arXiv:1401.3079v1 [math.NT] 14 Jan 2014

# Barnes-type Daehee polynomials

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MR Subject Classifications: 05A15, 05A40, 11B68, 11B75, 65Q05

**Abstract**

In this paper, we consider Barnes-type Daehee polynomials of the first kind and of the second kind. From the properties of Sheffer sequences of these polynomials arising from umbral calculus, we derive new and interesting identities.

*This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korean government (MOE) (No.2012R1A1A2003786).

†The present Research has been conducted by the Research Grant of Kwangwoon University in 2014.

‡The third author was supported in part by the Grant-in-Aid for Scientific research (C) (No.22540005), the Japan Society for the Promotion of Science.

# 1 Introduction

In this paper, we consider the polynomials $D_n(x|a_1,\ldots,a_r)$ and $\widehat{D}_n(x|a_1,\ldots,a_r)$ called the Barnes-type Daehee polynomials of the first kind and of the second kind, whose generating functions are given by

$$\prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)(1+t)^x=\sum_{n=0}^{\infty}D_n(x|a_1,\ldots,a_r)\frac{t^n}{n!}\,, \tag{1}$$

$$\prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right)(1+t)^x=\sum_{n=0}^{\infty}\widehat{D}_n(x|a_1,\ldots,a_r)\frac{t^n}{n!}\,, \tag{2}$$

respectively, where $a_1,\ldots,a_r\neq 0$. When $x=0$, $D_n(a_1,\ldots,a_r)=D_n(0|a_1,\ldots,a_r)$ and $\widehat{D}_n(a_1,\ldots,a_r)=\widehat{D}_n(0|a_1,\ldots,a_r)$ are called the Barnes-type Daehee numbers of the first kind and of the second kind, respectively.

Recall that the Daehee polynomials of the first kind and of the second kind of order $r$, denoted by $D_n^{(r)}(x)$ and $\widehat{D}_n^{(r)}(x)$, respectively, are given by the generating functions to be

$$\left(\frac{\ln(1+t)}{t}\right)^r(1+t)^x=\sum_{n=0}^{\infty}D_n^{(r)}(x)\frac{t^n}{n!}\,,$$

$$\left(\frac{(1+t)\ln(1+t)}{t}\right)^r(1+t)^x=\sum_{n=0}^{\infty}\widehat{D}_n^{(r)}(x)\frac{t^n}{n!}\,,$$

respectively. If $a_1=\cdots=a_r=1$, then $D_n^{(r)}(x)=D_n(x|\underbrace{1,\ldots,1}_{r})$ and $\widehat{D}_n^{(r)}(x)=\widehat{D}_n(x|\underbrace{1,\ldots,1}_{r})$. Dahee polynomials were defined by the second author [7] and have been investigated in [5, 12, 13].

In this paper, we consider Barnes-type Daehee polynomials of the first kind and of the second kind. From the properties of Sheffer sequences of these polynomials arising from umbral calculus, we derive new and interesting identities.

# 2 Umbral calculus

Let $\mathbb{C}$ be the complex number field and let $\mathcal{F}$ be the set of all formal power series in the variable $t$:

$$\mathcal{F}=\left\{f(t)=\sum_{k=0}^{\infty}\frac{a_k}{k!}t^k\,\middle|\,a_k\in\mathbb{C}\right\}. \tag{3}$$

Let $\mathbb{P}=\mathbb{C}[x]$ and let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. $\langle L|p(x)\rangle$ is the action of the linear functional $L$ on the polynomial $p(x)$, and we recall that the vector space operations on $\mathbb{P}^*$ are defined by $\langle L+M|p(x)\rangle=\langle L|p(x)\rangle+\langle M|p(x)\rangle$, $\langle cL|p(x)\rangle=$

$c\langle L|p(x)\rangle$, where $c$ is a complex constant in $\mathbb{C}$. For $f(t) \in \mathcal{F}$, let us define the linear functional on $\mathbb{P}$ by setting

$$\langle f(t)|x^n\rangle = a_n, \quad (n \geq 0). \tag{4}$$

In particular,

$$\left\langle t^k|x^n\right\rangle = n!\delta_{n,k} \quad (n, k \geq 0), \tag{5}$$

where $\delta_{n,k}$ is the Kronecker's symbol.

For $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L|x^k\rangle}{k!} t^k$, we have $\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle$. That is, $L = f_L(t)$. The map $L \mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ denotes both the algebra of formal power series in $t$ and the vector space of all linear functionals on $\mathbb{P}$, and so an element $f(t)$ of $\mathcal{F}$ will be thought of as both a formal power series and a linear functional. We call $\mathcal{F}$ the *umbral algebra* and the *umbral calculus* is the study of umbral algebra. The order $O\big(f(t)\big)$ of a power series $f(t)(\neq 0)$ is the smallest integer $k$ for which the coefficient of $t^k$ does not vanish. If $O\big(f(t)\big) = 1$, then $f(t)$ is called a *delta series*; if $O\big(f(t)\big) = 0$, then $f(t)$ is called an *invertible series*. For $f(t), g(t) \in \mathcal{F}$ with $O\big(f(t)\big) = 1$ and $O\big(g(t)\big) = 0$, there exists a unique sequence $s_n(x)$ ($\deg s_n(x) = n$) such that $\left\langle g(t)f(t)^k|s_n(x)\right\rangle = n!\delta_{n,k}$, for $n, k \geq 0$. Such a sequence $s_n(x)$ is called the *Sheffer sequence* for $\big(g(t), f(t)\big)$ which is denoted by $s_n(x) \sim \big(g(t), f(t)\big)$.

For $f(t), g(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$, we have

$$\langle f(t)g(t)|p(x)\rangle = \langle f(t)|g(t)p(x)\rangle = \langle g(t)|f(t)p(x)\rangle \tag{6}$$

and

$$f(t) = \sum_{k=0}^{\infty} \left\langle f(t)|x^k\right\rangle \frac{t^k}{k!}, \quad p(x) = \sum_{k=0}^{\infty} \left\langle t^k|p(x)\right\rangle \frac{x^k}{k!} \tag{7}$$

([14, Theorem 2.2.5]). Thus, by (7), we get

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k} \quad \text{and} \quad e^{yt}p(x) = p(x+y). \tag{8}$$

Sheffer sequences are characterized in the generating function ([14, Theorem 2.3.4]).

**Lemma 1** *The sequence $s_n(x)$ is Sheffer for $\big(g(t), f(t)\big)$ if and only if*

$$\frac{1}{g\big(\bar{f}(t)\big)} e^{y\bar{f}(t)} = \sum_{k=0}^{\infty} \frac{s_k(y)}{k!} t^k \quad (y \in \mathbb{C}),$$

*where $\bar{f}(t)$ is the compositional inverse of $f(t)$.*

For $s_n(x) \sim \big(g(t), f(t)\big)$, we have the following equations ([14, Theorem 2.3.7, Theorem

2.3.5, Theorem 2.3.9]):

$$f(t)s_n(x) = ns_{n-1}(x) \quad (n \geq 0), \tag{9}$$

$$s_n(x) = \sum_{j=0}^{n} \frac{1}{j!} \left\langle g\big(\bar{f}(t)\big)^{-1} \bar{f}(t)^j | x^n \right\rangle x^j, \tag{10}$$

$$s_n(x+y) = \sum_{j=0}^{n} \binom{n}{j} s_j(x) p_{n-j}(y)\,, \tag{11}$$

where $p_n(x) = g(t)s_n(x)$.

Assume that $p_n(x) \sim \big(1, f(t)\big)$ and $q_n(x) \sim \big(1, g(t)\big)$. Then the transfer formula ([14, Corollary 3.8.2]) is given by

$$q_n(x) = x \left(\frac{f(t)}{g(t)}\right)^n x^{-1} p_n(x) \quad (n \geq 1).$$

For $s_n(x) \sim \big(g(t), f(t)\big)$ and $r_n(x) \sim \big(h(t), l(t)\big)$, assume that

$$s_n(x) = \sum_{m=0}^{n} C_{n,m} r_m(x) \quad (n \geq 0)\,.$$

Then we have ([14, p.132])

$$C_{n,m} = \frac{1}{m!} \left\langle \frac{h\big(\bar{f}(t)\big)}{g\big(\bar{f}(t)\big)} l\big(\bar{f}(t)\big)^m \middle| x^n \right\rangle. \tag{12}$$

# 3 Main results

We now note that $D_n(x|a_1, \dots, a_r)$ is the Sheffer sequence for

$$g(t) = \prod_{j=1}^{r} \left(\frac{e^{a_j t} - 1}{t}\right) \quad \text{and} \quad f(t) = e^t - 1.$$

So,

$$D_n(x|a_1, \dots, a_r) \sim \left(\prod_{j=1}^{r} \left(\frac{e^{a_j t} - 1}{t}\right), e^t - 1\right). \tag{13}$$

$\widehat{D}_n(x|a_1, \dots, a_r)$ is the Sheffer sequences for

$$g(t) = \prod_{j=1}^{r} \left(\frac{e^{a_j t} - 1}{t e^{a_j t}}\right) \quad \text{and} \quad f(t) = e^t - 1.$$

So,

$$\widehat{D}_n(x|a_1, \dots, a_r) \sim \left(\prod_{j=1}^{r} \left(\frac{e^{a_j t} - 1}{t e^{a_j t}}\right), e^t - 1\right). \tag{14}$$

## 3.1 Explicit expressions

Recall that Barnes' multiple Bernoulli polynomials $B_n(x|a_1,\dots,a_r)$ are defined by the generating function as

$$\frac{t^r}{\prod_{j=1}^r(e^{a_j t}-1)}e^{xt}=\sum_{n=0}^\infty B_n(x|a_1,\dots,a_r)\frac{t^n}{n!}\,, \tag{15}$$

where $a_1,\dots,a_r\neq 0$ ([8, 9, 1]). Let $(n)_j=n(n-1)\cdots(n-j+1)$ $(j\ge 1)$ with $(n)_0=1$. The (signed) Stirling numbers of the first kind $S_1(n,m)$ are defined by

$$(x)_n=\sum_{m=0}^n S_1(n,m)x^m\,.$$

**Theorem 1**

$$\begin{aligned}D_n(x|a_1,\dots,a_r)&=\sum_{m=0}^n S_1(n,m)B_m(x|a_1,\dots,a_r) &(16)\\ &=\sum_{j=0}^n\left(\sum_{l=j}^n\binom{n}{l}S_1(l,j)D_{n-l}(a_1,\dots,a_r)\right)x^j &(17)\\ &=\sum_{m=0}^n\binom{n}{m}D_{n-m}(a_1,\dots,a_r)(x)_m\,, &(18)\end{aligned}$$

$$\begin{aligned}\widehat{D}_n(x|a_1,\dots,a_r)&=\sum_{m=0}^n S_1(n,m)B_m(x+a_1+\cdots+a_r|a_1,\dots,a_r) &(19)\\ &=\sum_{j=0}^n\left(\sum_{l=j}^n\binom{n}{l}S_1(l,j)\widehat{D}_{n-l}(a_1,\dots,a_r)\right)x^j &(20)\\ &=\sum_{m=0}^n\binom{n}{m}\widehat{D}_{n-m}(a_1,\dots,a_r)(x)_m\,. &(21)\end{aligned}$$

*Proof.* Since

$$\prod_{j=1}^r\left(\frac{e^{a_j t}-1}{t}\right)D_n(x|a_1,\dots,a_r)\sim(1,e^t-1) \tag{22}$$

and

$$(x)_n\sim(1,e^t-1)\,, \tag{23}$$

we have

$$D_n(x|a_1,\ldots,a_r) = \prod_{j=1}^{r}\left(\frac{t}{e^{a_j t}-1}\right)(x)_n$$
$$= \sum_{m=0}^{n} S_1(n,m)\prod_{j=1}^{r}\left(\frac{t}{e^{a_j t}-1}\right)x^m$$
$$= \sum_{m=0}^{n} S_1(n,m)B_m(x|a_1,\ldots,a_r)\,.$$

So, we get (16).
Similarly, by

$$\prod_{j=1}^{r}\left(\frac{e^{a_j t}-1}{te^{a_j t}}\right)\widehat{D}_n(x|a_1,\ldots,a_r) \sim (1, e^t - 1) \tag{24}$$

and (23), we have

$$\widehat{D}_n(x|a_1,\ldots,a_r) = \prod_{j=1}^{r}\left(\frac{te^{a_j t}}{e^{a_j t}-1}\right)(x)_n$$
$$= \sum_{m=0}^{n} S_1(n,m)\prod_{j=1}^{r}\left(\frac{te^{a_j t}}{e^{a_j t}-1}\right)x^m$$
$$= \sum_{m=0}^{n} S_1(n,m)e^{(a_1+\cdots+a_r)t}\prod_{j=1}^{r}\left(\frac{t}{e^{a_j t}-1}\right)x^m$$
$$= \sum_{m=0}^{n} S_1(n,m)B_m(x+a_1+\cdots+a_r|a_1,\ldots,a_r)\,.$$

So, we get (19).
By (10) with (13), we get

$$D_n(x|a_1,\ldots,a_r) = \sum_{j=0}^{n}\frac{1}{j!}\left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\big(\ln(1+t)\big)^j\Big| x^n\right\rangle x^j\,.$$

Since

$$\begin{aligned}
&\left\langle \prod_{j=1}^{r} \left( \frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) \left(\ln(1+t)\right)^j \middle| x^n \right\rangle \\
&= \left\langle \prod_{j=1}^{r} \left( \frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) \middle| \left(\ln(1+t)\right)^j x^n \right\rangle \\
&= \left\langle \prod_{j=1}^{r} \left( \frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) \middle| j! \sum_{l=j}^{\infty} S_1(l,j) \frac{t^l}{l!} x^n \right\rangle \\
&= j! \sum_{l=j}^{n} \binom{n}{l} S_1(l,j) \left\langle \prod_{j=1}^{r} \left( \frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) \middle| x^{n-l} \right\rangle \\
&= j! \sum_{l=j}^{n} \binom{n}{l} S_1(l,j) \left\langle \sum_{i=0}^{\infty} D_i(a_1,\dots,a_r) \frac{t^i}{i!} \middle| x^{n-l} \right\rangle \\
&= j! \sum_{l=j}^{n} \binom{n}{l} S_1(l,j) D_{n-l}(a_1,\dots,a_r)\,,
\end{aligned}$$

we obtain (17).
Similarly, by (10) with (14), we get

$$\widehat{D}_n(x|a_1,\dots,a_r) = \sum_{j=0}^{n} \frac{1}{j!} \left\langle \prod_{j=1}^{r} \left( \frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1} \right) \left(\ln(1+t)\right)^j \middle| x^n \right\rangle x^j\,.$$

Since

$$\begin{aligned}
&\left\langle \prod_{j=1}^{r} \left( \frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1} \right) \left(\ln(1+t)\right)^j \middle| x^n \right\rangle \\
&= \left\langle \prod_{j=1}^{r} \left( \frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1} \right) \middle| \left(\ln(1+t)\right)^j x^n \right\rangle \\
&= j! \sum_{l=j}^{n} \binom{n}{l} S_1(l,j) \left\langle \prod_{j=1}^{r} \left( \frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1} \right) \middle| x^{n-l} \right\rangle \\
&= j! \sum_{l=j}^{n} \binom{n}{l} S_1(l,j) \left\langle \sum_{i=0}^{\infty} \widehat{D}_i(a_1,\dots,a_r) \frac{t^i}{i!} \middle| x^{n-l} \right\rangle \\
&= j! \sum_{l=j}^{n} \binom{n}{l} S_1(l,j) \widehat{D}_{n-l}(a_1,\dots,a_r)\,,
\end{aligned}$$

we obtain (20).

Next, we obtain that

$$\begin{aligned}
D_n(y|a_1,\dots,a_r) &= \left\langle \sum_{i=0}^{\infty} D_i(y|a_1,\dots,a_r)\frac{t^i}{i!}\Big| x^n \right\rangle \\
&= \left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)(1+t)^y \Big| x^n \right\rangle \\
&= \left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right) \Big| \sum_{m=0}^{\infty}(y)_m \frac{t^m}{m!} x^n \right\rangle \\
&= \sum_{m=0}^{n}(y)_m \binom{n}{m} \left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right) \Big| x^{n-m} \right\rangle \\
&= \sum_{m=0}^{n}\binom{n}{m} D_{n-m}(a_1,\dots,a_r)(y)_m\,.
\end{aligned}$$

Thus, we get the identity (18).
Similarly,

$$\begin{aligned}
\widehat{D}_n(y|a_1,\dots,a_r) &= \left\langle \sum_{i=0}^{\infty} \widehat{D}_i(y|a_1,\dots,a_r)\frac{t^i}{i!}\Big| x^n \right\rangle \\
&= \left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right)(1+t)^y \Big| x^n \right\rangle \\
&= \left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right) \Big| \sum_{m=0}^{\infty}(y)_m \frac{t^m}{m!} x^n \right\rangle \\
&= \sum_{m=0}^{n}(y)_m \binom{n}{m} \left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right) \Big| x^{n-m} \right\rangle \\
&= \sum_{m=0}^{n}\binom{n}{m} \widehat{D}_{n-m}(a_1,\dots,a_r)(y)_m\,.
\end{aligned}$$

Thus, we get the identity (21). ∎

## 3.2 Sheffer identity

**Theorem 2**

$$D_n(x+y|a_1,\dots,a_r) = \sum_{j=0}^{n}\binom{n}{j} D_j(x|a_1,\dots,a_r)(y)_{n-j}\,, \tag{25}$$

$$\widehat{D}_n(x+y|a_1,\dots,a_r) = \sum_{j=0}^{n}\binom{n}{j} \widehat{D}_j(x|a_1,\dots,a_r)(y)_{n-j}\,. \tag{26}$$

*Proof.* By (13) with

$$\begin{aligned} p_n(x) &= \prod_{j=1}^{r} \left( \frac{e^{a_j t} - 1}{t} \right) D_n(x|a_1, \ldots, a_r) \\ &= (x)_n \sim (1, e^t - 1)\,, \end{aligned}$$

using (11), we have (25).
By (14) with

$$\begin{aligned} p_n(x) &= \prod_{j=1}^{r} \left( \frac{e^{a_j t} - 1}{t e^{a_j t}} \right) \widehat{D}_n(x|a_1, \ldots, a_r) \\ &= (x)_n \sim (1, e^t - 1)\,, \end{aligned}$$

using (11), we have (26). ∎

## 3.3 Difference relations

**Theorem 3**

$$D_n(x+1|a_1, \ldots, a_r) - D_n(x|a_1, \ldots, a_r) = n D_{n-1}(x|a_1, \ldots, a_r)\,, \tag{27}$$

$$\widehat{D}_n(x+1|a_1, \ldots, a_r) - \widehat{D}_n(x|a_1, \ldots, a_r) = n \widehat{D}_{n-1}(x|a_1, \ldots, a_r)\,. \tag{28}$$

*Proof.* By (9) with (13), we get

$$(e^t - 1) D_n(x|a_1, \ldots, a_r) = n D_{n-1}(x|a_1, \ldots, a_r)\,.$$

By (8), we have (27).
Similarly, by (9) with (14), we get

$$(e^t - 1) \widehat{D}_n(x|a_1, \ldots, a_r) = n \widehat{D}_{n-1}(x|a_1, \ldots, a_r)\,.$$

By (8), we have (28). ∎

### 3.4 Recurrence

**Theorem 4**

$$\begin{aligned}D_{n+1}(x|a_1,\dots,a_r) &= xD_n(x-1|a_1,\dots,a_r)\\ &\quad -\sum_{m=0}^n\left(\sum_{i=m}^n\sum_{l=i}^n\sum_{j=1}^r\frac{1}{i+1}\binom{n}{l}\binom{i+1}{m}S_1(l,i)\right.\\ &\quad \left.\times B_{i+1-m}(-a_j)^{i+1-m}D_{n-l}(a_1,\dots,a_r)\right)(x-1)^m\,, \qquad (29)\\ \widehat{D}_{n+1}(x|a_1,\dots,a_r) &= \left(x+\sum_{j=1}^r a_j\right)\widehat{D}_n(x-1|a_1,\dots,a_r)\\ &\quad -\sum_{m=0}^n\left(\sum_{i=m}^n\sum_{l=i}^n\sum_{j=1}^r\frac{1}{i+1}\binom{n}{l}\binom{i+1}{m}S_1(l,i)\right.\\ &\quad \left.\times B_{i+1-m}(-a_j)^{i+1-m}\widehat{D}_{n-l}(a_1,\dots,a_r)\right)(x-1)^m\,, \qquad (30)\end{aligned}$$

*where $B_n$ is the nth ordinary Bernoulli number.*

*Proof.* By applying

$$s_{n+1}(x)=\left(x-\frac{g'(t)}{g(t)}\right)\frac{1}{f'(t)}s_n(x) \qquad (31)$$

([14, Corollary 3.7.2]) with (13), we get

$$D_{n+1}(x|a_1,\dots,a_r)=xD_n(x-1|a_1,\dots,a_r)-e^{-t}\frac{g'(t)}{g(t)}D_n(x|a_1,\dots,a_r)\,.$$

Now,

$$\begin{aligned}\frac{g'(t)}{g(t)} &= (\ln g(t))'\\ &= \left(\sum_{j=1}^r\ln(e^{a_jt}-1)-r\ln t\right)'\\ &= \sum_{j=1}^r\frac{a_je^{a_jt}}{e^{a_jt}-1}-\frac{r}{t}\\ &= \frac{\sum_{j=1}^r\prod_{i\ne j}(e^{a_it}-1)(a_jte^{a_jt}-e^{a_jt}+1)}{t\prod_{j=1}^r(e^{a_jt}-1)}\,.\end{aligned}$$

Since

$$\sum_{j=1}^{r}\frac{a_jte^{a_jt}}{e^{a_jt}-1}-r=\frac{\sum_{j=1}^{r}\prod_{i\neq j}(e^{a_it}-1)(a_jte^{a_jt}-e^{a_jt}+1)}{\prod_{j=1}^{r}(e^{a_jt}-1)}$$
$$=\frac{\frac{1}{2}\left(\sum_{j=1}^{r}a_1\cdots a_{j-1}a_j^2a_{j+1}\cdots a_r\right)t^{r+1}+\cdots}{(a_1\cdots a_r)t^r+\cdots}$$
$$=\frac{1}{2}\left(\sum_{j=1}^{r}a_j\right)t+\cdots$$

is a series with order$\geq 1$, by (17) we have

$$\begin{aligned}
&D_{n+1}(x|a_1,\ldots,a_r)\\
&=xD_n(x-1|a_1,\ldots,a_r)-e^{-t}\frac{\sum_{j=1}^{r}\frac{a_jte^{a_jt}}{e^{a_jt}-1}-r}{t}D_n(x|a_1,\ldots,a_r)\\
&=xD_n(x-1|a_1,\ldots,a_r)-e^{-t}\frac{\sum_{j=1}^{r}\frac{a_jte^{a_jt}}{e^{a_jt}-1}-r}{t}\left(\sum_{i=0}^{n}\sum_{l=i}^{n}\binom{n}{l}S_1(l,i)D_{n-l}(a_1,\ldots,a_r)x^i\right)\\
&=xD_n(x-1|a_1,\ldots,a_r)\\
&\quad-\sum_{i=0}^{n}\sum_{l=i}^{n}\binom{n}{l}S_1(l,i)D_{n-l}(a_1,\ldots,a_r)e^{-t}\left(\sum_{j=1}^{r}\frac{a_jte^{a_jt}}{e^{a_jt}-1}-r\right)\frac{x^{i+1}}{i+1}\,.
\end{aligned}$$

Since

$$\begin{aligned}
e^{-t}\left(\sum_{j=1}^{r}\frac{a_jte^{a_jt}}{e^{a_jt}-1}-r\right)x^{i+1}&=e^{-t}\left(\sum_{j=1}^{r}\sum_{m=0}^{\infty}\frac{(-1)^mB_ma_j^m}{m!}t^m-r\right)x^{i+1}\\
&=e^{-t}\left(\sum_{j=1}^{r}\sum_{m=0}^{i+1}\binom{i+1}{m}B_m(-a_j)^mx^{i+1-m}-rx^{i+1}\right)\\
&=\sum_{j=1}^{r}\sum_{m=1}^{i+1}\binom{i+1}{m}B_m(-a_j)^m(x-1)^{i+1-m}\\
&=\sum_{j=1}^{r}\sum_{m=0}^{i}\binom{i+1}{m}B_{i+1-m}(-a_j)^{i+1-m}(x-1)^m\,,
\end{aligned}\tag{32}$$

we have

$$\begin{aligned}
D_{n+1}(x|a_1,\dots,a_r) &= xD_n(x-1|a_1,\dots,a_r)\\
&\quad -\sum_{i=0}^{n}\sum_{l=i}^{n}\sum_{j=1}^{r}\sum_{m=0}^{i}\frac{1}{i+1}\binom{n}{l}\binom{i+1}{m}S_1(l,i)\\
&\quad \times B_{i+1-m}(-a_j)^{i+1-m}D_{n-l}(a_1,\dots,a_r)(x-1)^m\\
&= xD_n(x-1|a_1,\dots,a_r)\\
&\quad -\sum_{m=0}^{n}\left(\sum_{i=m}^{n}\sum_{l=i}^{n}\sum_{j=1}^{r}\frac{1}{i+1}\binom{n}{l}\binom{i+1}{m}S_1(l,i)\right.\\
&\quad \left.\times B_{i+1-m}(-a_j)^{i+1-m}D_{n-l}(a_1,\dots,a_r)\right)(x-1)^m\,,
\end{aligned}$$

which is the identity (29).
Next, by applying (31) with (14), we get

$$\widehat{D}_{n+1}(x|a_1,\dots,a_r)=x\widehat{D}_n(x-1|a_1,\dots,a_r)-e^{-t}\frac{g'(t)}{g(t)}\widehat{D}_n(x|a_1,\dots,a_r)\,.$$

Now,

$$\begin{aligned}
\frac{g'(t)}{g(t)} &= (\ln g(t))'\\
&= \left(\sum_{j=1}^{r}\ln(e^{a_jt}-1)-r\ln t-\left(\sum_{j=1}^{r}a_j\right)t\right)'\\
&= \sum_{j=1}^{r}\frac{a_je^{a_jt}}{e^{a_jt}-1}-\frac{r}{t}-\sum_{j=1}^{r}a_j\,.
\end{aligned}$$

By (20) we have

$$
\begin{aligned}
&\widehat{D}_{n+1}(x|a_1,\ldots,a_r)\\
&=\left(x+\sum_{j=1}^{r}a_j\right)\widehat{D}_n(x-1|a_1,\ldots,a_r)-e^{-t}\frac{\sum_{j=1}^{r}\frac{a_jte^{a_jt}}{e^{a_jt}-1}-r}{t}\widehat{D}_n(x|a_1,\ldots,a_r)\\
&=\left(x+\sum_{j=1}^{r}a_j\right)\widehat{D}_n(x-1|a_1,\ldots,a_r)\\
&\quad-e^{-t}\frac{\sum_{j=1}^{r}\frac{a_jte^{a_jt}}{e^{a_jt}-1}-r}{t}\left(\sum_{i=0}^{n}\sum_{l=i}^{n}\binom{n}{l}S_1(l,i)\widehat{D}_{n-l}(a_1,\ldots,a_r)x^i\right)\\
&=\left(x+\sum_{j=1}^{r}a_j\right)\widehat{D}_n(x-1|a_1,\ldots,a_r)\\
&\quad-\sum_{i=0}^{n}\sum_{l=i}^{n}\binom{n}{l}S_1(l,i)\widehat{D}_{n-l}(a_1,\ldots,a_r)e^{-t}\left(\sum_{j=1}^{r}\frac{a_jte^{a_jt}}{e^{a_jt}-1}-r\right)\frac{x^{i+1}}{i+1}\,.
\end{aligned}
$$

By (32), we have the identity (30). ∎

## 3.5 Differentiation

**Theorem 5**

$$
\frac{d}{dx}D_n(x|a_1,\ldots,a_r)=n!\sum_{l=0}^{n-1}\frac{(-1)^{n-l-1}}{l!(n-l)}D_l(x|a_1,\ldots,a_r)\,, \tag{33}
$$

$$
\frac{d}{dx}\widehat{D}_n(x|a_1,\ldots,a_r)=n!\sum_{l=0}^{n-1}\frac{(-1)^{n-l-1}}{l!(n-l)}\widehat{D}_l(x|a_1,\ldots,a_r)\,. \tag{34}
$$

*Proof.* We shall use

$$
\frac{d}{dx}s_n(x)=\sum_{l=0}^{n-1}\binom{n}{l}\left\langle\bar{f}(t)|x^{n-l}\right\rangle s_l(x)
$$

(*Cf.* [14, Theorem 2.3.12]). Since

$$\begin{aligned}\left\langle \bar{f}(t)|x^{n-l}\right\rangle &= \left\langle \ln(1+t)|x^{n-l}\right\rangle \\ &= \left\langle \sum_{m=1}^{\infty}\frac{(-1)^{m-1}t^m}{m}\middle| x^{n-l}\right\rangle \\ &= \sum_{m=1}^{n-l}\frac{(-1)^{m-1}}{m}\left\langle t^m|x^{n-l}\right\rangle \\ &= \sum_{m=1}^{n-l}\frac{(-1)^{m-1}}{m}(n-l)!\delta_{m,n-l} \\ &= (-1)^{n-l-1}(n-l-1)!\,,\end{aligned}$$

with (13), we have

$$\begin{aligned}\frac{d}{dx}D_n(x|a_1,\dots,a_r) &= \sum_{l=0}^{n-1}\binom{n}{l}(-1)^{n-l-1}(n-l-1)!D_l(x|a_1,\dots,a_r) \\ &= n!\sum_{l=0}^{n-1}\frac{(-1)^{n-l-1}}{l!(n-l)}D_l(x|a_1,\dots,a_r)\,,\end{aligned}$$

which is the identity (33). Similarly, with (14), we have the identity (34). ∎

## 3.6 More relations

The classical Cauchy numbers $c_n$ are defined by

$$\frac{t}{\ln(1+t)} = \sum_{n=0}^{\infty} c_n\frac{t^n}{n!}$$

(see e.g. [3, 10]).

**Theorem 6**

$$
\begin{aligned}
D_n(x|a_1,\dots,a_r) &= xD_{n-1}(x-1|a_1,\dots,a_r) \\
&\quad + \frac{r}{n}\sum_{l=0}^{n}\binom{n}{l}c_l D_{n-l}(x-1|a_1,\dots,a_r) \\
&\quad - \frac{1}{n}\sum_{j=1}^{r}\sum_{l=0}^{n}\binom{n}{l}a_j c_l D_{n-l}(x+a_j-1|a_1,\dots,a_r,a_j)\,, \qquad (35)\\
\widehat{D}_n(x|a_1,\dots,a_r) &= \left(x+\sum_{j=1}^{r}a_j\right)\widehat{D}_{n-1}(x-1|a_1,\dots,a_r) \\
&\quad + \frac{r}{n}\sum_{l=0}^{n}\binom{n}{l}c_l \widehat{D}_{n-l}(x-1|a_1,\dots,a_r) \\
&\quad - \frac{1}{n}\sum_{j=1}^{r}\sum_{l=0}^{n}\binom{n}{l}a_j c_l \widehat{D}_{n-l}(x-1|a_1,\dots,a_r,a_j)\,. \qquad (36)
\end{aligned}
$$

*Proof.* For $n \ge 1$, we have

$$
\begin{aligned}
D_n(y|a_1,\dots,a_r) &= \left\langle \sum_{l=0}^{\infty} D_l(y|a_1,\dots,a_r)\frac{t^l}{l!}\middle| x^n\right\rangle \\
&= \left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)(1+t)^y \middle| x^n\right\rangle \\
&= \left\langle \partial_t\left(\prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)(1+t)^y\right)\middle| x^{n-1}\right\rangle \\
&= \left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\left(\partial_t(1+t)^y\right)\middle| x^{n-1}\right\rangle \\
&\quad + \left\langle \left(\partial_t\prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\right)(1+t)^y \middle| x^{n-1}\right\rangle \\
&= yD_{n-1}(y-1|a_1,\dots,a_r) \\
&\quad + \left\langle \left(\partial_t\prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\right)(1+t)^y \middle| x^{n-1}\right\rangle\,.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
&\partial_t \prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\\
&=\sum_{j=1}^{r}\prod_{i\neq j}\left(\frac{\ln(1+t)}{(1+t)^{a_i}-1}\right)\frac{\frac{1}{1+t}\big((1+t)^{a_j}-1\big)-\ln(1+t)\big(a_j(1+t)^{a_j-1}\big)}{\big((1+t)^{a_j}-1\big)^2}\\
&=\frac{1}{1+t}\prod_{i=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_i}-1}\right)\sum_{j=1}^{r}\left(\frac{1}{\ln(1+t)}-\frac{a_j(1+t)^{a_j}}{(1+t)^{a_j}-1}\right)\\
&=\frac{1}{1+t}\prod_{i=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_i}-1}\right)\frac{\sum_{j=1}^{r}\left(\frac{t}{\ln(1+t)}-\frac{a_j t(1+t)^{a_j}}{(1+t)^{a_j}-1}\right)}{t}.
\end{aligned}
$$

Since

$$
\sum_{j=1}^{r}\left(\frac{t}{\ln(1+t)}-\frac{a_j t(1+t)^{a_j}}{(1+t)^{a_j}-1}\right)=-\frac{1}{2}\left(\sum_{j=1}^{r}a_j\right)t+\cdots
$$

is a series with order($\geq 1$), we have

$$
\begin{aligned}
&\left\langle\left(\partial_t \prod_{i=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_i}-1}\right)\right)(1+t)^{y}\middle| x^{n-1}\right\rangle\\
&=\left\langle\prod_{i=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_i}-1}\right)(1+t)^{y-1}\middle|\frac{\sum_{j=1}^{r}\left(\frac{t}{\ln(1+t)}-\frac{a_j t(1+t)^{a_j}}{(1+t)^{a_j}-1}\right)}{t}x^{n-1}\right\rangle\\
&=\frac{1}{n}\sum_{j=1}^{r}\left\langle\prod_{i=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_i}-1}\right)(1+t)^{y-1}\middle|\left(\frac{t}{\ln(1+t)}-\frac{a_j t(1+t)^{a_j}}{(1+t)^{a_j}-1}\right)x^{n}\right\rangle\\
&=\frac{r}{n}\left\langle\prod_{i=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_i}-1}\right)(1+t)^{y-1}\middle|\frac{t}{\ln(1+t)}x^{n}\right\rangle\\
&\quad-\frac{1}{n}\sum_{j=1}^{r}a_j\left\langle\frac{\ln(1+t)}{(1+t)^{a_j}-1}\prod_{i=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_i}-1}\right)(1+t)^{y+a_j-1}\middle|\frac{t}{\ln(1+t)}x^{n}\right\rangle\\
&=\frac{r}{n}\left\langle\prod_{i=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_i}-1}\right)(1+t)^{y-1}\middle|\sum_{l=0}^{\infty}c_l\frac{t^l}{l!}x^{n}\right\rangle\\
&\quad-\frac{1}{n}\sum_{j=1}^{r}a_j\left\langle\frac{\ln(1+t)}{(1+t)^{a_j}-1}\prod_{i=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_i}-1}\right)(1+t)^{y+a_j-1}\middle|\sum_{l=0}^{\infty}c_l\frac{t^l}{l!}x^{n}\right\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{r}{n}\sum_{l=0}^{n} c_l \binom{n}{l} \left\langle \prod_{i=1}^{r} \left( \frac{\ln(1+t)}{(1+t)^{a_i}-1} \right) (1+t)^{y-1} \middle| x^{n-l} \right\rangle \\
&\quad - \frac{1}{n}\sum_{j=1}^{r} a_j \sum_{l=0}^{n} c_l \binom{n}{l} \left\langle \frac{\ln(1+t)}{(1+t)^{a_j}-1} \prod_{i=1}^{r} \left( \frac{\ln(1+t)}{(1+t)^{a_i}-1} \right) (1+t)^{y+a_j-1} \middle| x^{n-l} \right\rangle \\
&= \frac{r}{n}\sum_{l=0}^{n} \binom{n}{l} c_l D_{n-l}(y-1|a_1,\dots,a_r) \\
&\quad - \frac{1}{n}\sum_{j=1}^{r}\sum_{l=0}^{n} \binom{n}{l} a_j c_l D_{n-l}(y+a_j-1|a_1,\dots,a_r,a_j)\,.
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
D_n(x|a_1,\dots,a_r) &= xD_{n-1}(x-1|a_1,\dots,a_r) \\
&\quad + \frac{r}{n}\sum_{l=0}^{n} \binom{n}{l} c_l D_{n-l}(x-1|a_1,\dots,a_r) \\
&\quad - \frac{1}{n}\sum_{j=1}^{r}\sum_{l=0}^{n} \binom{n}{l} a_j c_l D_{n-l}(x+a_j-1|a_1,\dots,a_r,a_j)\,,
\end{aligned}
$$

which is the identity (35).

Next, for $n \ge 1$ we have

$$
\begin{aligned}
\widehat{D}_n(y|a_1,\dots,a_r) &= \left\langle \sum_{l=0}^{\infty} \widehat{D}_l(y|a_1,\dots,a_r)\frac{t^l}{l!} \middle| x^n \right\rangle \\
&= \left\langle \prod_{j=1}^{r} \left( \frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1} \right) (1+t)^{y} \middle| x^{n} \right\rangle \\
&= \left\langle \partial_t \left( \prod_{j=1}^{r} \left( \frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1} \right) (1+t)^{y} \right) \middle| x^{n-1} \right\rangle \\
&= \left\langle \prod_{j=1}^{r} \left( \frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1} \right) \left(\partial_t (1+t)^{y}\right) \middle| x^{n-1} \right\rangle \\
&\quad + \left\langle \left( \partial_t \prod_{j=1}^{r} \left( \frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1} \right) \right) (1+t)^{y} \middle| x^{n-1} \right\rangle \\
&= y\widehat{D}_{n-1}(y-1|a_1,\dots,a_r) \\
&\quad + \left\langle \left( \partial_t \prod_{j=1}^{r} \left( \frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1} \right) \right) (1+t)^{y} \middle| x^{n-1} \right\rangle .
\end{aligned}
$$

Observe that

$$
\begin{aligned}
&\partial_t \prod_{j=1}^{r} \left( \frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j}-1} \right) \\
&= \partial_t \left( \prod_{j=1}^{r} \left( \frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) \prod_{j=1}^{r} (1+t)^{a_j} \right) \\
&= \left( \partial_t \prod_{j=1}^{r} \left( \frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) \right) \prod_{j=1}^{r} (1+t)^{a_j} \\
&\quad + \prod_{j=1}^{r} \left( \frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) \left( \partial_t \prod_{j=1}^{r} (1+t)^{a_j} \right) \\
&= \frac{1}{1+t} \prod_{i=1}^{r} \left( \frac{(1+t)^{a_i} \ln(1+t)}{(1+t)^{a_i}-1} \right) \frac{\sum_{j=1}^{r} \left( \frac{t}{\ln(1+t)} - \frac{a_j t (1+t)^{a_j}}{(1+t)^{a_j}-1} \right)}{t} \\
&\quad + \frac{1}{1+t} \prod_{i=1}^{r} \left( \frac{(1+t)^{a_i} \ln(1+t)}{(1+t)^{a_i}-1} \right) \sum_{j=1}^{r} a_j \,.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&\left\langle \left( \partial_t \prod_{i=1}^{r} \left( \frac{(1+t)^{a_i} \ln(1+t)}{(1+t)^{a_i}-1} \right) \right) (1+t)^y \middle| x^{n-1} \right\rangle \\
&= \left\langle \prod_{i=1}^{r} \left( \frac{(1+t)^{a_i} \ln(1+t)}{(1+t)^{a_i}-1} \right) (1+t)^{y-1} \middle| \frac{\sum_{j=1}^{r} \left( \frac{t}{\ln(1+t)} - \frac{a_j t (1+t)^{a_j}}{(1+t)^{a_j}-1} \right)}{t} x^{n-1} \right\rangle \\
&\quad + \left( \sum_{j=1}^{r} a_j \right) \left\langle \prod_{i=1}^{r} \left( \frac{(1+t)^{a_i} \ln(1+t)}{(1+t)^{a_i}-1} \right) (1+t)^{y-1} \middle| x^{n-1} \right\rangle \\
&= \left( \sum_{j=1}^{r} a_j \right) \widehat{D}_{n-1}(y-1|a_1, \ldots, a_r) \\
&\quad + \frac{1}{n} \left\langle \prod_{i=1}^{r} \left( \frac{(1+t)^{a_i} \ln(1+t)}{(1+t)^{a_i}-1} \right) (1+t)^{y-1} \middle| \sum_{j=1}^{r} \left( \frac{t}{\ln(1+t)} - \frac{a_j t (1+t)^{a_j}}{(1+t)^{a_j}-1} \right) x^n \right\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\sum_{j=1}^{r} a_j\right) \widehat{D}_{n-1}(y-1|a_1,\dots,a_r) \\
&\quad + \frac{r}{n}\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}\ln(1+t)}{(1+t)^{a_i}-1}\right)(1+t)^{y-1}\Big|\frac{t}{\ln(1+t)}x^n\right\rangle \\
&\quad - \frac{1}{n}\sum_{j=1}^{r} a_j\left\langle \frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}\ln(1+t)}{(1+t)^{a_i}-1}\right)(1+t)^{y-1}\Big|\frac{t}{\ln(1+t)}x^n\right\rangle \\
&= \left(\sum_{j=1}^{r} a_j\right) \widehat{D}_{n-1}(y-1|a_1,\dots,a_r) \\
&\quad + \frac{r}{n}\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}\ln(1+t)}{(1+t)^{a_i}-1}\right)(1+t)^{y-1}\Big|\sum_{l=0}^{\infty}c_l\frac{t^l}{l!}x^n\right\rangle \\
&\quad - \frac{1}{n}\sum_{j=1}^{r} a_j\left\langle \frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}\ln(1+t)}{(1+t)^{a_i}-1}\right)(1+t)^{y-1}\Big|\sum_{l=0}^{\infty}c_l\frac{t^l}{l!}x^n\right\rangle \\
&= \left(\sum_{j=1}^{r} a_j\right) \widehat{D}_{n-1}(y-1|a_1,\dots,a_r) \\
&\quad + \frac{r}{n}\sum_{l=0}^{n}c_l\binom{n}{l}\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}\ln(1+t)}{(1+t)^{a_i}-1}\right)(1+t)^{y-1}\Big|x^{n-l}\right\rangle \\
&\quad - \frac{1}{n}\sum_{j=1}^{r} a_j\sum_{l=0}^{n}c_l\binom{n}{l}\left\langle \frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}\ln(1+t)}{(1+t)^{a_i}-1}\right)(1+t)^{y-1}\Big|x^{n-l}\right\rangle \\
&= \left(\sum_{j=1}^{r} a_j\right) \widehat{D}_{n-1}(y-1|a_1,\dots,a_r) + \frac{r}{n}\sum_{l=0}^{n}\binom{n}{l}c_l\widehat{D}_{n-l}(y-1|a_1,\dots,a_r) \\
&\quad - \frac{1}{n}\sum_{j=1}^{r}\sum_{l=0}^{n}\binom{n}{l}a_jc_l\widehat{D}_{n-l}(y-1|a_1,\dots,a_r,a_j)\,.
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
\widehat{D}_n(x|a_1,\dots,a_r) &= \left(x+\sum_{j=1}^{r} a_j\right) \widehat{D}_{n-1}(x-1|a_1,\dots,a_r) \\
&\quad + \frac{r}{n}\sum_{l=0}^{n}\binom{n}{l}c_l\widehat{D}_{n-l}(x-1|a_1,\dots,a_r) \\
&\quad - \frac{1}{n}\sum_{j=1}^{r}\sum_{l=0}^{n}\binom{n}{l}a_jc_l\widehat{D}_{n-l}(x-1|a_1,\dots,a_r,a_j)\,.
\end{aligned}
$$

which is the identity (36). ∎

### 3.7 Relations including the Stirling numbers of the first kind

**Theorem 7** *For $n-1\ge m\ge 1$, we have*

$$
\begin{aligned}
&\sum_{l=0}^{n-m}\binom{n}{l}S_1(n-l,m)D_l(a_1,\dots,a_r)\\
&=\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)D_l(-1|a_1,\dots,a_r)\\
&\quad+\frac{1}{n}\sum_{l=0}^{n-m-1}\binom{n}{l+1}S_1(n-l-1,m)\\
&\quad\times\left(r\sum_{i=0}^{l+1}\binom{l+1}{i}c_iD_{l+1-i}(-1|a_1,\dots,a_r)\right.\\
&\qquad\left.-\sum_{j=1}^{r}\sum_{i=0}^{l+1}\binom{l+1}{i}a_jc_iD_{l+1-i}(a_j-1|a_1,\dots,a_r,a_j)\right)\,,
\end{aligned}
\tag{37}
$$

$$
\begin{aligned}
&\sum_{l=0}^{n-m}\binom{n}{l}S_1(n-l,m)\widehat{D}_l(a_1,\dots,a_r)\\
&=\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)\widehat{D}_l(-1|a_1,\dots,a_r)\\
&\quad+\frac{1}{n}\sum_{l=0}^{n-m-1}\binom{n}{l+1}S_1(n-l-1,m)\\
&\quad\times\left(r\sum_{i=0}^{l+1}\binom{l+1}{i}c_i\widehat{D}_{l+1-i}(-1|a_1,\dots,a_r)\right.\\
&\qquad\left.-\sum_{j=1}^{r}\sum_{i=0}^{l+1}\binom{l+1}{i}a_jc_i\widehat{D}_{l+1-i}(-1|a_1,\dots,a_r,a_j)\right)\\
&\quad+\sum_{l=0}^{n-m-1}\binom{n-1}{l}S_1(n-l-1,m)\sum_{j=1}^{r}a_j\widehat{D}_l(-1|a_1,\dots,a_r)\,.
\end{aligned}
\tag{38}
$$

*Proof.* We shall compute

$$
\left\langle\prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\bigl(\ln(1+t)\bigr)^m\middle|x^n\right\rangle
$$

in two different ways. On the one hand,

$$\begin{aligned}
&\left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\big(\ln(1+t)\big)^m \middle| x^n \right\rangle\\
&= \left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right) \middle| \big(\ln(1+t)\big)^m x^n \right\rangle\\
&= \left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right) \middle| \sum_{l=0}^{\infty}\frac{m!}{(l+m)!}S_1(l+m,m)t^{l+m}x^n \right\rangle\\
&= \sum_{l=0}^{n-m}\frac{m!}{(l+m)!}S_1(l+m,m)(n)_{l+m}\left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right) \middle| x^{n-l-m} \right\rangle\\
&= \sum_{l=0}^{n-m} m!\binom{n}{l+m}S_1(l+m,m)D_{n-l-m}(a_1,\dots,a_r)\\
&= \sum_{l=0}^{n-m} m!\binom{n}{l}S_1(n-l,m)D_l(a_1,\dots,a_r)\,.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\big(\ln(1+t)\big)^m \middle| x^n \right\rangle\\
&= \left\langle \partial_t\left(\prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\big(\ln(1+t)\big)^m\right) \middle| x^{n-1} \right\rangle\\
&= \left\langle \left(\partial_t\prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\right)\big(\ln(1+t)\big)^m \middle| x^{n-1} \right\rangle\\
&\quad + \left\langle \prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\partial_t\left(\big(\ln(1+t)\big)^m\right) \middle| x^{n-1} \right\rangle\,. \qquad (39)
\end{aligned}$$

The second term of (39) is equal to

$$
\begin{aligned}
&\left\langle \prod_{j=1}^{r} \left( \frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) \partial_t \left( \left( \ln(1+t) \right)^m \right) \middle| x^{n-1} \right\rangle \\
&= m \left\langle \prod_{j=1}^{r} \left( \frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) (1+t)^{-1} \middle| \left( \ln(1+t) \right)^{m-1} x^{n-1} \right\rangle \\
&= m \left\langle \prod_{j=1}^{r} \left( \frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) (1+t)^{-1} \middle| \sum_{l=0}^{n-m} \frac{(m-1)!}{(l+m-1)!} S_1(l+m-1, m-1) t^{l+m-1} x^{n-1} \right\rangle \\
&= m \sum_{l=0}^{n-m} \frac{(m-1)!}{(l+m-1)!} S_1(l+m-1, m-1) (n-1)_{l+m-1} \\
&\quad \times \left\langle \prod_{j=1}^{r} \left( \frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) (1+t)^{-1} \middle| x^{n-l-m} \right\rangle \\
&= m! \sum_{l=0}^{n-m} \binom{n-1}{l+m-1} S_1(l+m-1, m-1) D_{n-l-m}(-1|a_1, \dots, a_r) \\
&= m! \sum_{l=0}^{n-m} \binom{n-1}{l} S_1(n-l-1, m-1) D_l(-1|a_1, \dots, a_r) \, .
\end{aligned}
$$

The first term of (39) is equal to

$$
\begin{aligned}
&\left\langle \left( \partial_t \prod_{j=1}^{r} \left( \frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) \right) \left( \ln(1+t) \right)^m \middle| x^{n-1} \right\rangle \\
&= \left\langle \partial_t \prod_{j=1}^{r} \left( \frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) \middle| \left( \ln(1+t) \right)^m x^{n-1} \right\rangle \\
&= \left\langle \partial_t \prod_{j=1}^{r} \left( \frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) \middle| \sum_{l=0}^{n-m-1} \frac{m!}{(l+m)!} S_1(l+m, m) t^{l+m} x^{n-1} \right\rangle \\
&= \sum_{l=0}^{n-m-1} \frac{m!}{(l+m)!} S_1(l+m, m) (n-1)_{l+m} \left\langle \partial_t \prod_{j=1}^{r} \left( \frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) \middle| x^{n-l-m-1} \right\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{l=0}^{n-m-1} m!\binom{n-1}{l+m} S_1(l+m,m) \\
&\quad \times \left\langle \prod_{i=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_i}-1}\right)(1+t)^{-1} \middle| \frac{\sum_{j=1}^{r}\left(\frac{t}{\ln(1+t)}-\frac{a_j t(1+t)^{a_j}}{(1+t)^{a_j}-1}\right)}{t} x^{n-l-m-1} \right\rangle \\
&= m!\sum_{l=0}^{n-m-1} \frac{1}{n-l-m}\binom{n-1}{l+m} S_1(l+m,m) \\
&\quad \times \left\langle \prod_{i=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_i}-1}\right)(1+t)^{-1} \middle| \sum_{j=1}^{r}\left(\frac{t}{\ln(1+t)}-\frac{a_j t(1+t)^{a_j}}{(1+t)^{a_j}-1}\right) x^{n-l-m} \right\rangle \\
&= \frac{m!}{n}\sum_{l=0}^{n-m-1} \binom{n}{l+1} S_1(n-1-l,m) \\
&\quad \times \left( r\left\langle \prod_{i=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_i}-1}\right)(1+t)^{-1} \middle| \frac{t}{\ln(1+t)} x^{l+1} \right\rangle \right. \\
&\qquad \left. - \left(\sum_{j=1}^{r} a_j\right)\left\langle \frac{\ln(1+t)}{(1+t)^{a_j}-1}(1+t)^{a_j-1}\prod_{i=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_i}-1}\right) \middle| \frac{t}{\ln(1+t)} x^{l+1} \right\rangle \right) \\
&= \frac{m!}{n}\sum_{l=0}^{n-m-1} \binom{n}{l+1} S_1(n-l-1,m) \\
&\quad \times \left( r\left\langle \prod_{i=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_i}-1}\right)(1+t)^{-1} \middle| \sum_{i=0}^{l+1} c_i \frac{t^i}{i!} x^{l+1} \right\rangle \right. \\
&\qquad \left. - \left(\sum_{j=1}^{r} a_j\right)\left\langle \frac{\ln(1+t)}{(1+t)^{a_j}-1}(1+t)^{a_j-1}\prod_{i=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_i}-1}\right) \middle| \sum_{i=0}^{l+1} c_i \frac{t^i}{i!} x^{l+1} \right\rangle \right) \\
&= \frac{m!}{n}\sum_{l=0}^{n-m-1} \binom{n}{l+1} S_1(n-l-1,m) \\
&\quad \times \left( r\sum_{i=0}^{l+1}\binom{l+1}{i} c_i D_{l+1-i}(-1|a_1,\dots,a_r) \right. \\
&\qquad \left. - \sum_{j=1}^{r} a_j \sum_{i=0}^{l+1}\binom{l+1}{i} c_i D_{l+1-i}(a_j-1|a_1,\dots,a_r,a_j) \right) .
\end{aligned}
$$

Therefore, we have, for $n-1 \geq m \geq 1$,

$$\begin{aligned}
& m!\sum_{l=0}^{n-m}\binom{n}{l}S_1(n-l,m)D_l(a_1,\dots,a_r) \\
&= m!\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)D_l(-1|a_1,\dots,a_r) \\
&\quad + \frac{m!}{n}\sum_{l=0}^{n-m-1}\binom{n}{l+1}S_1(n-l-1,m) \\
&\qquad \times \left(r\sum_{i=0}^{l+1}\binom{l+1}{i}c_{l+1-i}D_i(-1|a_1,\dots,a_r)\right. \\
&\qquad\quad \left. -\sum_{j=1}^{r}\sum_{i=0}^{l+1}a_j\binom{l+1}{i}c_{l+1-i}D_i(a_j-1|a_1,\dots,a_r,a_j)\right) .
\end{aligned}$$

Thus, we get (37).
Next, we shall compute

$$\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right)\big(\ln(1+t)\big)^m \middle| x^n\right\rangle$$

in two different ways. On the one hand,

$$\begin{aligned}
& \left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right)\big(\ln(1+t)\big)^m \middle| x^n\right\rangle \\
&= \left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right) \middle| \big(\ln(1+t)\big)^m x^n\right\rangle \\
&= \left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right) \middle| \sum_{l=0}^{\infty}\frac{m!}{(l+m)!}S_1(l+m,m)t^{l+m}x^n\right\rangle \\
&= \sum_{l=0}^{n-m}\frac{m!}{(l+m)!}S_1(l+m,m)(n)_{l+m}\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right) \middle| x^{n-l-m}\right\rangle \\
&= \sum_{l=0}^{n-m}m!\binom{n}{l+m}S_1(l+m,m)\widehat{D}_{n-l-m}(a_1,\dots,a_r) \\
&= \sum_{l=0}^{n-m}m!\binom{n}{l}S_1(n-l,m)\widehat{D}_l(a_1,\dots,a_r) .
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
&\left\langle \prod_{j=1}^r \left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right)\left(\ln(1+t)\right)^m \middle| x^n \right\rangle\\
&=\left\langle \partial_t\left(\prod_{j=1}^r \left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right)\left(\ln(1+t)\right)^m\right) \middle| x^{n-1} \right\rangle\\
&=\left\langle \left(\partial_t\prod_{j=1}^r \left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right)\right)\left(\ln(1+t)\right)^m \middle| x^{n-1} \right\rangle\\
&\quad+\left\langle \prod_{j=1}^r \left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right)\partial_t\left(\left(\ln(1+t)\right)^m\right) \middle| x^{n-1} \right\rangle . 
\end{aligned}
\tag{40}
$$

The second term of (40) is equal to

$$
\begin{aligned}
&\left\langle \prod_{j=1}^r \left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right)\partial_t\left(\left(\ln(1+t)\right)^m\right) \middle| x^{n-1} \right\rangle\\
&=m\left\langle \prod_{j=1}^r \left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right)(1+t)^{-1} \middle| \left(\ln(1+t)\right)^{m-1}x^{n-1} \right\rangle\\
&=m\left\langle \prod_{j=1}^r \left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right)(1+t)^{-1} \middle| \sum_{l=0}^{n-m}\frac{(m-1)!}{(l+m-1)!}S_1(l+m-1,m-1)t^{l+m-1}x^{n-1} \right\rangle\\
&=m\sum_{l=0}^{n-m}\frac{(m-1)!}{(l+m-1)!}S_1(l+m-1,m-1)(n-1)_{l+m-1}\\
&\quad\times\left\langle \prod_{j=1}^r \left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right)(1+t)^{-1} \middle| x^{n-l-m} \right\rangle\\
&=m!\sum_{l=0}^{n-m}\binom{n-1}{l+m-1}S_1(l+m-1,m-1)\widehat{D}_{n-l-m}(-1|a_1,\dots,a_r)\\
&=m!\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)\widehat{D}_l(-1|a_1,\dots,a_r)\,.
\end{aligned}
$$

The first term of (40) is equal to

$$\begin{aligned}
&\left\langle \left( \partial_t \prod_{j=1}^{r} \left( \frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j}-1} \right) \right) \big(\ln(1+t)\big)^m \middle| x^{n-1} \right\rangle \\
&= \left\langle \partial_t \prod_{j=1}^{r} \left( \frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j}-1} \right) \middle| \big(\ln(1+t)\big)^m x^{n-1} \right\rangle \\
&= \left\langle \partial_t \prod_{j=1}^{r} \left( \frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j}-1} \right) \middle| \sum_{l=0}^{n-m-1} \frac{m!}{(l+m)!} S_1(l+m,m) t^{l+m} x^{n-1} \right\rangle \\
&= \sum_{l=0}^{n-m-1} \frac{m!}{(l+m)!} S_1(l+m,m)(n-1)_{l+m} \left\langle \partial_t \prod_{j=1}^{r} \left( \frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j}-1} \right) \middle| x^{n-l-m-1} \right\rangle .
\end{aligned}$$

From the proof of (36), we recall

$$\begin{aligned}
\partial_t \prod_{j=1}^{r} \left( \frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j}-1} \right) &= \frac{1}{1+t} \prod_{i=1}^{r} \left( \frac{(1+t)^{a_i} \ln(1+t)}{(1+t)^{a_i}-1} \right) \frac{\sum_{j=1}^{r} \left( \frac{t}{\ln(1+t)} - \frac{a_j t (1+t)^{a_j}}{(1+t)^{a_j}-1} \right)}{t} \\
&\quad + \frac{1}{1+t} \prod_{i=1}^{r} \left( \frac{(1+t)^{a_i} \ln(1+t)}{(1+t)^{a_i}-1} \right) \sum_{j=1}^{r} a_j .
\end{aligned}$$

Hence, the first term of (39) is equal to

$$\sum_{l=0}^{n-m-1} m!\binom{n-1}{l+m} S_1(l+m,m)$$
$$\times \left( \left\langle \prod_{i=1}^{r} \left( \frac{(1+t)^{a_i}\ln(1+t)}{(1+t)^{a_i}-1} \right)(1+t)^{-1} \middle| \frac{\sum_{j=1}^{r}\left( \frac{t}{\ln(1+t)} - \frac{a_j t(1+t)^{a_j}}{(1+t)^{a_j}-1} \right)}{t} x^{n-l-m-1} \right\rangle \right.$$
$$\left. + \left( \sum_{j=1}^{r} a_j \right) \left\langle \prod_{i=1}^{r} \left( \frac{(1+t)^{a_i}\ln(1+t)}{(1+t)^{a_i}-1} \right)(1+t)^{-1} \middle| x^{n-l-m-1} \right\rangle \right)$$
$$= m! \sum_{l=0}^{n-m-1} \binom{n-1}{l} S_1(n-l-1,m)$$
$$\times \left( \left\langle \prod_{i=1}^{r} \left( \frac{(1+t)^{a_i}\ln(1+t)}{(1+t)^{a_i}-1} \right)(1+t)^{-1} \middle| \frac{\sum_{j=1}^{r}\left( \frac{t}{\ln(1+t)} - \frac{a_j t(1+t)^{a_j}}{(1+t)^{a_j}-1} \right)}{t} x^{l} \right\rangle \right.$$
$$\left. + \left( \sum_{j=1}^{r} a_j \right) \left\langle \prod_{i=1}^{r} \left( \frac{(1+t)^{a_i}\ln(1+t)}{(1+t)^{a_i}-1} \right)(1+t)^{-1} \middle| x^{l} \right\rangle \right)$$
$$= m! \sum_{l=0}^{n-m-1} \binom{n-1}{l} S_1(n-l-1,m)$$
$$\times \left( \frac{1}{l+1} \left\langle \prod_{i=1}^{r} \left( \frac{(1+t)^{a_i}\ln(1+t)}{(1+t)^{a_i}-1} \right)(1+t)^{-1} \middle| \sum_{j=1}^{r}\left( \frac{t}{\ln(1+t)} - \frac{a_j t(1+t)^{a_j}}{(1+t)^{a_j}-1} \right) x^{l+1} \right\rangle \right.$$
$$\left. + \left( \sum_{j=1}^{r} a_j \right) \left\langle \prod_{i=1}^{r} \left( \frac{(1+t)^{a_i}\ln(1+t)}{(1+t)^{a_i}-1} \right)(1+t)^{-1} \middle| x^{l} \right\rangle \right)$$
$$= m! \sum_{l=0}^{n-m-1} \binom{n-1}{l} S_1(n-l-1,m)$$
$$\times \left( \frac{r}{l+1} \left\langle \prod_{i=1}^{r} \left( \frac{(1+t)^{a_i}\ln(1+t)}{(1+t)^{a_i}-1} \right)(1+t)^{-1} \middle| \frac{t}{\ln(1+t)} x^{l+1} \right\rangle \right.$$
$$- \frac{1}{l+1} \left( \sum_{j=1}^{r} a_j \right) \left\langle \frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1} \prod_{i=1}^{r} \left( \frac{(1+t)^{a_i}\ln(1+t)}{(1+t)^{a_i}-1} \right)(1+t)^{-1} \middle| \frac{t}{\ln(1+t)} x^{l+1} \right\rangle$$
$$\left. + \left( \sum_{j=1}^{r} a_j \right) \left\langle \prod_{i=1}^{r} \left( \frac{(1+t)^{a_i}\ln(1+t)}{(1+t)^{a_i}-1} \right)(1+t)^{-1} \middle| x^{l} \right\rangle \right)$$

$$
\begin{aligned}
&= m! \sum_{l=0}^{n-m-1} \binom{n-1}{l} S_1(n-l-1,m) \\
&\quad \times \left( \frac{r}{l+1} \left\langle \prod_{i=1}^{r} \left( \frac{(1+t)^{a_i}\ln(1+t)}{(1+t)^{a_i}-1} \right) (1+t)^{-1} \middle| \sum_{i=0}^{l+1} c_i \frac{t^i}{i!} x^{l+1} \right\rangle \right. \\
&\quad - \frac{1}{l+1} \left( \sum_{j=1}^{r} a_j \right) \left\langle \frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1} \prod_{i=1}^{r} \left( \frac{(1+t)^{a_i}\ln(1+t)}{(1+t)^{a_i}-1} \right) (1+t)^{-1} \middle| \sum_{i=0}^{l+1} c_i \frac{t^i}{i!} x^{l+1} \right\rangle \\
&\quad \left. + \left( \sum_{j=1}^{r} a_j \right) \left\langle \prod_{i=1}^{r} \left( \frac{(1+t)^{a_i}\ln(1+t)}{(1+t)^{a_i}-1} \right) (1+t)^{-1} \middle| x^l \right\rangle \right) \\
&= m! \sum_{l=0}^{n-m-1} \binom{n-1}{l} S_1(n-l-1,m) \\
&\quad \times \left( \frac{r}{l+1} \sum_{i=0}^{l+1} \binom{l+1}{i} c_i \widehat{D}_{l+1-i}(-1|a_1,\dots,a_r) \right. \\
&\quad \left. - \frac{1}{l+1} \sum_{j=1}^{r} a_j \sum_{i=0}^{l+1} \binom{l+1}{i} c_i \widehat{D}_{l+1-i}(-1|a_1,\dots,a_r,a_j) + \sum_{j=1}^{r} a_j \widehat{D}_l(-1|a_1,\dots,a_r) \right) \\
&= \frac{m!}{n} \sum_{l=0}^{n-m-1} \binom{n}{l+1} S_1(n-l-1,m) \\
&\quad \times \left( r \sum_{i=0}^{l+1} \binom{l+1}{i} c_i \widehat{D}_{l+1-i}(-1|a_1,\dots,a_r) \right. \\
&\quad \left. - \sum_{j=1}^{r} \sum_{i=0}^{l+1} \binom{l+1}{i} a_j c_i \widehat{D}_{l+1-i}(-1|a_1,\dots,a_r,a_j) \right) \\
&\quad + m! \sum_{l=0}^{n-m-1} \binom{n-1}{l} S_1(n-l-1,m) \sum_{j=1}^{r} a_j \widehat{D}_l(-1|a_1,\dots,a_r) \,.
\end{aligned}
$$

Therefore, we get (38). ∎

## 3.8 Relations with the falling factorials

**Theorem 8**

$$
D_n(x|a_1,\dots,a_r) = \sum_{m=0}^{n} \binom{n}{m} D_{n-m}(a_1,\dots,a_r)(x)_m \,, \tag{41}
$$

$$
\widehat{D}_n(x|a_1,\dots,a_r) = \sum_{m=0}^{n} \binom{n}{m} \widehat{D}_{n-m}(a_1,\dots,a_r)(x)_m \,. \tag{42}
$$

*Proof.* For (13) and (23), assume that $D_n(x|a_1,\dots,a_r)=\sum_{m=0}^n C_{n,m}(x)_m$. By (12), we have

$$\begin{aligned}C_{n,m}&=\frac{1}{m!}\left\langle\frac{1}{\prod_{j=1}^r\left(\frac{e^{a_j\ln(1+t)}-1}{\ln(1+t)}\right)}t^m\middle|x^n\right\rangle\\&=\frac{1}{m!}\left\langle\prod_{j=1}^r\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\middle|t^m x^n\right\rangle\\&=\binom{n}{m}\left\langle\prod_{j=1}^r\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\middle|x^{n-m}\right\rangle\\&=\binom{n}{m}D_{n-m}(a_1,\dots,a_r)\,.\end{aligned}$$

Thus, we get the identity (41).
Similarly, for (13) and (23), assume that $\widehat{D}_n(x|a_1,\dots,a_r)=\sum_{m=0}^n C_{n,m}(x)_m$. By (12), we have

$$\begin{aligned}C_{n,m}&=\frac{1}{m!}\left\langle\frac{1}{\prod_{j=1}^r\left(\frac{e^{a_j\ln(1+t)}-1}{e^{a_j\ln(1+t)}\ln(1+t)}\right)}t^m\middle|x^n\right\rangle\\&=\frac{1}{m!}\left\langle\prod_{j=1}^r\left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right)\middle|t^m x^n\right\rangle\\&=\binom{n}{m}\left\langle\prod_{j=1}^r\left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right)\middle|x^{n-m}\right\rangle\\&=\binom{n}{m}\widehat{D}_{n-m}(a_1,\dots,a_r)\,.\end{aligned}$$

Thus, we get the identity (42). ∎

### 3.9 Relations with higher-order Frobenius-Euler polynomials

For $\lambda\in\mathbb{C}$ with $\lambda\neq 1$, the Frobenius-Euler polynomials of order $r$, $H_n^{(r)}(x|\lambda)$ are defined by the generating function

$$\left(\frac{1-\lambda}{e^t-\lambda}\right)^r e^{xt}=\sum_{n=0}^\infty H_n^{(r)}(x|\lambda)\frac{t^n}{n!}$$

(see e.g. [4, 6]).

**Theorem 9**

$$D_n(x|a_1,\dots,a_r)=\sum_{m=0}^{n}\left(\sum_{j=0}^{n-m}\sum_{l=0}^{n-m-j}\binom{s}{j}\binom{n-j}{l}(n)_j\right.$$
$$\left.\times(1-\lambda)^{-j}S_1(n-j-l,m)D_l(a_1,\dots,a_r)\right)H_m^{(s)}(x|\lambda)\,, \tag{43}$$
$$\widehat{D}_n(x|a_1,\dots,a_r)=\sum_{m=0}^{n}\left(\sum_{j=0}^{n-m}\sum_{l=0}^{n-m-j}\binom{s}{j}\binom{n-j}{l}(n)_j\right.$$
$$\left.\times(1-\lambda)^{-j}S_1(n-j-l,m)\widehat{D}_l(a_1,\dots,a_r)\right)H_m^{(s)}(x|\lambda)\,. \tag{44}$$

*Proof.* For (13) and

$$H_n^{(s)}(x|\lambda)\sim\left(\left(\frac{e^t-\lambda}{1-\lambda}\right)^s,t\right)\,, \tag{45}$$

assume that $D_n(x|a_1,\dots,a_r)=\sum_{m=0}^{n}C_{n,m}H_m^{(s)}(x|\lambda)$. By (12), similarly to the proof of (37), we have

$$\begin{aligned}C_{n,m}&=\frac{1}{m!}\left\langle\frac{\left(\frac{e^{\ln(1+t)}-\lambda}{1-\lambda}\right)^s}{\prod_{j=1}^{r}\left(\frac{e^{a_j\ln(1+t)}-1}{\ln(1+t)}\right)}\bigl(\ln(1+t)\bigr)^m\middle|x^n\right\rangle\\
&=\frac{1}{m!(1-\lambda)^s}\left\langle\prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\bigl(\ln(1+t)\bigr)^m(1-\lambda+t)^s\middle|x^n\right\rangle\\
&=\frac{1}{m!(1-\lambda)^s}\left\langle\prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\bigl(\ln(1+t)\bigr)^m\middle|\sum_{i=0}^{\min\{s,n\}}\binom{s}{i}(1-\lambda)^{s-i}t^ix^n\right\rangle\\
&=\frac{1}{m!(1-\lambda)^s}\sum_{i=0}^{n-m}\binom{s}{i}(1-\lambda)^{s-i}(n)_i\left\langle\prod_{j=1}^{r}\left(\frac{\ln(1+t)}{(1+t)^{a_j}-1}\right)\middle|\bigl(\ln(1+t)\bigr)^m x^{n-i}\right\rangle\\
&=\frac{1}{m!(1-\lambda)^s}\sum_{i=0}^{n-m}\binom{s}{i}(1-\lambda)^{s-i}(n)_i\sum_{l=0}^{n-m-i}m!\binom{n-i}{l}S_1(n-i-l,m)D_l(a_1,\dots,a_r)\\
&=\sum_{i=0}^{n-m}\sum_{l=0}^{n-m-i}\binom{s}{i}\binom{n-i}{l}(n)_i(1-\lambda)^{-i}S_1(n-i-l,m)D_l(a_1,\dots,a_r)\,.\end{aligned}$$

Thus, we get the identity (43).
Next, for (14) and (45), assume that $\widehat{D}_n(x|a_1,\dots,a_r)=\sum_{m=0}^{n}C_{n,m}H_m^{(s)}(x|\lambda)$. By (12),

similarly to the proof of (38), we have

$$\begin{aligned}
C_{n,m} &= \frac{1}{m!}\left\langle \frac{\left(\frac{e^{\ln(1+t)}-\lambda}{1-\lambda}\right)^s}{\prod_{j=1}^r\left(\frac{e^{a_j\ln(1+t)}-1}{e^{a_j\ln(1+t)}\ln(1+t)}\right)}\bigl(\ln(1+t)\bigr)^m \middle| x^n\right\rangle\\
&= \frac{1}{m!(1-\lambda)^s}\left\langle \prod_{j=1}^r\left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right)\bigl(\ln(1+t)\bigr)^m \middle| (1-\lambda+t)^s x^n\right\rangle\\
&= \frac{1}{m!(1-\lambda)^s}\left\langle \prod_{j=1}^r\left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right)\bigl(\ln(1+t)\bigr)^m \middle| \sum_{i=0}^{\min\{s,n\}}\binom{s}{i}(1-\lambda)^{s-i}t^i x^n\right\rangle\\
&= \frac{1}{m!(1-\lambda)^s}\sum_{i=0}^{n-m}\binom{s}{i}(1-\lambda)^{s-i}(n)_i\left\langle \prod_{j=1}^r\left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right) \middle| \bigl(\ln(1+t)\bigr)^m x^{n-i}\right\rangle\\
&= \frac{1}{m!(1-\lambda)^s}\sum_{i=0}^{n-m}\binom{s}{i}(1-\lambda)^{s-i}(n)_i\sum_{l=0}^{n-m-i} m!\binom{n-i}{l}S_1(n-i-l,m)\widehat{D}_l(a_1,\dots,a_r)\\
&= \sum_{i=0}^{n-m}\sum_{l=0}^{n-m-i}\binom{s}{i}\binom{n-i}{l}(n)_i(1-\lambda)^{-i}S_1(n-i-l,m)\widehat{D}_l(a_1,\dots,a_r)\,.
\end{aligned}$$

Thus, we get the identity (44). ∎

## 3.10 Relations with higher-order Bernoulli polynomials

Bernoulli polynomials $\mathfrak{B}_n^{(r)}(x)$ of order $r$ are defined by

$$\left(\frac{t}{e^t-1}\right)^r e^{xt} = \sum_{n=0}^\infty \frac{\mathfrak{B}_n^{(r)}(x)}{n!}t^n$$

(see e.g. [14, Section 2.2]). In addition, Cauchy numbers of the first kind $\mathfrak{C}_n^{(r)}$ of order $r$ are defined by

$$\left(\frac{t}{\ln(1+t)}\right)^r = \sum_{n=0}^\infty \frac{\mathfrak{C}_n^{(r)}}{n!}t^n$$

(see e.g. [2, (2.1)], [11, (6)]).

**Theorem 10**

$$\begin{aligned}
&D_n(x|a_1,\dots,a_r)\\
&= \sum_{m=0}^n\left(\sum_{i=0}^{n-m}\sum_{l=0}^{n-m-i}\binom{n}{i}\binom{n-i}{l}\mathfrak{C}_i^{(s)}S_1(n-i-l,m)D_l(a_1,\dots,a_r)\right)\mathfrak{B}_m^{(s)}(x)\,, \qquad (46)\\
&\widehat{D}_n(x|a_1,\dots,a_r)\\
&= \sum_{m=0}^n\left(\sum_{i=0}^{n-m}\sum_{l=0}^{n-m-i}\binom{n}{i}\binom{n-i}{l}\mathfrak{C}_i^{(s)}S_1(n-i-l,m)\widehat{D}_l(a_1,\dots,a_r)\right)\mathfrak{B}_m^{(s)}(x)\,. \qquad (47)
\end{aligned}$$

*Proof.* For (13) and

$$\mathfrak{B}_n^{(s)}(x) \sim \left( \left( \frac{e^t - 1}{t} \right)^s, t \right), \tag{48}$$

assume that $D_n(x|a_1, \dots, a_r) = \sum_{m=0}^{n} C_{n,m} \mathfrak{B}_m^{(s)}(x)$. By (12), similarly to the proof of (37), we have

$$\begin{aligned}
C_{n,m} &= \frac{1}{m!} \left\langle \frac{\left( \frac{e^{\ln(1+t)} - 1}{\ln(1+t)} \right)^s}{\prod_{j=1}^{r} \left( \frac{e^{a_j \ln(1+t)} - 1}{\ln(1+t)} \right)} \big(\ln(1+t)\big)^m \middle| x^n \right\rangle \\
&= \frac{1}{m!} \left\langle \prod_{j=1}^{r} \left( \frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \big(\ln(1+t)\big)^m \middle| \left( \frac{t}{\ln(1+t)} \right)^s x^n \right\rangle \\
&= \frac{1}{m!} \left\langle \prod_{j=1}^{r} \left( \frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \big(\ln(1+t)\big)^m \middle| \sum_{i=0}^{\infty} \mathfrak{C}_i^{(s)} \frac{t^i}{i!} x^n \right\rangle \\
&= \frac{1}{m!} \sum_{i=0}^{n-m} \mathfrak{C}_i^{(s)} \binom{n}{i} \left\langle \prod_{j=1}^{r} \left( \frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \big(\ln(1+t)\big)^m \middle| x^{n-i} \right\rangle \\
&= \frac{1}{m!} \sum_{i=0}^{n-m} \mathfrak{C}_i^{(s)} \binom{n}{i} \sum_{l=0}^{n-m-i} m! \binom{n-i}{l} S_1(n-i-l, m) D_l(a_1, \dots, a_r) \\
&= \sum_{i=0}^{n-m} \sum_{l=0}^{n-m-i} \binom{n}{i} \binom{n-i}{l} \mathfrak{C}_i^{(s)} S_1(n-i-l, m) D_l(a_1, \dots, a_r).
\end{aligned}$$

Thus, we get the identity (46).
Next, for (13) and (48), assume that $\widehat{D}_n(x|a_1, \dots, a_r) = \sum_{m=0}^{n} C_{n,m} \mathfrak{B}_m^{(s)}(x)$. By (12),

similarly to the proof of (38), we have

$$\begin{aligned}
C_{n,m} &= \frac{1}{m!}\left\langle \frac{\left(\frac{e^{\ln(1+t)}-1}{\ln(1+t)}\right)^s}{\prod_{j=1}^r\left(\frac{e^{a_j\ln(1+t)}-1}{e^{a_j\ln(1+t)}\ln(1+t)}\right)}\left(\ln(1+t)\right)^m \middle| x^n\right\rangle \\
&= \frac{1}{m!}\left\langle \prod_{j=1}^r\left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right)\left(\ln(1+t)\right)^m \middle| \left(\frac{t}{\ln(1+t)}\right)^s x^n\right\rangle \\
&= \frac{1}{m!}\left\langle \prod_{j=1}^r\left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right)\left(\ln(1+t)\right)^m \middle| \sum_{i=0}^\infty \mathfrak{C}_i^{(s)}\frac{t^i}{i!}x^n\right\rangle \\
&= \frac{1}{m!}\sum_{i=0}^{n-m}\mathfrak{C}_i^{(s)}\binom{n}{i}\left\langle \prod_{j=1}^r\left(\frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\right)\left(\ln(1+t)\right)^m \middle| x^{n-i}\right\rangle \\
&= \frac{1}{m!}\sum_{i=0}^{n-m}\mathfrak{C}_i^{(s)}\binom{n}{i}\sum_{l=0}^{n-m-i} m!\binom{n-i}{l}S_1(n-i-l,m)\widehat{D}_l(a_1,\dots,a_r) \\
&= \sum_{i=0}^{n-m}\sum_{l=0}^{n-m-i}\binom{n}{i}\binom{n-i}{l}\mathfrak{C}_i^{(s)}S_1(n-i-l,m)\widehat{D}_l(a_1,\dots,a_r)\,.
\end{aligned}$$

Thus, we get the identity (47). ∎

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