14),

$$\begin{aligned}\mathbb{E}(X_p-\mathbb{E}\,X_p)^n=\mathbb{E}\Bigl(X_p-\frac{p}{1-p}\Bigr)^n&=\frac{P_{n,-p/(1-p)}(p)}{(1-p)^n}\\&=\Bigl(\frac{p}{1-p}\Bigr)^nP_{n,1/(1-p)}\Bigl(\frac{1}{p}\Bigr).\end{aligned} \tag{A.37}$$

In particular, for $p=1/2$ we have the moments $\mathbb{E}\,X_{1/2}^n=2^{n-1}P_{n,1}(1/2)$ ($n\geqslant 1$) [65, A000670] (*numbers of preferential arrangements*, also called *surjection numbers*; see further e.g. [39, Exercise 7.44] and [30, II.3.1]) and the central moments $\mathbb{E}(X_{1/2}-1)^n=P_{n,2}(2)$ [65, A052841].

Remark A.5. Benoumhani [4] studied polynomials related to Euler–Frobenius polynomials. His $F_m(n,x)$ can be expressed as

$$F_m(n,x)=m^n(1+x)^nP_{n,1/m}\left(\frac{x}{1+x}\right)=\sum_{k=0}^{n}m^nA_{n,k,1/m}x^k(1+x)^{n-k}. \tag{A.38}$$

Appendix B. Splines

As said in Section 3, the density function f_n is continuous for $n\geqslant 2$, while f_1 has jumps at 0 and 1. More generally, f_n is $n-2$ times continuously differentiable, while $f_n^{(n-1)}$ has jumps at the integer points $0,\dots,n$; furthermore, f_n is a polynomial of degree $n-1$ in any interval $(k-1,k)$. Such functions are called *splines* of degree $n-1$, with knots at the integers, see e.g. [77; 78; 80]. Hence f_n is a spline of degree $n-1$, with knots at the integers, which moreover vanishes outside $[0,n]$; in this context, f_n is known as a *B-spline*, see e.g. [78, Lecture 2]. Here “B” stands for basis, since translates of f_n form a basis in the linear space $\mathcal{S}_{n-1}$ of all splines of degree $n-1$ with integer knots [77; 78]; in other words, every spline $g\in\mathcal{S}_n$ can be written

$$g(x)=\sum_{k=-\infty}^{\infty}c_kf_{n+1}(x-k) \tag{B.1}$$

for a unique sequence $(c_k)_{-\infty}^{\infty}$ of complex numbers (or real numbers, if we consider real splines), and conversely, every such sum gives a spline in $\mathcal{S}_n$. (The sum converges trivially pointwise.) The interpretation of the B-spline as the density function of S_n was observed already by Schoenberg [77, 3.17].

By Theorem 3.2, the values of the B-spline are given by the Euler–Frobenius numbers; equivalently, the B-splines satisfy the recursion formula (3.10), which is well-known in this setting [80, (4.52)–(4.53)].

A related construction, see e.g. [78], is the *exponential spline*, defined by taking $c_k = t^k$ in (B.1) for some complex $t \neq 0$, i.e.,

$$\Phi_n(x;t) := \sum_{k=-\infty}^{\infty} t^k f_{n+1}(x-k) = \sum_{k=-\infty}^{\infty} t^{-k} f_{n+1}(x+k), \tag{B.2}$$

which is a spline of degree n satisfying

$$\Phi_n(x+1;t) = t\Phi_n(x;t) \tag{B.3}$$

(and, up to a constant factor, the only such spline). By (B.3), $\Phi_n(x;t)$ is determined by its restriction to [0,1], and by (B.2), (3.4), (1.3) and (A.14) we have for $0 \leqslant x \leqslant 1$,

$$\Phi_n(x;t) = \frac{1}{n!} \sum_{k=-\infty}^{\infty} A_{n,k,x} t^{-k} = \frac{1}{n!} P_{n,x}(t^{-1}) = \frac{t^{-n}}{n!} P_{n,1-x}(t). \tag{B.4}$$

Note that here we rather consider $P_{n,1-x}(t)$ as a polynomial in x, with a parameter t, instead of the opposite as we usually do.

These and other relations betweens splines and Euler–Frobenius polynomials have been known and used for a long time, see e.g. [44; 59; 60; 61; 68; 71; 72; 73; 78; 81; 82; 92; 93; 96]. We give a few further examples; see the references just given for details and further results.

First, consider the following interpolation problem: Let $\lambda \in [0,1]$ be given and find a spline $g \in \mathcal{S}_n$ such that

$$g(k+\lambda) = a_k, \qquad k \in \mathbb{Z}, \tag{B.5}$$

for a given sequence $(a_k)_{-\infty}^{\infty}$. It is not difficult to see that this problem always has a solution, and that the space of solutions has dimension n if $\lambda \in (0,1)$ and $n-1$ if $\lambda \in \{0,1\}$. (If $0 < \lambda < 1$, we may e.g. choose $c_{-n+1}, \dots, c_0$ arbitrarily, and then choose $c_1, c_2, \dots$ and $c_{-n}, c_{-n-1}, \dots$ recursively so that (B.5) holds; the case $\lambda = 0$ or 1 is similar.) Moreover, the null space, i.e., the space of splines $g \in \mathcal{S}_n$ such that $g(k+\lambda) = 0$ for all integers k, contains by (B.3) every exponential spline $\Phi_n(x;t)$ such that $\Phi_n(\lambda, t) = 0$; by (B.4) this is equivalent to $P_{n,1-\lambda}(t) = 0$. Since $\Phi_{n,1-\lambda}$ has n non-zero roots if $\lambda \in (0,1)$ and $n-1$ if $\lambda \in \{0,1\}$, see Remark A.2, the exponential splines $\Phi_n(x;t_i)$, where t_i is a non-zero root of $P_{n,1-\lambda}$, form a basis of the null space of (B.5). Note that these roots t_i are real and negative by Remark A.2.

The cases $\lambda = 0$ and $\lambda = 1/2$ are particularly important, which explains the importance of $P_{n,1}(x)$ and $P_{n,1/2}(x)$ and the corresponding Eulerian numbers $A_{n,k,1}$ and the Eulerian numbers of type B $B_{n,k} = 2^n A_{n,k,1/2}$ in spline theory.

Similarly, one may consider the periodic interpolation problem, considering only functions and sequences with a given period N. By simple Fourier analysis, if $\omega_N = \exp(2\pi\mathrm{i}/N)$, then the exponential splines $\Phi(x, \omega_N^j)$, $j = 1, \dots, N$, form a basis of the N-dimensional space of periodic splines in $\mathcal{S}_n$; here $\omega_N^j = \exp(2\pi\mathrm{i}j/N)$ ranges over the N:th unit roots. Moreover, (B.5) has a unique periodic solution for every periodic sequence $(a_k)_{-\infty}^{\infty}$ if and only if none of these exponential splines vanishes at λ, i.e., if and only if $P_{n,1-\lambda}(\omega_N^j) \neq 0$ for all j. Since the roots of $P_{n,1-\lambda}$ lie in $(-\infty, 0]$, the only possible problem is for -1, so this holds always if N is odd, and if N is even unless $P_{n,1-\lambda}(-1) = 0$; by (A.25) and standard properties of the Euler polynomials [64, §24.12(ii), see also (24.4.26), (24.4.28), (24.4.35)], the periodic interpolation problem (B.5) thus has a unique solution except if either N even, n even and $\lambda \in \{0, 1\}$, or N even, n odd and $\lambda = 1/2$.

References

[1] Michel L. Balinski & Svetlozar T. Rachev, Rounding proportions: rules of rounding. *Numer. Funct. Anal. Optim.* **14** (1993), no. 5-6, 475–501.

[2] Michel L. Balinski & H. Peyton Young, *Fair Representation.* 2nd ed., Brookings Institution Press, Washington, D.C., 2001.

[3] Edward A. Bender, Central and local limit theorems applied to asymptotic enumeration. *J. Combinatorial Theory Ser. A* **15** (1973), 91–111.

[4] Moussa Benoumhani, On some numbers related to Whitney numbers of Dowling lattices. *Adv. in Appl. Math.*, **19** (1997), no. 1, 106–116.

[5] Francesco Brenti, q-Eulerian polynomials arising from Coxeter groups. *European J. Combin.* **15** (1994), no. 5, 417–441.

[6] P. L. Butzer & M. Hauss, Eulerian numbers with fractional order parameters. *Aequationes Math.* **46** (1993), no. 1-2, 119–142.

[7] L. Carlitz, q-Bernoulli and Eulerian numbers. *Trans. Amer. Math. Soc.* **76** (1954), 332–350.

[8] L. Carlitz, Eulerian numbers and polynomials. *Mathematics Magazine* **32** (1959), 247–260.

[9] L. Carlitz, Eulerian numbers and polynomials of higher order. *Duke Math. J.* **27** (1960), 401–423.

[10] L. Carlitz, A combinatorial property of q-Eulerian numbers. *Amer. Math. Monthly* **82** (1975), 51–54.

[11] L. Carlitz, D. C. Kurtz, R. Scoville & O. P. Stackelberg, Asymptotic properties of Eulerian numbers. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **23** (1972), 47–54.

[12] L. Carlitz & Richard Scoville, Generalized Eulerian numbers: combinatorial applications. *J. Reine Angew. Math.* **265** (1974), 110–137.

[13] Don Chakerian & Dave Logothetti, Cube slices, pictorial triangles, and probability. *Math. Mag.* **64** (1991), no. 4, 219–241.

[14] Chak-On Chow & Ira M. Gessel, On the descent numbers and major indices for the hyperoctahedral group. *Adv. in Appl. Math.* **38** (2007), no. 3, 275–301.

[15] Sylvie Corteel & Sandrine Dasse-Hartaut, Statistics on staircase tableaux, Eulerian and Mahonian statistics. In *23rd International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2011)*, Discrete Math. Theor. Comput. Sci. Proc., AO:245–255, 2011.

[16] Harald Cramér, Sur un noveau théorème-limite de la théorie des probabilités. *Les sommes et les fonctions de variables aléatoires*, Actualités Scientifiques et Industrielles 736, Hermann, Paris, 1938, pp. 5–23.

[17] Sandrine Dasse-Hartaut & Paweł Hitczenko, Greek letters in random staircase tableaux. *Random Struct. Algorithms*, to appear.

[18] F. N. David & D. E. Barton, *Combinatorial Chance.* Charles Griffin & co., London, 1962.

[19] Amir Dembo and Ofer Zeitouni, *Large Deviations Techniques and Applications.* 2nd ed., Springer, New York, 1998.

[20] Persi Diaconis & David Freedman, On rounding percentages. *J. Amer. Statist. Assoc.* **74** (1979), no. 366, part 1, 359–364.

[21] Dominique Dumont, Une généralisation trivariée symétrique des nombres eulériens. *J. Combin. Theory Ser. A* **28** (1980), no. 3, 30–320.

[22] R. Ehrenborg & M. Readdy & E. Steingrímsson, Mixed volumes and slices of the cube. *J. Combin. Theory Ser. A* **81** (1998), no. 1, 121–126.

[23] Carl-Gustav Esseen, On the application of the theory of probability to two combinatorial problems involving permutations. *Proceedings of the Seventh Conference on Probability Theory (Braşov, 1982)*, 137–147, VNU Sci. Press, Utrecht, 1985.

[24] Leonhard Euler, Methodus universalis series summandi ulterius promota. *Commentarii academiae scientiarum imperialis Petropolitanae* **8** (1736), St. Petersburg, 1741, pp. 147–158. http://www.math.dartmouth.edu/~euler/pages/E055.html

[25] Leonhard Euler, *Institutiones calculi differentialis cum eius usu in analysi finitorum ac doctrina serierum. Vol I.* St. Petersburg, 1755. http://www.math.dartmouth.edu/~euler/pages/E212.html

[26] L. Euler. Remarques sur un beau rapport entre les séries des puissances tant direct que réciproques, *Memoires de l'Académie Royale des Sciences et des Belles-Lettres* **17**, in *Histoire de l'Académie Royale des Sciences et des Belles-Lettres de Berlin 1761*, Berlin, Haude et Spener, 1768, pp. 83–106. (Paper read to the academy in 1749.) http://www.math.dartmouth.edu/~euler/pages/E352.html

[27] Jutta Faldey & Wolfgang Gawronski, On the limit distributions of the zeros of Jonquière polynomials and generalized classical orthogonal polynomials. *J. Approx. Theory* **81** (1995), no. 2, 231–249.

[28] William Feller, *An Introduction to Probability Theory and its Applications, Volume I.* 2nd ed., Wiley, New York, 1957.

[29] William Feller, *An Introduction to Probability Theory and its Applications, Volume II.* 2nd ed., Wiley, New York, 1971.

[30] Philippe Flajolet & Robert Sedgewick, *Analytic Combinatorics.* Cambridge Univ. Press, Cambridge, UK, 2009.

[31] Dominique Foata, Distributions eulériennes et mahoniennes sur le groupe des permutations. *Higher Combinatorics (Proc. NATO Advanced Study Inst., Berlin, 1976)*, Reidel, Dordrecht, 1977, pp. 27–49.

[32] Dominique Foata, Eulerian polynomials: from Euler's time to the present. *The Legacy of Alladi Ramakrishnan in the Mathematical Sciences*, 253–273, Springer, New York, 2010.

[33] Dominique Foata & Marcel-P. Schützenberger, *Théorie géométrique des polynômes eulériens.* Lecture Notes in Math. **138**, Springer-Verlag, Berlin-New York, 1970.

[34] Ghislain R. Franssens. On a number pyramid related to the binomial, Deleham, Eulerian, MacMahon and Stirling number triangles. *J. Integer Seq.* **9** (2006), no. 4, Article 06.4.1, 34 pp.

[35] G. Frobenius, Über die Bernoullischen Zahlen und die Eulerschen Polynome. *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften, 1910*, Berlin, 1910, 809–847.

[36] Wolfgang Gawronski, On the asymptotic distribution of the zeros of Hermite, Laguerre, and Jonquière polynomials. *J. Approx. Theory* **50** (1987), no. 3, 214–231.

[37] Wolfgang Gawronski & Thorsten Neuschel, Euler–Frobenius numbers. *Integral Transforms and Special Functions*, to appear. Published online 22 Jan 2013.

[38] Wolfgang Gawronski & Ulrich Stadtmüller, On the zeros of Lerch's transcendental function with real parameters. *J. Approx. Theory* **53** (1988), no. 3, 354–364.

[39] Ronald L. Graham, Donald E. Knuth & Oren Patashnik, *Concrete Mathematics.* 2nd ed., Addison-Wesley, Reading, MA, 1994.

[40] Max Happacher, The discrepancy distribution of stationary multiplier rules for rounding probabilities. *Metrika* **53** (2001), no. 2, 171–181.

[41] Max Happacher & Friedrich Pukelsheim, Rounding probabilities: unbiased multipliers. *Statist. Decisions* **14** (1996), no. 4, 373–382.

[42] Max Happacher & Friedrich Pukelsheim, Rounding probabilities: maximum probability and minimum complexity multipliers. *J. Statist. Plann. Inference* **85** (2000), no. 1-2, 145–158.

[43] G. H. Hardy, J. E. Littlewood & G. Pólya, *Inequalities.* 2nd ed., Cambridge, at the University Press, 1952.

[44] Tian-Xiao He, Eulerian polynomials and B-splines. *J. Comput. Appl. Math.* **236** (2012), no. 15, 3763–3773.

[45] Douglas Hensley, Eulerian numbers and the unit cube. *Fibonacci Quart.* **20** (1982), no. 4, 344–348.

[46] Friedrich Hirzebruch, Eulerian polynomials. *Münster J. of Math.* **1** (2008), 9–14.

[47] Paweł Hitczenko & Svante Janson, Weighted random staircase tableaux. Preprint, 2012. arXiv:1212.5498.

[48] Svante Janson, *Gaussian Hilbert Spaces.* Cambridge Univ. Press, Cambridge, 1997.

[49] Svante Janson, Rounding of continuous random variables and oscillatory asymptotics. *Ann. Probab.* **34** (2006), no. 5, 1807–1826.

[50] Svante Janson, Asymptotic bias of some election methods. *Annals of Operations Research.* Published online April 2012.

[51] Olav Kallenberg, *Foundations of Modern Probability.* 2nd ed., Springer, New York, 2002.

[52] A. C. Kimber, Eulerian numbers and links with some statistical procedures. *Utilitas Math.* **31** (1987), 57–65.

[53] Klaus Kopfermann, *Mathematische Aspekte der Wahlverfahren.* Wissenschaftsverlag, Mannheim, 1991.

[54] Pierre-Simon de Laplace, *Théorie analytique des probabilités.* 3rd ed., Courcier, Paris 1820; *Œuvres completes de Laplace, vol. 7*, Gauthier-Villars, Paris, 1886. http://gallica.bnf.fr/ark:/12148/bpt6k775950

[55] M. Lerch, Note sur la fonction $\mathfrak{K}(w,x,s)=\sum_{k=0}^{\infty}\frac{e^{2k\pi ix}}{(w+k)^s}$. *Acta Mathematica* **11** (1887), 19–24.

[56] Léonce Lesieur & Jean-Louis Nicolas, Double interpolation des nombres euleriens. *European J. Combin.* **16** (1995), no. 1, 41–57.

[57] Lily L. Liu & Yi Wang, A unified approach to polynomial sequences with only real zeros. *Adv. Appl. Math.*, **38** (2007), no. 4, 542–560.

[58] P. A. MacMahon, The divisors of numbers. *Proc. London Math. Soc. Ser. 2* **19** (1920), no. 1, 305–340.

[59] Günter Meinardus & Gerhard Merz, Zur periodischen Spline-Interpolation. *Spline-Funktionen (Proceedings, Oberwolfach, 1973)*, Bibliographisches Inst., Mannheim, 1974, pp. 177–195.

[60] Gerhard Merz, Interpolation mit periodischen Spline-Funktionen I. *J. Approx. Theory* **30** (1980), no. 1, 11–19.

[61] Gerhard Merz, Interpolation mit periodischen Spline-Funktionen II. *J. Approx. Theory* **30** (1980), no. 1, 20–28.

[62] Frederick Mosteller, Cleo Youtz & Douglas Zahn, The distribution of sums of rounded percentages. *Demography* **4** (1967), 850–858.

[63] Jean-Louis Nicolas, An integral representation for Eulerian numbers. *Sets, Graphs and Numbers (Budapest, 1991)*, 513–527, Colloq. Math. Soc. János Bolyai, 60, North-Holland, Amsterdam, 1992.

[64] *NIST Digital Library of Mathematical Functions.* http://dlmf.nist.gov/

[65] *The On-Line Encyclopedia of Integer Sequences.* oeis.org

[66] Valentin V. Petrov, *Sums of Independent Random Variables.* Springer-Verlag, Berlin, 1975.

[67] Alexander Peyerimhoff, *Lectures on Summability.* Lecture Notes in Math. **107**, Springer-Verlag, Berlin-New York, 1969.

[68] Gerlind Plonka, Periodic spline interpolation with shifted nodes. *J. Approx. Theory* **76** (1994), no. 1, 1–20.

[69] Georg Pólya, Berechnung eines bestimmten Integrals. *Math. Ann.* **74** (1913), no. 2, 204–212.

[70] Friedrich Pukelsheim, *Proportional Representation – Rigorous Methodology.* In preparation.

[71] M. Reimer, Extremal spline bases. *J. Approx. Theory* **36** (1982), no. 2, 91–98.

[72] M. Reimer, The main roots of the Euler–Frobenius polynomials. *J. Approx. Theory* **45** (1985), no. 4, 358–362.

[73] M. Reimer & D. Siepmann, An elementary algebraic representation of polynomial spline interpolants for equidistant lattices and its condition. *Numer. Math.* **49** (1986), no. 1, 55–65.

[74] John Riordan, *An Introduction to Combinatorial Analysis.* Wiley, New York, 1958.

[75] Vladimir N. Sachkov, *Probabilistic Methods in Combinatorial Analysis.* (Russian.) Nauka, Moscow, 1978. English translation and revision: Cambridge University Press, Cambridge, 1997.

[76] Frank Schmidt & Rodica Simion, Some geometric probability problems involving the Eulerian numbers. *Electron. J. Combin.* **4** (1997), no. 2, Research Paper 18, 13 pp.

[77] I. J. Schoenberg, Contributions to the problem of approximation of equidistant data by analytic functions. *Quart. Appl. Math.* **4** (1946), 45–99 and 112–141.

[78] I. J. Schoenberg, *Cardinal Spline Interpolation.* CBMS Regional Conference Series in Applied Mathematics **12**. SIAM, Philadelphia, PA, 1973.

[79] I. J. Schoenberg, Cardinal spline interpolation and the exponential Euler splines. *Functional Analysis and its Applications (Proc., Madras, 1973)*, Lecture Notes in Math. **399**, Springer, Berlin, 1974, pp. 477–489.

[80] Larry L. Schumaker, *Spline Functions: Basic Theory.* Wiley, New York, 1981.

[81] Dietmar Siepmann, *Kardinale Spline-Interpolation bezüglich äquidistant verteilter Knoten.* Dissertation, Dortmund, 1984.

[82] Dietmar Siepmann, Cardinal interpolation by polynomial splines: interpolation of data with exponential growth. *J. Approx. Theory* **53** (1988), no. 2, 167–183.

[83] Yilmas Simsek, Generating functions for generalized Stirling type numbers, Array type polynomials, Eulerian type polynomials and their applications. `arXiv:1111.3848.`

[84] Y. Simsek, T. Kim, D. W. Park, Y. S. Ro, L. C. Jang & S. H. Rim, An explicit formula for the multiple Frobenius-Euler numbers and polynomials. *JP J. Algebra Number Theory Appl.* **4** (2004), no. 3, 519–529.

[85] S. H. Siraždinov, Asymptotic expression for Euler numbers. (Russian.) *Izv. Akad. Nauk UzSSR Ser. Fiz.-Mat. Nauk* **1979**, no. 6, 39–43, 92.

[86] David Slepian, On the volume of certain polytopes. Technical memorandum, Bell Telephone Laboratories, 1956.

[87] Richard P. Stanley, Eulerian partitions of a unit hypercube. *Higher Combinatorics (Proc. NATO Advanced Study Inst., Berlin, 1976)*, Reidel, Dordrecht, 1977, p. 49.

[88] Richard P. Stanley, *Enumerative Combinatorics, Volume I.* Cambridge Univ. Press, Cambridge, 1997.

[89] Einar Steingrímsson, Permutation statistics of indexed permutations. *European J. Combin.* **15** (1994), no. 2, 187–205.

[90] J. J. Sylvester, Sur une propriété des nombres premiers qui se rattache au théorème de Fermat. *Comptes rendus hebdomadaires des séances de l'Académie des sciences* **52** (1861), 161–163.

[91] S. Tanny, A probabilistic interpretation of Eulerian numbers. *Duke Math. J.* **40** (1973), 717–722. Corrigenda, *ibid* **41** (1974), 689.

[92] Hennie ter Morsche, On the existence and convergence of interpolating periodic spline functions of arbitrary degree. *Spline-Funktionen (Proceedings, Oberwolfach, 1973)*, Bibliographisches Inst., Mannheim, 1974, pp. 197–214.

[93] Hennie ter Morsche, On the relations between finite differences and derivatives of cardinal spline functions. *Spline Functions*, Lecture Notes in Mathematics **501**, Springer, 1976, pp. 210–219.

[94] C. Truesdell, On a function which occurs in the theory of the structure of polymers. *Ann. of Math. (2)* **46** (1945). 144–157.

[95] Hirofumi Tsumura, On a p-adic interpolation of the generalized Euler numbers and its applications. *Tokyo J. Math.* **10** (1987), no. 2, 281–293.

[96] Ren-Hong Wang, Yan Xu & Zhi-Qiang Xu, Eulerian numbers: a spline perspective. *J. Math. Anal. Appl.* **370** (2010), no. 2, 486–490.

[97] Yi Wang & Yeong-Nan Yeh, Polynomials with real zeros and Pólya frequency sequences. *J. Combin. Theory Ser. A*, **109** (2005), no. 1, 63–74.

[98] Steven R. Weller, W. Moran, Brett Ninness & A. D. Pollington, Sampling zeros and the Euler–Frobenius polynomials. *IEEE Trans. Automat. Control* **46** (2001), no. 2, 340–343.

[99] Tingyao Xiong, Hung-ping Tsao, Jonathan I. Hall, General Eulerian Numbers and Eulerian Polynomials. `arXiv:1207.0430.`

[100] Yan Xu & Ren-hong Wang, Asymptotic properties of B-splines, Eulerian numbers and cube slicing. *J. Comput. Appl. Math.* **236** (2011), no. 5, 988–995.

Department of Mathematics, Uppsala University, PO Box 480, SE-751 06 Uppsala, Sweden

E-mail address: `svante.janson@math.uu.se`

URL: `http://www2.math.uu.se/∼svante/`