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Some Umbral Calculus Presentations of the Chan-Chyan-Srivastava Polynomials and the Erkuş-Srivastava Polynomials

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Abstract

In their recent investigation involving differential operators for the generalized Lagrange polynomials, Chan et. al. [3] *encountered and proved a certain summation identity and several other results for the Lagrange polynomials in several variables, which are popularly known in the literature as the Chan-Chyan-Srivastava polynomials. These multivariable polynomials have been studied systematically and extensively in the literature ever since then (see, for example,* [1], [4], [9], [11], [12] *and* [13]*). In the present paper, we investigate umbral calculus presentations of the Chan-Chyan-Srivastava polynomials and also of their substantially more general form, the Erkuş-Srivastava polynomials* [9]*. Some other closely-related results are also considered.*

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1. Introduction, Definitions and Notations

The familiar (two-variable) polynomials $g_n^{(\alpha,\beta)}(x,y)$, which are generated by

$$(1-xz)^{-\alpha}\,(1-yz)^{-\beta}=\sum_{n=0}^{\infty}g_n^{(\alpha,\beta)}(x,y)z^n$$

$$\left(\alpha,\beta\in\mathbf{C};\ |z|<\min\{|x|^{-1},|y|^{-1}\}\right), \tag{1.1}$$

are known as the Lagrange polynomials which occur in certain problems in statistics (*cf.*, *e.g.*, Erdélyi *et al.* [8, p. 267]; see also [15, p. 441 *et seq.*]).

The (three-variable) Lagrange polynomials $g_n^{(\alpha,\beta,\gamma)}(x,y,z)$, which are defined by means of the generating function:

$$(1-xt)^{-\alpha}(1-yt)^{-\beta}(1-zt)^{-\gamma}=\sum_{n=0}^{\infty}g_n^{(\alpha,\beta,\gamma)}(x,y,z)\,\frac{t^n}{n!} \tag{1.2}$$

$$\left(\alpha,\beta,\gamma\in\mathbf{C};\ |t|<\min\{|x|^{-1},|y|^{-1},|z|^{-1}\}\right),$$

were studied recently by Khan and Shukla [10]. Subsequently, Chan et al. [3] introduced and investigated the multivariable extension of the classical Lagrange polynomials $g_n^{(\alpha,\beta)}(x,y)$ generated by (1.1).

These multivariable Lagrange polynomials $g_n^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)$, which are popularly known in the literature as the *Chan-Chyan-Srivastava polynomials*, are generated by (see, for details, [3]; see also [4], [12] and [13])

$$\prod_{j=1}^{r}\{(1-x_jz)^{-\alpha_j}\}=\sum_{r=0}^{\infty}g_n^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)z^n \tag{1.3}$$

$$\left(\alpha_j\in\mathbf{C}\quad(j=1,\cdots,r);\ |z|<\min\{|x_1|^{-1},\cdots,|x_r|^{-1}\}\right),$$

so that, upon comparison with the generating function (1.2), we have the following relationship:

$$g_n^{(\alpha,\beta,\gamma)}(x,y,z)=n!\,g_n^{(\alpha,\beta,\gamma)}(x,y,z)$$

$$(n\in\mathbf{N}_0:=\mathbf{N}\cup\{0\};\ \mathbf{N}:=\{1,2,3,\cdots\})$$

with the (three-variable) Lagrange polynomials $g_n^{(\alpha,\beta,\gamma)}(x,y,z)$ studied by Khan and Shukla [10].

Clearly, the defining generating function (1.3) yields the explicit representation given by [3, p. 140, Eq. (6)]

$$g_n^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r) = \sum_{\substack{k_1,\cdots,k_r\in\mathbf{N}_0 \\ (k_1+\cdots+k_r=n)}} (\alpha_1)_{k_1}\cdots(\alpha_r)_{k_r}\frac{x_1^{k_1}}{k_1!}\cdots\frac{x_r^{k_r}}{k_r!}$$

$$= \sum_{k_1,\cdots,k_{r-1}=0}^{k_1+\cdots+k_{r-1}\leqq n} \frac{(\alpha_1)_{n-k_1-\cdots-k_{r-1}}(\alpha_2)_{k_1}\cdots(\alpha_r)_{k_{r-1}}}{(n-k_1-\cdots-k_{r-1})!\,k_1!\cdots k_{r-1}!}$$

$$\cdot\, x_1^{n-k_1-\cdots-k_{r-1}}\, x_2^{k_1}\cdots x_r^{k_{r-1}} \tag{1.4}$$

or, equivalently, by [11, p. 522, Eq. (17)]

$$g_n^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r) = \sum_{n_{r-1}=0}^{n}\sum_{n_{r-2}=0}^{n_{r-1}}\cdots\sum_{n_1=0}^{n_2}\frac{(\alpha_1)_{n_1}(\alpha_2)_{n_2-n_1}\cdots(\alpha_r)_{n-n_{r-1}}}{n_1!(n_2-n_1)!\cdots(n-n_{r-1})!}$$

$$\cdot x_1^{n_1}x_2^{n_2-n_1}\cdots x_r^{n-n_{r-1}}$$

$$= \sum_{k_1=0}^{n}\sum_{k_2=0}^{k_1}\cdots\sum_{k_{r-1}=0}^{k_{r-2}}\frac{(\alpha_1)_{n-k_1}(\alpha_2)_{k_2-k_1}\cdots(\alpha_r)_{k_{r-1}}}{(n-k_1)!(k_2-k_1)!\cdots k_{r-1}!}$$

$$\cdot\, x_1^{n-k_1}\, x_2^{k_2-k_1}\cdots x_r^{k_{r-1}}, \tag{1.5}$$

where, as usual, $(\lambda)_n$ denotes the Pochhammer symbol given by

$$(\lambda)_0 := 1 \qquad \text{and} \qquad (\lambda)_n := \lambda(\lambda+1)\cdots(\lambda+n-1) \qquad (n\in\mathbf{N}).$$

Altın and Erkuş [1] presented a multivariable extension of the so-called Lagrange-Hermite polynomials generated by (see [1, p. 239, Eq. (1.2)])

$$\prod_{j=1}^{r}\{(1-x_jz^j)^{-\alpha_j}\} = \sum_{n=0}^{\infty} h_n^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)z^n \tag{1.6}$$

$$\left(\alpha_j\in\mathbf{C}\quad (j=1,\cdots,r);\,|z|<\min_{j\in\{1,\cdots,r\}}\{|x_j|^{-1/j}\}\right).$$

The case $r=2$ of the polynomials given by (1.6) corresponds to the familiar (two-variable) Lagrange-Hermite polynomials considered by Dattoli *et al.* [6].

The multivariable (Erkuş-Srivastava) polynomials $\mathcal{U}_{n;\ell_1,\cdots,\ell_r}^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)$, which are defined by the following generating function [9, p. 268, Eq. (3)]:

$$\prod_{j=1}^{r}\{(1-x_j z^{\ell_j})^{-\alpha_j}\} = \sum_{n=0}^{\infty} \mathcal{U}_{n;\ell_1,\cdots,\ell_r}^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)z^n \tag{1.7}$$

$$\Big(\alpha_j \in \mathbf{C} \quad (j=1,\cdots,r);\ \ell_j \in \mathbf{N} \quad (j=1,\cdots,r);$$
$$|z| < \min\{|x_1|^{-1/\ell_1},\cdots,|x_r|^{-1/\ell_r}\}\Big),$$

are a unification (*and* generalization) of several known families of multivariable polynomials including (for example) the Chan-Chyan-Srivastava polynomials $g_n^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)$ defined by (1.3) (see, for details, [9]). Obviously, the Chan-Chyan-Srivastava polynomials $g_n^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)$ follow as the special case of the Erkuş-Srivastava polynomials $\mathcal{U}_{n;\ell_1,\cdots,\ell_r}^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)$ when $\ell_j = 1\ \ (j=1,\cdots,r)$.

Moreover, the Lagrange-Hermite polynomials $h_n^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)$ follow as a special case of the Erkuş-Srivastava polynomials $\mathcal{U}_{n;\ell_1,\cdots,\ell_r}^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)$ when $\ell_j = j\ \ (j=1,\cdots,r)$.

The generating function (1.7) yields the following *explicit* representation [9, p. 268, Eq. (4)]:

$$\mathcal{U}_{n;\ell_1,\cdots,\ell_r}^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r) = \sum_{\substack{k_1,\cdots,k_r\in\mathbf{N}_0 \\ (\ell_1 k_1+\cdots+\ell_r k_r=n)}} (\alpha_1)_{k_1}\cdots(\alpha_r)_{k_r}\frac{x_1^{k_1}}{k_1!}\cdots\frac{x_r^{k_r}}{k_r!}, \tag{1.8}$$

which, in the special case when $\ell_j = 1\ \ (j=1,\cdots,r)$, corresponds to the *first* expression in (1.4).

Each of the above families of multivariable polynomials has been investigated systematically and extensively in the literature ever since the publication of the work by Chan *et al.* [9] (see, for example, [1], [4], [9], [11], [12] and [13]). The main objective of the present paper is to derive umbral calculus presentations of the Chan-Chyan-Srivastava polynomials $g_n^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)$ generated by (1.3) and also of the substantially more general Erkuş-Srivastava polynomials $\mathcal{U}_{n;\ell_1,\cdots,\ell_r}^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)$ generated by (1.7). Upon suitable specialization, this last umbral calculus presentation is shown to yield the corresponding result for the polynomials $h_n^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)$ generated by (1.6).

2. Umbral Calculus Presentations

The Chan-Chyan-Srivastava polynomials in (1.3) exhibits a structure which, according to the prescription provided in [7], can be viewed as the umbral image of ordinary monomials. Indeed, by using the generating function in (1.3) and an elementary integral identity in the form [2] :

$$\kappa^{-\nu} = \frac{1}{\Gamma(\nu)} \int_0^\infty e^{-\kappa t}\, t^{\nu-1} dt \qquad \Big(\min\{\Re(\kappa), \Re(\nu)\} > 0\Big), \tag{2.1}$$

we can easily derive the following integral representation of the Chan-Chyan-Srivastava polynomials:

$$g_n^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r) = \frac{1}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_r)} \int_0^\infty \cdots \int_0^\infty e^{-(\xi_1+\cdots+\xi_r)}$$

$$\cdot\, \xi_1^{\alpha_1-1} \cdots \xi_r^{\alpha_r-1} \mathcal{P}_n(x_1\xi_1,\cdots,x_r\xi_r)\, d\xi_1 \;\cdots\; d\xi_r, \tag{2.2}$$

where

$$\mathcal{P}_n(x_1,\cdots,x_r) := \frac{(x_1+\cdots+x_r)^n}{n!}$$

$$= \frac{1}{n!} \sum_{\substack{k_1,\cdots,k_r \in \mathbf{N}_0 \\ (k_1+\cdots+k_r=n)}} \begin{pmatrix} n \\ k_1, ..., k_r \end{pmatrix} x_1^{k_1} \cdots x_r^{k_r} \tag{2.3}$$

in terms of the multinomial coefficients given by

$$\begin{pmatrix} n \\ k_1,\cdots,k_r \end{pmatrix} := \frac{n!}{k_1!\cdots k_r!} \qquad (n, k_1,\cdots,k_r \in \mathbf{N}_0).$$

Alternatively, the multinomial theorem (see, for example, [15, p. 87, Problem 5]) used in (2.3) can indeed be restated as follows:

$$\mathcal{P}_n(x_1,\cdots,x_r) = \frac{1}{n!} \sum_{k_1=0}^{n} \sum_{k_2=0}^{n-k_1} \cdots \sum_{k_{r-1}=0}^{n-k_1-\cdots-k_{r-2}} \binom{n}{k_1} \binom{n-k_1}{k_2}$$

$$\cdots \binom{n-k_1-\cdots-k_{r-2}}{k_{r-1}}$$

$$\cdot\, x_1^{n-k_1-\cdots-k_{r-1}}\, x_2^{k_1} \;\cdots\; x_r^{k_{r-1}}. \tag{2.4}$$

It is immediately seen from the integral representation (2.2) that [see Eq. (1.4)]

$$g_n^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r) = \sum_{\substack{k_1,\cdots,k_r\in\mathbf{N}_0 \\ (k_1+\cdots+k_r=n)}} \binom{\alpha_1+k_1-1}{k_1}\cdots\binom{\alpha_r+k_r-1}{k_r} x_1^{k_1}\cdots x_r^{k_r}$$

$$= \sum_{k_1=0}^{n}\sum_{k_2=0}^{n-k_1}\cdots\sum_{k_{r-1}=0}^{n-k_1-\cdots-k_{r-2}} \frac{(\alpha_1)_{n-k_1-\cdots-k_{r-1}}(\alpha_2)_{k_1}\cdots(\alpha_r)_{k_{r-1}}}{(n-k_1-\cdots-k_{r-1})!\,k_1!\cdots k_{r-2}!}$$

$$\cdot\, x_1^{n-k_1-\cdots-k_{r-1}} x_2^{k_1}\cdots x_r^{k_{r-1}}, \tag{2.5}$$

which follows also from the *second* explicit expression in (1.4).

We now define the umbral quantities $\langle\alpha_1 x_1\rangle,\cdots,\langle\alpha_r x_r\rangle$ together with their properties given by (see, for details, [5] ; see also [14])

$$\langle\alpha_j x_j\rangle^n = (\alpha_j)_n\, x_j^n \qquad (j=1,\cdots,r;\; n\in\mathbf{N}_0)$$

and the pairs of operators $\hat{X}_j$ and $\hat{P}_{x_j}$ $(j=1,\cdots,r)$ such that

$$\hat{X}_j\left\{\langle\alpha_j x_j\rangle^n\right\} = \langle\alpha_j x_j\rangle^{n+1} \qquad \text{and} \qquad \hat{P}_{x_j}\left\{\langle\alpha_j x_j\rangle^n\right\} = n\,\langle\alpha_j x_j\rangle^{n-1}$$

$$(j=1,\cdots,r),$$

so that, obviously,

$$\hat{X}_j\hat{P}_{x_j}\left\{\langle\alpha_j x_j\rangle^n\right\} = n\,\langle\alpha_j x_j\rangle^n \quad \text{and} \quad \hat{P}_{x_j}\hat{X}_j\left\{\langle\alpha_j x_j\rangle^n\right\} = (n+1)\,\langle\alpha_j x_j\rangle^n$$

$$(j=1,\cdots,r).$$

We thus find that the Chan-Chyan-Srivastava polynomials $g_n^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)$ satisfy the following operational formula:

$$\left(\hat{X}_1\hat{P}_{x_1}+\cdots+\hat{X}_r\hat{P}_{x_r}\right)\left\{g_n^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)\right\}$$

$$= n g_n^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r), \tag{2.6}$$

which, in the special case when

$$r = 2, \quad \alpha_1 = \alpha, \quad \alpha_2 = \beta, \quad x_1 = x \qquad \text{and} \qquad x_2 = y,$$

would provide the *corrected* version of a result stated by Dattoli *et al.* [6, p. 182, Eq. (10)].

It is also clear that the umbral image of the generating function (1.3) is given by

$$e^{[\langle\alpha_1 x_1\rangle + \cdots + \langle\alpha_r x_r\rangle]z} = \sum_{n=0}^{\infty} g_n^{(\alpha_1,\cdots,\alpha_r)}(x_1, \cdots, x_r) z^n. \tag{2.7}$$

In our next section (Section 3), we will show that the above notion and formalism of *monoumbrality* (that is, *monomiality* together with *umbrality*) can be extended to a large family of polynomials as well as of functions and that this principle of monoumbrality will provide a powerful tool for simplifying calculations (see also [5], [6] and [7]).

3. Applications of the Principle of Monoumbrality

In the preceding section, we have noted that the umbral image of the generating function (1.3) of the Chan-Chyan-Srivastava polynomials $g_n^{(\alpha_1,\cdots,\alpha_r)}(x_1, \cdots, x_r)$ is, in fact, of the exponential type given by (2.7). Here, in this section, we begin by recalling the following easily understandable decomposition rules for the umbral algebra (see [5], [6] and [7]):

$$e^{\langle\alpha_1 x_1\rangle + \cdots + \langle\alpha_r x_r\rangle} = e^{\langle\alpha_1 x_1\rangle} \cdots e^{\langle\alpha_r x_r\rangle}, \tag{3.1}$$

and

$$e^{[\langle\alpha(x_1 + \cdots + x_r)\rangle]} \neq e^{\langle\alpha x_1\rangle} \cdots e^{\langle\alpha x_r\rangle}, \tag{3.2}$$

that is, more explicitly,

$$e^{[\langle\alpha(x_1 + \cdots + x_r)\rangle]} \neq \left(\sum_{k_1=0}^{\infty} (\alpha)_{k_1} \frac{x_1^{k_1}}{k_1!}\right) \cdots \left(\sum_{k_r=0}^{\infty} (\alpha)_{k_r} \frac{x_r^{k_r}}{k_r!}\right). \tag{3.3}$$

We also note that

$$\begin{aligned} e^{\langle\alpha_1 x_1\rangle z} &= e^{\langle\alpha_1 x_1\rangle z + \cdots + \langle\alpha_r x_r\rangle z} \cdot e^{\langle -\alpha_2 x_2\rangle z} \cdots e^{\langle -\alpha_r x_r\rangle z} \\ &= e^{[\langle\alpha_1 x_1\rangle + \cdots + \langle\alpha_r x_r\rangle]z} \cdot e^{[\langle -\alpha_2 x_2\rangle + \cdots + \langle -\alpha_r x_r\rangle]z}. \end{aligned} \tag{3.4}$$

By applying this last umbral relationship (3.4) in conjunction with (2.5) and (2.7), we find that

$$\sum_{n=0}^{\infty} (\alpha_1)_n \frac{(x_1 z)^n}{n!} = \left(\sum_{n=0}^{\infty} g_n^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r) z^n\right) \left(\sum_{n=0}^{\infty} g_n^{(-\alpha_2,\cdots,-\alpha_r)}(x_2,\cdots,x_r) z^n\right)$$

$$= \sum_{n=0}^{\infty} z^n \sum_{k=0}^{n} g_k^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r) g_{n-k}^{(-\alpha_2,\cdots,-\alpha_r)}(x_2,\cdots,x_r), \tag{3.5}$$

so that, upon equating the coefficients of z^n from both sides in (3.5), we finally obtain

$$x_1^n = \frac{1}{(\alpha_1)_n} \sum_{k_1=0}^{n} \sum_{k_2=0}^{n-k_1} \cdots \sum_{k_{r-1}=0}^{n-k_1-\cdots-k_{r-2}} k_1! \binom{n}{k_1} \binom{n-k_1}{k_2} \cdots \binom{n-k_1-\cdots-k_{r-2}}{k_{r-1}}$$
$$\cdot g_{k_1}^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r) \langle -\alpha_2 x_2 \rangle^{n-k_1-\cdots-k_{r-1}} \langle -\alpha_3 x_3 \rangle^{k_2} \cdots \langle -\alpha_r x_r \rangle^{k_{r-1}}, \tag{3.6}$$

where, just as in the preceding section,

$$\langle -\alpha_2 x_2 \rangle^{n-k_1-\cdots-k_{r-1}} = \frac{\Gamma(-\alpha_2 + n - k_1 - \cdots - k_{r-1})}{\Gamma(-\alpha_2)} x_2^{n-k_1-\cdots-k_{r-1}}$$

$$= (-\alpha_2)_{n-k_1-\cdots-k_{r-1}} x_2^{n-k_1-\cdots-k_{r-1}} \tag{3.7}$$

and

$$\langle -\alpha_j x_j \rangle^{k_{j-1}} = \frac{\Gamma(-\alpha_j + k_{j-1})}{\Gamma(-\alpha_j)} x_j^{k_{j-1}} = (-\alpha_j)_{k_{j-1}} x_j^{k_{j-1}} \qquad (j = 3,\cdots,r). \tag{3.8}$$

Formula (3.6) provides the expansion of an ordinary monomial x_1^n in terms of the Chan-Chyan-Srivastava polynomials

$$g_n^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)$$

involving such umbral quantities as those specified in (3.7) and (3.8).

Next, for Erkuş-Srivastava polynomials $\mathcal{U}_{n;\ell_1,\cdots,\ell_r}^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)$ generated by (1.7), it is easily seen that

$$\mathcal{U}_{n;\ell_1,\cdots,\ell_r}^{(\alpha_1+\beta_1,\cdots,\alpha_r+\beta_r)}(x_1,\cdots,x_r)$$

$$= \sum_{k=0}^{n} \mathcal{U}_{n-k;\ell_1,\cdots,\ell_r}^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)\, \mathcal{U}_{k;\ell_1,\cdots,\ell_r}^{(\beta_1,\cdots,\beta_r)}(x_1,\cdots,x_r), \tag{3.9}$$

which, for $\ell_j = 1 \ (j = 1,\cdots,r)$, was given by Chan *et al.* [3, p. 147, Eq. (35)]. Moreover, the generating function (1.7) together with the integral formula (2.1) would yield the following analogue of the integral representation (2.2):

$$\mathcal{U}_{n;\ell_1,\cdots,\ell_r}^{(\alpha_1+\beta_1,\cdots,\alpha_r+\beta_r)}(x_1,\cdots,x_r) = \frac{1}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_r)} \int_0^\infty \cdots \int_0^\infty e^{-(\xi_1+\cdots+\xi_r)}$$

$$\cdot\, \xi_1^{\alpha_1-1}\cdots\xi_r^{\alpha_r-1}\mathcal{Q}_n(x_1\xi_1,\cdots,x_r\xi_r)\, d\xi_1 \ \cdots\ d\xi_r, \tag{3.10}$$

where the polynomials $\mathcal{Q}_n(x_1,\cdots,x_r)$ are essentially the same as the multivariable Hermite-Kampé de Fériet polynomials given by

$$\mathcal{Q}_n(x_1,\cdots,x_r) := \frac{1}{n!} \sum_{\substack{k_1,\cdots,k_r\in\mathbf{N}_0 \\ (\ell_1k_1+\cdots+\ell_rk_r=n)}} \binom{n}{k_1,\cdots,k_r} x_1^{k_1}\cdots x_r^{k_r} \tag{3.11}$$

in terms of the multinomial coefficients involved also in (2.3).

If we now make use of the principle of monoumbrality as detailed above, we can show similarly that

$$x_1^n = \frac{n!}{(\alpha_1)_n} \sum_{k_1=0}^{\ell_1 n} \sum_{\substack{k_2,\cdots,k_r\in\mathbf{N}_0 \\ (\ell_2k_2+\cdots+\ell_rk_r=\ell_1n-k_1)}} \mathcal{U}_{k_1;\ell_1,\cdots,\ell_r}^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)$$

$$\cdot\, \frac{\langle -\alpha_2x_2\rangle^{k_2}}{k_2!}\cdots\frac{\langle -\alpha_rx_r\rangle^{k_r}}{k_r!}, \tag{3.12}$$

which, in the special case when $\ell_j = 1 \quad (j = 1, \cdots, r)$, the monoumbral expansion given by (3.6). Moreover, if we simply assume that $\ell_r = 1$, we can easily rewrite the monoumbral expansion (3.12) as follows:

$$x_1^n = \frac{n!}{(\alpha_1)_n} \sum_{k_1=0}^{\ell_1 n} \sum_{k_2=0}^{\left[\frac{\ell_1 n - k_1}{\ell_2}\right]} \cdots \sum_{k_{r-1}=0}^{\left[\frac{\ell_1 n - k_1 - \ell_2 k_2 - \cdots - \ell_{r-2} k_{r-2}}{\ell_{r-1}}\right]} \cdot \mathcal{U}_{k_1;\ell_1,\cdots,\ell_{r-1},1}^{(\alpha_1,\cdots,\alpha_r)}(x_1, \cdots, x_r)$$

$$\cdot \frac{\langle -\alpha_2 x_2 \rangle^{\ell_1 n - k_1 - \ell_2 k_2 - \cdots - \ell_{r-1} k_{r-1}}}{(\ell_1 n - k_1 - \ell_2 k_2 - \cdots - \ell_{r-1} k_{r-1})!} \frac{\langle -\alpha_3 x_3 \rangle^{k_2}}{k_2!} \cdots \frac{\langle -\alpha_r x_r \rangle^{k_{r-1}}}{k_{r-1}!} \tag{3.13}$$

or, equivalently,

$$x_1^n = \frac{n!}{(\alpha_1)_n} \sum_{k_1=0}^{\ell_1 n} \sum_{k_2=0}^{\left[\frac{\ell_1 n - k_1}{\ell_2}\right]} \cdots \sum_{k_{r-1}=0}^{\left[\frac{\ell_1 n - k_1 - \ell_2 k_2 - \cdots - \ell_{r-2} k_{r-2}}{\ell_{r-1}}\right]} \cdot \mathcal{U}_{k_1;\ell_1,\cdots,\ell_{r-1},1}^{(\alpha_1,\cdots,\alpha_r)}(x_1, \cdots, x_r)$$

$$\cdot \frac{k_1!(\ell_2 k_2)! \cdots (\ell_{r-1} k_{r-1})!}{k_2! \cdots k_{r-1}!} \binom{\ell_1 n}{k_1} \binom{\ell_1 n - k_1}{\ell_2 k_2} \cdots \binom{\ell_1 n - k_1 - \ell_2 k_2 - \cdots - \ell_{r-2} k_{r-2}}{\ell_{r-1} k_{r-1}}$$

$$\cdot \langle -\alpha_2 x_2 \rangle^{\ell_1 n - k_1 - \ell_2 k_2 - \cdots - \ell_{r-1} k_{r-1}} \langle -\alpha_3 x_3 \rangle^{k_2} \cdots \langle -\alpha_r x_r \rangle^{k_{r-1}} \tag{3.14}$$

in terms of the notations and conventions given by (3.7) and (3.8). Indeed, when we *further* set $\ell_j = 1 \quad (j = 1, \cdots, r-1)$, this last result (3.14) would correspond *precisely* to the monoumbral expansion (3.6).

Finally, upon setting $\ell_j = j \quad (j = 1, \cdots, r)$ in our general result (3.12), we are led at once to the following monoumbral expansion for the multivariable Lagrange-Hermite polynomials $h_n^{(\alpha_1,\cdots,\alpha_r)}(x_1, \cdots, x_r)$ in (1.6), which were studied by (for example) Altın and Erkuş [1] :

$$x_1^n = \frac{n!}{(\alpha_1)_n} \sum_{k_1=0}^{n} \sum_{\substack{k_2,\cdots,k_r \in \mathbf{N}_0 \\ (2k_2+\cdots+rk_r=n-k_1)}} h_{k_1}^{(\alpha_1,\cdots,\alpha_r)}(x_1, \cdots, x_r)$$

$$(3.15) \qquad \cdot \frac{\langle -\alpha_2 x_2 \rangle^{k_2}}{k_2!} \cdots \frac{\langle -\alpha_r x_r \rangle^{k_r}}{k_r!}.$$

4. Concluding Remarks and Observations

In this work, we have investigated the umbral calculus presentations of the Chan-Chyan-Srivastava polynomials $g_n^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)$ generated by (1.3) and also of the substantially more general Erkuş-Srivastava polynomials $\mathcal{U}_{n;\ell_1,\cdots,\ell_r}^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)$ generated by (1.7). One of our main monoumbral expansions asserted by (3.12) has been shown to yield the corresponding monoumbral expansion (3.15) for the multivariable Lagrange-Hermite polynomials $h_n^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)$ in (1.6), which were studied by (for example) Altın and Erkuş [1].

We need hardly emphasize upon the fact that the notion and formalism of monoumbrality can be extended to a large family of polynomials as well as of functions and that the underlying principle of monoumbrality would provide a powerful tool for simplifying calculations.

We conclude our present investigation by remarking further that, in the special case when $\ell_1 = 1$, we find from (1.8) that

$$\mathcal{U}_{n;1,\ell_2,\cdots,\ell_r}^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r) = \sum_{\substack{k_1,\cdots,k_r \in \mathbf{N}_0 \\ (k_1+\ell_2 k_2+\cdots+\ell_r k_r = n)}} \frac{\langle \alpha_1 x_1 \rangle^{k_1}}{k_1!} \cdots \frac{\langle \alpha_r x_r \rangle^{k_r}}{k_r!}$$

$$= \sum_{k_2,\ldots,k_r=0}^{l_2k_2+\ldots+l_rk_r \leqq n} \frac{\langle \alpha_1 x_1 \rangle^{n-\ell_2 k_2-\cdots-\ell_r k_r}}{(n-\ell_2 k_2-\cdots-\ell_r k_r)!} \frac{\langle \alpha_2 x_2 \rangle^{k_2}}{k_2!} \cdots \frac{\langle \alpha_r x_r \rangle^{k_r}}{k_r!}$$

$$= \sum_{k_1=0}^{\left[\frac{n}{\ell_2}\right]} \sum_{k_2=0}^{\left[\frac{n-\ell_2 k_1}{\ell_3}\right]} \cdots \sum_{k_{r-1}=0}^{\left[\frac{n-\ell_2 k_1-\cdots-\ell_{r-1}k_{r-2}}{\ell_r}\right]}$$

$$(4.1) \qquad \cdot \frac{\langle \alpha_1 x_1 \rangle^{n-\ell_2 k_1-\cdots-\ell_r k_{r-1}}}{(n-\ell_2 k_1-\cdots-\ell_r k_{r-1})!} \frac{\langle \alpha_2 x_2 \rangle^{k_1}}{k_1!} \cdots \frac{\langle \alpha_r x_r \rangle^{k_{r-1}}}{k_{r-1}!},$$

which readily yields the following generalization of the operational formula (2.6):

$$\left(\hat{X}_1 \hat{P}_{x_1} + \ell_2 \hat{X}_2 \hat{P}_{x_2} + \cdots + \ell_r \hat{X}_r \hat{P}_{x_r}\right) \left\{\mathcal{U}_{n;1,\ell_2,\cdots,\ell_r}^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)\right\}$$

$$= n\mathcal{U}_{n;1,\ell_2,\cdots,\ell_r}^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r). \tag{4.2}$$

In particular, if we set $\ell_j = j \ \ (j = 2, \cdots, r)$ in this last result (4.2), we immediately obtain

$$\left(\hat{X}_1\hat{P}_{x_1} + 2\hat{X}_2\hat{P}_{x_2} + \cdots + r\hat{X}_r\hat{P}_{x_r}\right)\left\{h_n^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)\right\}$$
$$= nh_n^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r) \tag{4.3}$$

for the multivariable Lagrange-Hermite polynomials $h_n^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)$ in (1.6), which were studied by (for example) Altın and Erkuş [1]. Indeed, in its *further* special case when

$$r = 2, \quad \alpha_1 = \alpha, \quad \alpha_2 = \beta, \quad x_1 = x \qquad \text{and} \qquad x_2 = y,$$

the operational formula (4.3) would provide the *corrected* version of another result stated by Dattoli *et al.* [6, p. 184, Eq. (22)].

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