HIGHER-ORDER DAEHEE NUMBERS AND POLYNOMIALS

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Abstract. Recently, Daehee numbers and polynomials are introduced by the authors. In this paper, we consider the Daehee numbers and polynomials of order $k\,(\in\mathbb{N})$ and give some relation between Daehee polynomials of order $k(\in\mathbb{N})$ and special polynomials.

1. Introduction

For $\alpha\in\mathbb{N}$, as is well known, the Bernoulli polynomials of order α are defined by the generating function to be

$$\left(\frac{t}{e^t-1}\right)^{\alpha}e^{xt}=\sum_{n=0}^{\infty}B_n^{(\alpha)}(x)\frac{t^n}{n!},\tag{1}$$

(see [1-14]).

When $x=0$, $B_n^{(\alpha)}=B_n^{(\alpha)}(0)$ are the Bernoulli numbers of order α. In [?, ?, ?], the Daehee polynomials are defined by the generating function to be

$$\left(\frac{\log(1+t)}{t}\right)(1+t)^x=\sum_{n=0}^{\infty}D_n(x)\frac{t^n}{n!}.\tag{2}$$

When $x=0$, $D_n=D_n(0)$ are called the Daehee numbers.

Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic numbers and the completion of algebraic closure of $\mathbb{Q}_p$. The p-adic norm $|\cdot|_p$ is normalized as $|p|_p=\frac{1}{p}$. Let $\mathrm{UD}(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f\in\mathrm{UD}(\mathbb{Z}_p)$, the p-adic invariant integral on $\mathbb{Z}_p$ is defined by

$$I(f)=\int_{\mathbb{Z}_p}f(x)\,d\mu(x)=\lim_{n\to\infty}\frac{1}{p^n}\sum_{x=0}^{p^n-1}f(x),\tag{3}$$

(see [?]).

Let $f_1(x)=f(x+1)$. Then, by (3), we get

$$I(f_1)-I(f)=f'(0),\ \text{where } f'(0)=\left.\frac{df(x)}{dx}\right|_{x=0}.\tag{4}$$

The signed Stirling numbers of the first kind $S_1(n,l)$ are defined by

$$\begin{aligned}(x)_n&=x(x-1)\cdots(x-n+1)\\&=\sum_{l=0}^{\infty}S_1(n,l)\,x^l,\end{aligned}\tag{5}$$

(see [?, ?, ?]).

From (5), we note that

$$\begin{aligned}x^{(n)}&=x(x+1)\cdots(x+n-1)=(-1)^n(-x)_n\\&=\sum_{l=0}^{n}(-1)^{n-l}S_1(n,l)\,x^l,\end{aligned}$$

(see [?, ?, ?]).

The Stirling numbers of the second kind $S_2(l,n)$ are defined by the generating function to be

$$(6)\qquad \left(e^t-1\right)^n = n!\sum_{l=n}^{\infty} S_2\left(l,n\right)\frac{t^l}{l!} = \sum_{l=0}^{\infty}\frac{n!}{(l+n)!}S_2\left(l+n,n\right)t^{l+n}.$$

In this paper, we study the higher-order Daehee numbers and polynomials and give some relations between Daehee polynomials and special polynomials.

2. Higher-order Daehee polynomials

In this section, we assume that $t\in\mathbb{C}_p$ with $|t|_p < p^{\frac{-1}{p-1}}$.

For $k\in\mathbb{N}$, let us consider the Daehee numbers of the first kind of order k :

$$(7)\qquad D_n^{(k)} = \underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{k-\text{times}}\left(x_1+x_2+\cdots+x_k\right)_n d\mu\left(x_1\right)\cdots d\mu\left(x_k\right),$$

where $n\in\mathbb{Z}_{\geq 0}$.

From (7), we can derive the generating function of $D_n^{(k)}$ as follows :

$$(8)\qquad \sum_{n=0}^{\infty} D_n^{(k)}\frac{t^n}{n!} = \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\sum_{n=0}^{\infty}\binom{x_1+\cdots+x_k}{n}t^n d\mu\left(x_1\right)\cdots d\mu\left(x_k\right) = \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left(1+t\right)^{x_1+\cdots+x_k} d\mu\left(x_1\right)\cdots d\mu\left(x_k\right).$$

By (4), we easily see that

$$(9)\qquad \int_{\mathbb{Z}_p}\left(1+t\right)^x d\mu\left(x\right) = \frac{\log\left(1+t\right)}{t}.$$

Thus, by (8) and (9), we get

$$(10)\qquad \sum_{n=0}^{\infty} D_n^{(k)}\frac{t^n}{n!} = \left(\frac{\log\left(1+t\right)}{t}\right)^k.$$

Now, we observe that

$$(11)\qquad \left(\frac{\log\left(1+t\right)}{t}\right)^k = \frac{k!}{t^k}\sum_{l=k}^{\infty} S_1\left(t,k\right)\frac{t^l}{l!} = \sum_{n=0}^{\infty} S_1\left(n+k,k\right)\frac{k!}{(n+k)!}t^n = \sum_{n=0}^{\infty}\frac{S_1\left(n+k,k\right)}{\binom{n+k}{k}}\frac{t^n}{n!}.$$

Therefore, by (10) and (11), we obtain the following theorem.

Theorem 1. *For $n \in \mathbb{Z}_{\geq 0}$, $k \in \mathbb{N}$, we have*

$$D_n^{(k)} = \frac{S_1(n+k,k)}{\binom{n+k}{k}}.$$

It is easy to show that

$$\left(\frac{\log(1+t)}{t}\right)^k = \sum_{n=0}^{\infty} B_n^{(n+k+1)}(1)\frac{t^n}{n!}. \tag{12}$$

Threfore, we obtain the following corollary.

Corollary 2. *For $n \in \mathbb{Z}_{\geq 0}$, $k \in \mathbb{N}$, we have*

$$D_n^{(k)} = \frac{S_1(n+k,k)}{\binom{n+k}{k}} = B_n^{(n+k+1)}(1).$$

From (7), we note that

$$\begin{aligned} D_n^{(k)} &= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+\cdots+x_k)_n\,d\mu(x_1)\cdots d\mu(x_k) \\ &= \sum_{l=0}^{n} S_1(n,l)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+\cdots+x_k)^l\,d\mu(x_1)\cdots d\mu(x_k) \\ &= \sum_{l=0}^{n} S_1(n,l)\,B_l^{(k)}. \end{aligned} \tag{13}$$

Therefore, by (13), we obtain the following theorem.

Theorem 3. *For $n \in \mathbb{Z}_{\geq 0}$, $k \in \mathbb{N}$, we have*

$$\begin{aligned} D_n^{(k)} &= \sum_{l_1+\cdots+l_k=n}\binom{n}{l_1,\cdots,l_k}D_{l_1}\cdots D_{l_k} \\ &= \sum_{l=0}^{n} S_1(n,l)\,B_l^{(k)}. \end{aligned}$$

From (10), we can derive

$$\sum_{n=0}^{\infty} D_n^{(k)}\frac{(e^t-1)^n}{n!} = \left(\frac{t}{e^t-1}\right)^k = \sum_{n=0}^{\infty} B_n^{(k)}\frac{t^n}{n!}, \tag{14}$$

and

$$\sum_{n=0}^{\infty} D_n^{(k)}\frac{(e^t-1)^n}{n!} = \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m} D_n^{(k)}S_2(n,m)\right)\frac{t^n}{m!}. \tag{15}$$

Therefore, by (14) and (15), we obtain the following theorem.

Theorem 4. *For $m \in \mathbb{Z}_{\geq 0}$, $k \in \mathbb{N}$, we have*

$$B_m^{(k)} = \sum_{n=0}^{m} D_n^{(k)}S_2(m,n).$$

Now, we consider the higher-order Daehee polynomials as follows :

$$D_n^{(k)}(x) = \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+\cdots+x_k+x)_n\,d\mu(x_1)\cdots d\mu(x_k). \tag{16}$$

Thus, by (16), we get

$$D_n^{(k)}(x) = \sum_{l=0}^{n} S_1(n,l) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_k + x)^l \, d\mu(x_1) \cdots d\mu(x_k) = \sum_{l=0}^{n} S_1(n,l) B_l^{(k)}(x). \tag{17}$$

Therefore, by (17), we obtain the following theorem.

Theorem 5. *For $n \in \mathbb{Z}_{\geq 0}$, $k \in \mathbb{N}$, we have*

$$D_n^{(k)}(x) = \sum_{l=0}^{n} S_1(n,l) B_l^{(k)}(x).$$

From (16), we derive the generating function of $D_n^{(k)}(x)$:

$$\begin{aligned} \sum_{n=0}^{\infty} D_n^{(k)}(x) \frac{t^n}{n!} &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} \binom{x_1 + \cdots + x_k + x}{n} t^n d\mu(x_1) \cdots d\mu(x_k) \\ &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+t)^{x_1 + \cdots + x_k + x} \, d\mu(x_1) \cdots d\mu(x_k) \\ &= \left(\frac{\log(1+t)}{t} \right)^k (1+t)^x. \end{aligned} \tag{18}$$

It is easy to show that

$$\left(\frac{\log(1+t)}{t} \right)^k (1+t)^x = \sum_{n=0}^{\infty} B_n^{(n+k+1)}(x+1) \frac{t^n}{n!}. \tag{19}$$

Therefore, by (18) and (19), we obtain the following theorem.

Theorem 6. *For $n \in \mathbb{Z}_{\geq 0}$, $k \in \mathbb{N}$,*

$$\begin{aligned} D_n^{(k)}(x) &= B_n^{(n+k+1)}(x+1) \\ &= \sum_{l=0}^{n} \binom{n}{l} B_l^{(n+k+1)}(x+1)^{n-l}. \end{aligned}$$

In (18), we note that

$$\sum_{n=0}^{\infty} D_n^{(k)}(x) \frac{(e^t - 1)^n}{n!} = \sum_{m=0}^{\infty} \left(\sum_{n=0}^{m} S_2(n,m) D_n^{(k)}(x) \right) \frac{t^m}{m!} \tag{20}$$

and

$$\begin{aligned} \sum_{n=0}^{\infty} D_n^{(k)}(x) \frac{(e^t - 1)^n}{n!} &= \left(\frac{t}{e^t - 1} \right)^k e^{xt} \\ &= \sum_{m=0}^{\infty} B_m^{(k)}(x) \frac{t^m}{m!}. \end{aligned} \tag{21}$$

Therefore, by (20) and (21), we obtain the following theorem.

Theorem 7. *For $m \in \mathbb{Z}_{\geq 0}$, $k \in \mathbb{N}$, we have*

$$B_m^{(k)}(x) = \sum_{n=0}^{m} S_2(m,n) D_n^{(k)}(x).$$

Now, we define Daehee numbers of the second kind of order $k(\in \mathbb{N})$:

$$\begin{aligned} \widehat{D}_n^{(k)} &= (-1)^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (-x_1 - x_2 - \cdots - x_k)_n \, d\mu(x_1) \cdots d\mu(x_k) \\ &= (-1)^n \sum_{l=0}^{n} (-1)^{n-l} S_1(n,l) B_l^{(k)} = \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix} B_l^{(k)}, \end{aligned} \tag{22}$$

where $\begin{bmatrix} n \\ l \end{bmatrix} = (-1)^{n-l} S_1(n,l)$.

Thus, by (22), we get

$$\begin{aligned} \widehat{D}_n^{(k)} &= (-1)^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (-x_1 - x_2 - \cdots - x_k)_n \, d\mu(x_1) \cdots d\mu(x_k) \\ &= (-1)^n \sum_{l=0}^{n} S_1(n,l)(-1)^l \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + x_2 + \cdots + x_k)^l \, d\mu(x_1) \cdots d\mu(x_k) \\ &= \sum_{l=0}^{n} (-1)^{n-l} S_1(n,l) B_l^{(k)} = \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix} B_l^{(k)}, \end{aligned} \tag{23}$$

where $\begin{bmatrix} n \\ l \end{bmatrix} = (-1)^{n-l} S_1(n,l)$.

Therefore, by (23), we obtain the following theorem.

Theorem 8. *For $n \in \mathbb{Z}_{\geq 0}$, $k \in \mathbb{N}$, we have*

$$\widehat{D}_n^{(k)} = \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix} B_l^{(k)}.$$

From (22), we derive the generating function of $\widehat{D}_n^{(k)}$:

$$\begin{aligned} \sum_{n=0}^{\infty} \widehat{D}_n^{(k)} \frac{t^n}{n!} &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} \binom{x_1 + \cdots + x_k + n - 1}{n} t^n d\mu(x_1) \cdots d\mu(x_k) \\ &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1-t)^{-x_1 - \cdots - x_k} \, d\mu(x_1) \cdots d\mu(x_k) \\ &= \left(\frac{(1-t)\log(1-t)}{-t} \right)^k. \end{aligned} \tag{24}$$

By (24), we get

$$\begin{aligned} \sum_{n=0}^{\infty} \widehat{D}_n^{(k)} \frac{(1-e^{-t})^n}{n!} &= \left(\frac{e^{-t}(-t)}{e^{-t}-1} \right)^k = \left(\frac{t}{e^t - 1} \right)^k \\ &= \sum_{m=0}^{\infty} B_m^{(k)} \frac{t^m}{m!}, \end{aligned} \tag{25}$$

and

$$\sum_{n=0}^{\infty} \widehat{D}_n^{(k)} \frac{\left(1-e^{-t}\right)^n}{n!} = \sum_{m=0}^{\infty} \left(\sum_{n=0}^{m} \widehat{D}_n^{(k)} (-1)^{m-n} S_2(m,n) \right) \frac{t^m}{m!}. \tag{26}$$

Thererfore, by (25) and (26), we obtain the following theorem.

Theorem 9. *For $m \in \mathbb{Z}_{\geq 0}$, $k \in \mathbb{N}$, we have*

$$B_m^{(k)} = \sum_{n=0}^{m} \widehat{D}_n^{(k)} (-1)^{n-m} S_2(m,n).$$

Now, we consider the higher-order Daehee polynomials of the second kind :

$$\widehat{D}_n^{(k)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left(x_1 + x_2 + \cdots + x_k - x\right)^{(n)} d\mu(x_1) \cdots d\mu(x_k). \tag{27}$$

Thus, by (27), we get

(28)

$$\begin{aligned}
&\widehat{D}_n^{(k)}(x) \\
=&(-1)^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left(-x_1 - x_2 - \cdots - x_k + x\right)_n d\mu(x_1) \cdots d\mu(x_k) \\
=&(-1)^n \sum_{l=0}^{n} S_1(n,l) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left(-x_1 - x_2 - \cdots - x_k + x\right)^l d\mu(x_1) \cdots d\mu(x_k) \\
=&(-1)^n \sum_{l=0}^{n} S_1(n,l) \sum_{m=0}^{l} \binom{l}{m} x^{l-m} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left(-x_1 - x_2 - \cdots - x_k\right)^m d\mu(x_1) \cdots d\mu(x_k) \\
=&(-1)^n \sum_{l=0}^{n} S_1(n,l) \sum_{m=0}^{l} \binom{l}{m} (-1)^m x^{l-m} B_m^{(k)} \\
=&\sum_{l=0}^{n} (-1)^{n-l} S_1(n,l) B_l^{(k)}(-x).
\end{aligned}$$

Thus, by (28), we get

$$\widehat{D}_n^{(k)}(x) = \sum_{l=0}^{n} (-1)^{n-l} S_1(n,l) B_l^{(k)}(-x). \tag{29}$$

Let us consider the generating function of $D_n^{(k)}(x)$ as follows :

$$\begin{aligned}
&\sum_{n=0}^{\infty} \widehat{D}_n^{(k)}(x) \frac{t^n}{n!} \\
&= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} \binom{x_1 + \cdots + x_k - x + n - 1}{n} t^n d\mu(x_1) \cdots d\mu(x_k) \\
&= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1-t)^{-x_1 - \cdots - x_k + x} d\mu(x_1) \cdots d\mu(x_k) \\
&= \left(\frac{(1-t)\log(1-t)}{-t} \right)^k (1-t)^x.
\end{aligned} \tag{30}$$

From (30), we have

$$\begin{aligned}(31)\qquad &\sum_{n=0}^{\infty}\widehat{D}_n^{(k)}(x)(-1)^n\frac{t^n}{n!}\\ &=\left(\frac{\log(1+t)}{t}\right)^k(1+t)^{x+k}\\ &=\sum_{n=0}^{\infty}B_n^{(n+k+1)}(x+k+1)\frac{t^n}{n!}.\end{aligned}$$

Therefore, by (31), we obtain the following theorem.

Theorem 10. *For $n\in\mathbb{Z}_{\geq 0}$, $k\in\mathbb{N}$, we have*

$$(-1)^n\widehat{D}_n^{(k)}(x)=B_n^{(n+k+1)}(x+k+1).$$

By (30), we get

$$\begin{aligned}(32)\qquad \sum_{n=0}^{\infty}\widehat{D}_n^{(k)}(x)\frac{(1-e^{-t})^n}{n!} &= e^{-tx}\left(\frac{t}{e^t-1}\right)^k\\ &=\sum_{m=0}^{\infty}B_m^{(k)}(-x)\frac{t^m}{m!},\end{aligned}$$

and

$$\begin{aligned}(33)\qquad &\sum_{n=0}^{\infty}\widehat{D}_n^{(k)}(x)\frac{1}{n!}\left(1-e^{-t}\right)^n\\ &=\sum_{m=0}^{\infty}\left(\sum_{n=0}^{m}\widehat{D}_n^{(k)}(x)(-1)^{m-n}S_2(m,n)\right)\frac{t^m}{m!}.\end{aligned}$$

Therefore, by (32) and (12), we obtain the following theorem.

Theorem 11. *For $m\in\mathbb{Z}_{\geq 0}$, $k\in\mathbb{N}$, we have*

$$B_m^{(k)}(-x)=\sum_{n=0}^{m}\widehat{D}_n^{(k)}(x)(-1)^{m-n}S_2(m,n).$$

Now, we observe that

$$\begin{aligned}(34)\qquad &(-1)^n\frac{D_n^{(k)}(x)}{n!}\\ &=(-1)^n\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\binom{x_1+\cdots+x_k+x}{n}d\mu(x_1)\cdots d\mu(x_k)\\ &=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\binom{-(x_1+\cdots+x_k)-x+n-1}{n}d\mu(x_1)\cdots d\mu(x_k)\\ &=\sum_{m=0}^{n}\binom{n-1}{n-m}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\binom{-(x_1+\cdots+x_k)-x}{m}d\mu(x_1)\cdots d\mu(x_k)\\ &=\sum_{m=0}^{n}\frac{\binom{n-1}{n-m}}{m!}m!\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\binom{-(x_1+\cdots+x_k)-x}{m}d\mu(x_1)\cdots d\mu(x_k)\\ &=\sum_{m=1}^{n}\frac{\binom{n-1}{n-m}}{m!}(-1)^m\widehat{D}_m^{(k)}(-x).\end{aligned}$$

Therefore, by (34), we obtain the following theorem.

Theorem 12. *For $n \in \mathbb{Z}_{\geq 0}$, $k \in \mathbb{N}$, we have*

$$(-1)^n \frac{D_n^{(k)}(x)}{n!} = \sum_{m=1}^{n} \frac{\binom{n-1}{n-m}}{m!} (-1)^m \widehat{D}_m^{(k)}(-x).$$

By the same method as Theorem 12, we get

$$\begin{aligned}
\frac{\widehat{D}_n^{(k)}(x)}{n!} \\
&= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \binom{x_1 + \cdots + x_k - x + n - 1}{n} d\mu(x_1) \cdots d\mu(x_k) \\
&= \sum_{m=0}^{n} \binom{n-1}{n-m} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \binom{x_1 + \cdots + x_k - x}{m} d\mu(x_1) \cdots d\mu(x_k) \\
&= \sum_{m=0}^{n} \frac{\binom{n-1}{n-m}}{m!} m! \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \binom{x_1 + \cdots + x_k - x}{m} d\mu(x_1) \cdots d\mu(x_k) \\
&= \sum_{m=1}^{n} \frac{\binom{n-1}{n-m}}{m!} D_m^{(k)}(-x).
\end{aligned} \tag{35}$$

Thus, by (35), we get

$$\frac{\widehat{D}_n^{(k)}(x)}{n!} = \sum_{m=1}^{n} \frac{\binom{n-1}{n-m}}{m!} D_m^{(k)}(-x). \tag{36}$$

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