# DAEHEE NUMBERS AND POLYNOMIALS

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Abstract. We consider the Witt-type formula for Daehee numbrers and polynomials and investigate some properties of those numbers and polynomials. In particular, Daehee numbers are closely related to higher-order Bernoulli numbers and Bernoulli numbers of the second kind.

## 1. Introduction

As is known, the $n$-th Daehee polynomials are defined by the generating function to be

$$\left(\frac{\log(1+t)}{t}\right)(1+t)^x=\sum_{n=0}^{\infty}D_n(x)\frac{t^n}{n!},\ (\text{see [5,6,8,9,10,11]}). \tag{1.1}$$

In the special case, $x=0$, $D_n=D_n(0)$ are called the Daehee numbers.

Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the rings of $p$-adic integers, the fields of $p$-adic numbers and the completion of algebraic closure of $\mathbb{Q}_p$. The $p-$adic norm $|\cdot|_p$ is normalized by $|p|_p={}^1\!/p$. Let $\mathrm{UD}(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f\in \mathrm{UD}(\mathbb{Z}_p)$, the $p$-adic invariant integral on $\mathbb{Z}_p$ is defined by

$$I(f)=\int_{\mathbb{Z}_p}f(x)\,d\mu_0(x)=\lim_{n\to\infty}\frac{1}{p^n}\sum_{x=0}^{p^n-1}f(x),\ (\text{see [6]}). \tag{1.2}$$

Let $f_1$ be the translation of $f$ with $f_1(x)=f(x+1)$. Then, by (1.2), we get

$$I(f_1)=I(f)+f'(0),\ \text{where } f'(0)=\left.\frac{df(x)}{dx}\right|_{x=0}. \tag{1.3}$$

As is known, the Stirling number of the first kind is defined by

$$(x)_n=x(x-1)\cdots(x-n+1)=\sum_{l=0}^{n}S_1(n,l)x^l, \tag{1.4}$$

and the Stirling number of the second kind is given by the generating function to be

$$\left(e^t-1\right)^m=m!\sum_{l=m}^{\infty}S_2(l,m)\frac{t^l}{l!},\ (\text{see [2,3,4]}). \tag{1.5}$$

For $\alpha\in\mathbb{Z}$, the Bernoulli polynomials of order $\alpha$ are defined by the generating function to be

$$\left(\frac{t}{e^t-1}\right)^{\alpha}e^{xt}=\sum_{n=0}^{\infty}B_n^{(\alpha)}(x)\frac{t^n}{n!},\ (\text{see [1,2,8]}). \tag{1.6}$$

When $x=0$, $B_n^{(\alpha)}=B_n^{(\alpha)}(0)$ are called the Bernoulli numbers of order $\alpha$.

In this paper, we give a $p$-adic integral representation of Daehee numbers and polynomials, which are called the Witt-type formula for Daehee numbers and polynomials. From our integral representation, we can derive some interesting properties related to Daehee numbers and polynomials.

## 2. Witt-type formula for Daehee numbers and polynomials

First, we consider the following integral representation associated with falling factorial sequences :

$$\int_{\mathbb{Z}_p} (x)_n \, d\mu_0(x), \quad \text{where } n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}. \tag{2.1}$$

By (2.1), we get

$$\begin{aligned} \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} (x)_n \, d\mu_0(x) \frac{t^n}{n!} &= \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} \binom{x}{n} t^n d\mu_0(x) \\ &= \int_{\mathbb{Z}_p} (1+t)^x \, d\mu_0(x), \end{aligned} \tag{2.2}$$

where $t \in \mathbb{C}_p$ with $|t|_p < p^{-\frac{1}{p-1}}$.

For $t \in \mathbb{C}_p$ with $|t|_p < p^{-\frac{1}{p-1}}$, let us take $f(x) = (1+t)^x$. Then, from (1.3), we have

$$\int_{\mathbb{Z}_p} (1+t)^x \, d\mu_0(x) = \frac{\log(1+t)}{t}. \tag{2.3}$$

By (1.1) and (2.3), we see that

$$\begin{aligned} \sum_{n=0}^{\infty} D_n \frac{t^n}{n!} &= \frac{\log(1+t)}{t} \\ &= \int_{\mathbb{Z}_p} (1+t)^x \, d\mu_0(x) \\ &= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} (x)_n \, d\mu_0(x) \frac{t^n}{n!}. \end{aligned} \tag{2.4}$$

Therefore, by (2.4), we obtain the following theorem.

**Theorem 1.** *For $n \geq 0$, we have*

$$\int_{\mathbb{Z}_p} (x)_n \, d\mu_0(x) = D_n.$$

For $n \in \mathbb{Z}$, it is known that

$$\left(\frac{t}{\log(1+t)}\right)^n (1+t)^{x-1} = \sum_{k=0}^{\infty} B_k^{(k-n+1)}(x) \frac{t^k}{k!}, \text{ (see [2,3,4])}. \tag{2.5}$$

Thus, by (2.5), we get

$$D_k = \int_{\mathbb{Z}_p} (x)_k \, d\mu_0(x) = B_k^{(k+2)}(1), \quad (k \geq 0), \tag{2.6}$$

where $B_k^{(n)}(x)$ are the Bernoulli polynomials of order $n$.

In the special case, $x = 0$, $B_k^{(n)} = B_k^{(n)}(0)$ are called the $n$-th Bernoulli numbers of order $n$.

From (2.4), we note that

$$\begin{aligned} (1+t)^x \int_{\mathbb{Z}_p} (1+t)^y \, d\mu_0(y) &= \left(\frac{\log(1+t)}{t}\right)(1+t)^x \\ &= \sum_{n=0}^{\infty} D_n(x) \frac{t^n}{n!}. \end{aligned} \tag{2.7}$$

Thus, by (2.7), we get

$$\int_{\mathbb{Z}_p} (x+y)_n \, d\mu_0(y) = D_n(x), \quad (n \geq 0), \tag{2.8}$$

and, from (2.5), we have

$$D_n(x) = B_n^{(n+2)}(x+1). \tag{2.9}$$

Therefore, by (2.8) and (2.9), we obtain the following theorem.

**Theorem 2.** *For $n \geq 0$, we have*

$$D_n(x) = \int_{\mathbb{Z}_p} (x+y)_n \, d\mu_0(y),$$

*and*

$$D_n(x) = B_n^{(n+2)}(x+1).$$

By Theorem 1, we easily see that

$$D_n = \sum_{l=0}^{n} S_1(n,l) B_l, \tag{2.10}$$

where $B_l$ are the ordinary Bernoulli numbers.

From Theorem 2, we have

$$\begin{aligned} D_n(x) &= \int_{\mathbb{Z}_p} (x+y)_n \, d\mu_0(y) \\ &= \sum_{l=0}^{n} S_1(n,l) B_l(x), \end{aligned} \tag{2.11}$$

where $B_l(x)$ are the Bernoulli polynomials defined by generating function to be

$$\frac{t}{e^t-1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.$$

Therefore, by (2.10) and (2.11), we obtain the following corollary.

**Corollary 3.** *For $n \geq 0$, we have*

$$D_n(x) = \sum_{l=0}^{n} S_1(n,l) B_l(x).$$

In (2.4), we have

$$\begin{aligned} \frac{t}{e^t-1} &= \sum_{n=0}^{\infty} D_n \frac{1}{n!} \left(e^t-1\right)^n \\ &= \sum_{n=0}^{\infty} D_n \frac{1}{n!} n! \sum_{m=n}^{\infty} S_2(m,n) \frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty} \left( \sum_{n=0}^{m} D_n S_2(m,n) \right) \frac{t^m}{m!} \end{aligned} \tag{2.12}$$

and

$$\frac{t}{e^t-1} = \sum_{m=0}^{\infty} B_m \frac{t^m}{m!}. \tag{2.13}$$

Therefore, by (2.12) and (2.13), we obtain the following theorem.

**Theorem 4.** *For $m \geq 0$, we have*

$$B_m = \sum_{n=0}^{m} D_n S_2(m,n).$$

*In particular,*

$$\int_{\mathbb{Z}_p} x^m d\mu_0(x) = \sum_{n=0}^{m} D_n S_2(m,n).$$

*Remark.* For $m \geq 0$, by (2.11), we have

$$\int_{\mathbb{Z}_p} (x+y)^m d\mu_0(y) = \sum_{n=0}^{m} D_n(x) S_2(m,n).$$

For $n \in \mathbb{Z}_{\geq 0}$, the rising factorial sequence is defined by

$$x^{(n)} = x(x+1)\cdots(x+n-1). \tag{2.14}$$

Let us define the Daehee numbers of the second kind as follows :

$$\widehat{D}_n = \int_{\mathbb{Z}_p} (-x)_n \, d\mu_0(x), \; (n \in \mathbb{Z}_{\geq 0}). \tag{2.15}$$

By (2.15), we get

$$x^{(n)} = (-1)^n (-x)_n = \sum_{l=0}^{n} S_1(n,l)(-1)^{n-l} x^l. \tag{2.16}$$

From (2.15) and (2.16), we have

$$\begin{aligned} \widehat{D}_n &= \int_{\mathbb{Z}_p} (-x)_n \, d\mu_0(x) = \int_{\mathbb{Z}_p} x^{(n)} (-1)^n d\mu_0(x) \\ &= \sum_{l=0}^{n} S_1(n,l)(-1)^l B_l. \end{aligned} \tag{2.17}$$

Therefore, by (2.17), we obtain the following theorem.

**Theorem 5.** *For $n \geq 0$, we have*

$$\widehat{D}_n = \sum_{l=0}^{n} S_1(n,l)(-1)^l B_l.$$

Let us consider the generating function of the Daehee numbers of the second kind as follows :

$$\begin{aligned} \sum_{n=0}^{\infty} \widehat{D}_n \frac{t^n}{n!} &= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} (-x)_n \, d\mu_0(x) \frac{t^n}{n!} \\ &= \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} \binom{-x}{n} t^n d\mu_0(x) \\ &= \int_{\mathbb{Z}_p} (1+t)^{-x} d\mu_0(x). \end{aligned} \tag{2.18}$$

From (1.3), we can derive the following equation :

$$\int_{\mathbb{Z}_p} (1+t)^{-x} d\mu_0(x) = \frac{(1+t)\log(1+t)}{t}, \tag{2.19}$$

where $|t|_p < p^{-\frac{1}{p}}$.

By (2.18) and (2.19), we get

$$\frac{1}{t}(1+t)\log(1+t) = \int_{\mathbb{Z}_p}(1+t)^{-x}\,d\mu_0(x) = \sum_{n=0}^{\infty}\widehat{D}_n\frac{t^n}{n!}. \tag{2.20}$$

Let us consider the Daehee polynomials of the second kind as follows :

$$\frac{(1+t)\log(1+t)}{t}\frac{1}{(1+t)^x} = \sum_{n=0}^{\infty}\widehat{D}_n(x)\frac{t^n}{n!}. \tag{2.21}$$

Then, by (2.21), we get

$$\int_{\mathbb{Z}_p}(1+t)^{-x-y}\,d\mu_0(y) = \sum_{n=0}^{\infty}\hat{D}_n(x)\frac{t^n}{n!}. \tag{2.22}$$

From (2.22), we get

$$\widehat{D}_n(x) = \int_{\mathbb{Z}_p}(-x-y)_n\,d\mu_0(y), \quad (n\geq 0) = \sum_{l=0}^{n}(-1)^l S_1(n,l)B_l(x). \tag{2.23}$$

Therefore, by (2.23), we obtain the following theorem.

**Theorem 6.** *For $n\geq 0$, we have*

$$\widehat{D}_n(x) = \int_{\mathbb{Z}_p}(-x-y)_n\,d\mu_0(y) = \sum_{l=0}^{n}(-1)^l S_1(n,l)B_l(x).$$

From (2.21) and (2.22), we have

$$\left(\frac{t}{e^t-1}\right)e^{(1-x)t} = \sum_{n=0}^{\infty}\widehat{D}_n(x)\frac{1}{n!}\left(e^t-1\right)^n = \sum_{n=0}^{\infty}\widehat{D}_n(x)\frac{1}{n!}n!\sum_{m=n}^{\infty}S_2(m,n)\frac{t^m}{m!} = \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m}\widehat{D}_n(x)S_2(m,n)\right)\frac{t^n}{m!}, \tag{2.24}$$

and

$$\int_{\mathbb{Z}_p}e^{-(x+y)t}d\mu_0(y) = \sum_{n=0}^{\infty}\widehat{D}_n(x)\frac{\left(e^t-1\right)^n}{n!} = \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m}\widehat{D}_n(x)S_2(m,n)\right)\frac{t^m}{m!}. \tag{2.25}$$

Therefore, by (2.24) and (2.25), we obtain the follwoing theorem.

**Theorem 7.** *For $m\geq 0$, we have*

$$B_m(1-x) = (-1)^m\int_{\mathbb{Z}_p}(x+y)^m\,d\mu_0(y) = \sum_{n=0}^{m}\widehat{D}_n(x)S_2(m,n).$$

*In particular,*

$$B_m(1-x) = (-1)^m B_m(x) = \sum_{n=0}^{m} \widehat{D}_m(x) S_2(m,n).$$

*Remark.* By (2.5), (2.20) and (2.21), we see that

$$\widehat{D}_n = B_n^{(n+2)}(2), \quad \widehat{D}_n(x) = B_n^{(n+2)}(2-x).$$

From Theorem 1 and (2.15), we have

$$\begin{aligned}(-1)^n \frac{D_n}{n!} &= (-1)^n \int_{\mathbb{Z}_p} \binom{x}{n} d\mu_0(x) \\ &= \int_{\mathbb{Z}_p} \binom{-x+n-1}{n} d\mu_0(x) \\ &= \sum_{m=0}^{n} \binom{n-1}{n-m} \int_{\mathbb{Z}_p} \binom{-x}{m} d\mu_0(x) \\ &= \sum_{m=0}^{n} \binom{n-1}{n-m} \frac{\widehat{D}_m}{m!} = \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{\widehat{D}_m}{m!},\end{aligned} \tag{2.26}$$

and

$$\begin{aligned}(-1)^n \frac{\widehat{D}_n}{n!} &= (-1)^n \int_{\mathbb{Z}_p} \binom{-x}{n} d\mu_0(x) = \int_{\mathbb{Z}_p} \binom{x+n-1}{n} d\mu_0(x) \\ &= \sum_{m=0}^{n} \binom{n-1}{n-m} \int_0^1 \binom{x}{m} d\mu_0(x) \\ &= \sum_{m=0}^{n} \binom{n-1}{m-1} \frac{D_m}{m!} = \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{D_m}{m!}.\end{aligned} \tag{2.27}$$

Therefore, by (2.26) and (2.27), we obtain the following theorem.

**Theorem 8.** *For $n \in \mathbb{N}$, we have*

$$(-1)^n \frac{D_n}{n!} = \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{\widehat{D}_m}{m!},$$

*and*

$$(-1)^n \frac{\widehat{D}_n}{n!} = \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{D_m}{m!}.$$

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