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Some properties of higher-order Daehee polynomials of the second kind arising from umbral calculus

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Abstract

In this paper, we study the higher-order Daehee polynomials of the second kind from the umbral calculus viewpoint and give various identities of the higher-order Daehee polynomials of the second kind arising from umbral calculus.

1 Introduction

Let $k \in \mathbb{Z}_{\geq 0}$. The Daehee polynomials of the second kind of order k are defined by the generating function to be

$$\left(\frac{(1+t)\log(1+t)}{t} \right)^k (1+t)^x = \sum_{n=0}^{\infty} \hat{D}_n^{(k)}(x) \frac{t^n}{n!} \quad (1)$$

(see [1]).

When $x = 0$, $\hat{D}_n^{(k)} = \hat{D}_n^{(k)}(0)$ are called the Daehee numbers of the second kind of order k . The Stirling number of the first kind is defined by the falling factorial to be

$$(x)_n = x(x-1) \cdots (x-n+1) = \sum_{l=0}^n S_1(n, l) x^l. \quad (2)$$

Thus, by (2), we get

$$(\log(1+t))^m = m! \sum_{l=m}^{\infty} S_1(l, m) \frac{t^l}{l!} \quad (3)$$

(see [2–4]), where $m \in \mathbb{Z}_{\geq 0}$.

For $\lambda \in \mathbb{C}$ with $\lambda \neq 1$, the Frobenius-Euler polynomials of order s ($\in \mathbb{N}$) are given by

$$\left(\frac{1-\lambda}{e^t - \lambda} \right)^s e^{xt} = \sum_{n=0}^{\infty} H_n^{(s)}(x|\lambda) \frac{t^n}{n!} \quad (4)$$

(see [1–18]).

When $x = 0$, $H_n^{(s)}(\lambda) = H_n^{(s)}(\lambda|0)$ are called the Frobenius-Euler numbers of order s .

As is well known, the Bernoulli polynomials of order $k \in \mathbb{N}$ are defined by the generating function to be

$$\left(\frac{t}{e^t - 1}\right)^k e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!} \tag{5}$$

(see [1–18]).

When $x = 0$, $B_n^{(k)} = B_n^{(k)}(0)$ are called the Bernoulli numbers of order k .

In this paper, we study the higher-order Daehee polynomials of the second kind with umbral calculus viewpoint and give various identities of the higher-order Daehee polynomials of the second kind arising from umbral calculus.

2 Umbral calculus

Let \mathbb{C} be the complex number field and let \mathcal{F} be the set of all formal power series

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \mid a_k \in \mathbb{C} \right\}.$$

Let $\mathbb{P} = \mathbb{C}[x]$, and let \mathbb{P}^* be the vector space of all linear functionals on \mathbb{P} . $\langle L|p(x) \rangle$ indicates the action of the linear functional L on the polynomial $p(x)$. Then the vector space operations on \mathbb{P}^* are given by $\langle L + M|p(x) \rangle = \langle L|p(x) \rangle + \langle M|p(x) \rangle$, and $\langle cL|p(x) \rangle = c\langle L|p(x) \rangle$, where c is a complex constant in \mathbb{C} . For $f(t) \in \mathcal{F}$, the linear functional on \mathbb{P} is defined by $\langle f(t)|x^n \rangle = a_n$. Then, in particular, we have

$$\langle t^k|x^n \rangle = n! \delta_{n,k} \quad (n, k \geq 0) \tag{6}$$

(see [3, 18]), where $\delta_{n,k}$ is the Kronecker symbol.

Let $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L|x^k \rangle}{k!} t^k$. By (6), we get $\langle f_L(t)|x^n \rangle = \langle L|x^n \rangle$. That is, $L = f_L(t)$. The map $L \mapsto f_L(t)$ is a vector space isomorphism from \mathbb{P}^* onto \mathcal{F} . Henceforth, \mathcal{F} denotes both the algebra of the formal power series in t and the vector space of all linear functionals on \mathbb{P} , and so an element $f(t)$ of \mathcal{F} will be thought of as both a formal power series and a linear functional. We call \mathcal{F} the umbral algebra and the umbral calculus is the study of the umbral algebra. The order $o(f(t))$ of the power series $f(t) (\neq 0)$ is the smallest integer for which the coefficient of t^k does not vanish. If $o(f(t)) = 0$, then $f(t)$ is called an invertible series; if $o(f(t)) = 1$, then $f(t)$ is called a delta series.

Let $f(t), g(t) \in \mathcal{F}$ with $o(f(t)) = 1$ and $o(g(t)) = 0$. Then there exists a unique sequence $s_n(x)$ ($\deg s_n(x) = n$) such that $\langle g(t)f(t)^k|s_n(x) \rangle = n! \delta_{n,k}$, for $n, k \geq 0$. The sequence $s_n(x)$ is called the Sheffer sequence for $(g(t), f(t))$ which is denoted by $s_n(x) \sim (g(t), f(t))$. For $f(t), g(t) \in \mathcal{F}$, we have

$$\langle f(t)g(t)|p(x) \rangle = \langle f(t)|g(t)p(x) \rangle = \langle g(t)|f(t)p(x) \rangle. \tag{7}$$

From (6), we note that

$$f(t) = \sum_{k=0}^{\infty} \langle f(t)|x^k \rangle \frac{t^k}{k!}, \quad p(x) = \sum_{k=0}^{\infty} \langle t^k|p(x) \rangle \frac{x^k}{k!} \tag{8}$$

and, by (8), we get

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k} \quad \text{and} \quad e^{yt} p(x) = p(x+y) \tag{9}$$

(see [3, 18]).

For $s_n(x) \sim (g(t), f(t))$, we have

$$\frac{ds_n(x)}{dx} = \sum_{l=0}^{n-1} \binom{n}{l} \langle \bar{f}(t) | x^{n-l} \rangle s_l(x), \tag{10}$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$ with $\bar{f}(f(t)) = t$. We have

$$\frac{1}{g(\bar{f}(t))} e^{x\bar{f}(t)} = \sum_{n=0}^{\infty} s_n(x) \frac{t^n}{n!}, \quad \text{for all } x \in \mathbb{C}, \tag{11}$$

$$f(t)s_n(x) = ns_{n-1}(x) \quad (n \geq 1), \quad s_n(x) = \sum_{j=0}^n \frac{1}{j!} \langle g(\bar{f}(t))^{-1} \bar{f}(t)^j | x^n \rangle x^j, \tag{12}$$

$$s_n(x+y) = \sum_{j=0}^n \binom{n}{j} s_j(x) p_{n-j}(y), \tag{13}$$

where $p_n(x) = g(t)s_n(x)$.

$$\langle f(t) | xp(x) \rangle = \langle \partial_t f(t) | p(x) \rangle, \tag{14}$$

with $\partial_t f(t) = \frac{df(t)}{dt}$, and

$$s_{n+1}(x) = \left(x - \frac{g'(t)}{g(t)} \right) \frac{1}{f'(t)} s_n(x) \quad (n \geq 0) \tag{15}$$

(see [3, 18]).

Let us assume that $s_n(x) \sim (g(t), f(t))$ and $r_n(x) \sim (h(t), l(t))$. Then we see that

$$s_n(x) = \sum_{m=0}^n C_{n,m} r_m(x) \quad (n \geq 0), \tag{16}$$

where

$$C_{n,m} = \frac{1}{m!} \left\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))} l(\bar{f}(t))^m | x^n \right\rangle \tag{17}$$

(see [3, 18]).

3 Higher-order Daehee polynomials of the second kind

By (1), we see that

$$\hat{D}_n^{(k)}(x) \sim \left(\left(\frac{e^t - 1}{te^t} \right)^k, e^t - 1 \right). \tag{18}$$

From (18), we have

$$\left(\frac{e^t - 1}{te^t}\right)^k \hat{D}_n^{(k)}(x) \sim (1, e^t - 1) \quad \text{and} \quad (x)_n \sim (1, e^t - 1). \tag{19}$$

By (19), we get

$$\begin{aligned} \hat{D}_n^{(k)}(x) &= \left(\frac{te^t}{e^t - 1}\right)^k (x)_n \\ &= \sum_{m=0}^n S_1(n, m) \left(\frac{te^t}{e^t - 1}\right)^k x^m \\ &= \sum_{m=0}^n S_1(n, m) e^{kt} B_n^{(k)}(x) \\ &= \sum_{m=0}^n S_1(n, m) B_m^{(k)}(x + k). \end{aligned} \tag{20}$$

From (12) and (18), we have

$$\hat{D}_n^{(k)}(x) = \sum_{j=0}^n \frac{1}{j!} \left\langle \left(\frac{(1+t)\log(1+t)}{t}\right)^k (\log(1+t))^j \middle| x^n \right\rangle x^j, \tag{21}$$

where

$$\begin{aligned} &\left\langle \left(\frac{(1+t)\log(1+t)}{t}\right)^k (\log(1+t))^j \middle| x^n \right\rangle \\ &= \left\langle \left(\frac{\log(1+t)}{t}\right)^{k+j} (1+t)^k \middle| t^j x^n \right\rangle \\ &= (n)_j \left\langle \left(\frac{\log(1+t)}{t}\right)^{k+j} \middle| \sum_{m=0}^{\min\{k, n-j\}} \binom{k}{m} t^m x^{n-j} \right\rangle \\ &= (n)_j \sum_{m=0}^{n-j} \binom{k}{m} (n-j)_m \sum_{l=0}^{\infty} \frac{(k+j)!}{(l+k+j)!} S_1(l+k+j, k+j) (t^l | x^{n-j-m}) \\ &= (n)_j \sum_{m=0}^{n-j} \binom{k}{m} (n-j)_m \frac{(k+j)!}{(n+k-m)!} S_1(n+k-m, k+j) (n-j-m)! \\ &= (n)_j \sum_{m=0}^{n-j} \binom{k}{m} (n-j)_m \frac{S_1(n+k-m, k+j)}{\binom{n+k-m}{k+j}}. \end{aligned} \tag{22}$$

Therefore, by (21) and (22), we obtain the following theorem.

Theorem 1 For $n \in \mathbb{Z}_{\geq 0}$ and $k \geq 1$, we have

$$\hat{D}_n^{(k)}(x) = \sum_{j=0}^n \left\{ \binom{n}{j} \sum_{m=0}^{n-j} \binom{k}{m} (n-j)_m \frac{S_1(n+k-m, k+j)}{\binom{n+k-m}{k+j}} \right\} x^j.$$

By (1) and (6), we get

$$\begin{aligned}
 \hat{D}_n^{(k)}(y) &= \left\langle \sum_{l=0}^{\infty} \hat{D}_l^{(k)}(y) \frac{t^l}{l!} \middle| x^n \right\rangle \\
 &= \left\langle \left(\frac{\log(1+t)}{t} \right)^k (1+t)^y \middle| (1+t)^k x^n \right\rangle \\
 &= \sum_{0 \leq r \leq \min\{k,n\}} \binom{k}{r} (n)_r \left\langle \left(\frac{\log(1+t)}{t} \right)^k (1+t)^y \middle| x^{n-r} \right\rangle \\
 &= \sum_{0 \leq r \leq \min\{k,n\}} \binom{k}{r} (n)_r \sum_{0 \leq m \leq n-r} \binom{y}{m} (n-r)_m \\
 &\quad \times \sum_{0 \leq l \leq n-r-m} \frac{k! S_1(l+k, k)}{(l+k)!} \langle t^l \middle| x^{n-r-m} \rangle \\
 &= \sum_{0 \leq r \leq n} \sum_{0 \leq m \leq n-r} \frac{(n)_r \binom{k}{r} \binom{n-r}{m}}{\binom{n-r-m+k}{k}} S_1(n-r-m+k, k) (y)_m. \tag{23}
 \end{aligned}$$

Therefore, by (23), we obtain the following theorem.

Theorem 2 For $n \geq 0$, we have

$$\begin{aligned}
 \hat{D}_n^{(k)}(x) &= \sum_{0 \leq m \leq n} \left\{ \sum_{0 \leq r \leq n-m} \frac{(n)_r \binom{k}{r} \binom{n-r}{m}}{\binom{n-r-m+k}{k}} S_1(n-r-m+k, k) \right\} (x)_m \\
 &= \sum_{0 \leq m \leq n} \left\{ \sum_{0 \leq r \leq n-m} \frac{(n)_r \binom{k}{r} \binom{n-r}{n-m}}{\binom{m-r+k}{k}} S_1(m-r+k, k) \right\} (x)_{n-m}.
 \end{aligned}$$

From (12) and (18), we have

$$(e^t - 1) \hat{D}_n^{(k)}(x) = n \hat{D}_{n-1}^{(k)}(x) \tag{24}$$

and

$$(e^t - 1) \hat{D}_n^{(k)}(x) = \hat{D}_n^{(k)}(x+1) - \hat{D}_n^{(k)}(x).$$

Thus, by (24), we get

$$\hat{D}_n^{(k)}(x+1) - \hat{D}_n^{(k)}(x) = n \hat{D}_{n-1}^{(k)}(x) \quad (n \geq 1). \tag{25}$$

From (15) and (18), we derive the following equation:

$$\begin{aligned}
 \hat{D}_{n+1}^{(k)}(x) &= \left(x + k \frac{e^t - 1 - t}{t(e^t - 1)} \right) e^{-t} \hat{D}_n^{(k)}(x) \\
 &= x \hat{D}_n^{(k)}(x-1) + k e^{-t} \frac{e^t - 1 - t}{t(e^t - 1)} \hat{D}_n^{(k)}(x), \tag{26}
 \end{aligned}$$

where

$$\begin{aligned}
 & e^{-t} \frac{e^t - 1 - t}{t(e^t - 1)} \hat{D}_n^{(k)}(x) \\
 &= e^{-t} \frac{e^t - 1 - t}{t(e^t - 1)} \sum_{0 \leq j \leq n} \left\{ \binom{n}{j} \sum_{0 \leq m \leq n-j} \frac{m! \binom{k}{m} \binom{n-j}{m}}{\binom{n+k-m}{k+j}} \right. \\
 &\quad \left. \times S_1(n+k-m, k+j) \right\} x^j \\
 &= \sum_{0 \leq j \leq n} \binom{n}{j} \sum_{0 \leq m \leq n-j} \frac{m! \binom{k}{m} \binom{n-j}{m}}{\binom{n+k-m}{k+j}} \\
 &\quad \times S_1(n+k-m, k+j) e^{-t} \frac{e^t - 1 - t}{t(e^t - 1)} x^j \\
 &= \sum_{0 \leq j \leq n} \binom{n}{j} \sum_{0 \leq m \leq n-j} \frac{m! \binom{m+k}{m} \binom{n-j}{m}}{\binom{n+k-m}{k+j}} \\
 &\quad \times S_1(n+k-m, k+j) e^{-t} \left(\frac{e^t - 1 - t}{e^t - 1} \right) \frac{x^{j+1}}{j+1} \\
 &= \sum_{0 \leq j \leq n} \binom{n}{j} \sum_{0 \leq m \leq n-j} \frac{m! \binom{m+k}{m} \binom{n-j}{m}}{\binom{n+k-m}{k+j}} \\
 &\quad \times \frac{S_1(n+k-m, k+j)}{j+1} e^{-t} (x^{j+1} - B_{j+1}(x)) \\
 &= \sum_{0 \leq j \leq n} \binom{n}{j} \sum_{0 \leq m \leq n-j} \frac{m! \binom{m+k}{m} \binom{n-j}{m}}{\binom{n+k-m}{k+j}} \\
 &\quad \times \frac{S_1(n+k-m, k+j)}{j+1} e^{-t} ((x-1)^{j+1} - B_{j+1}(x-1)). \tag{27}
 \end{aligned}$$

Therefore, from (26) and (27), we obtain the following theorem.

Theorem 3 For $n \geq 0, k \geq 1$, we have

$$\begin{aligned}
 & \hat{D}_{n+1}^{(k)}(x) \\
 &= x \hat{D}_n^{(k)}(x-1) + k \sum_{0 \leq j \leq n} \binom{n}{j} \sum_{0 \leq m \leq n-j} \frac{m! \binom{m+k}{m} \binom{n-j}{m}}{\binom{n+k-m}{k+j}} \\
 &\quad \times \frac{S_1(n+k-m, k+j)}{j+1} \{(x-1)^{j+1} - B_{j+1}(x-1)\}.
 \end{aligned}$$

Now, we observe that

$$\begin{aligned}
 & e^{-t} \frac{e^t - 1 - t}{t(e^t - 1)} \hat{D}_n^{(k)}(x) \\
 &= \sum_{j=0}^n \frac{\binom{n}{j}}{\binom{n+k}{j+k}} S_1(n+k, j+k) e^{-t} \frac{e^t - 1 - t}{t(e^t - 1)} (x+k)^j
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{j=0}^n \frac{\binom{n}{j}}{\binom{n+k}{j+k}} S_1(n+k, j+k) e^{(k-1)t} \frac{e^t - 1 - t}{t(e^t - 1)} x^j \\
 &= \sum_{j=0}^n \frac{\binom{n}{j}}{\binom{n+k}{j+k}} \frac{S_1(n+k, j+k)}{j+1} e^{(k-1)t} (x^{j+1} - B_{j+1}(x)) \\
 &= \sum_{j=0}^n \frac{\binom{n}{j}}{\binom{n+k}{j+k}} \frac{S_1(n+k, j+k)}{j+1} ((x+k-1)^{j+1} - B_{j+1}(x+k-1)). \tag{28}
 \end{aligned}$$

Thus, by (28), we get

$$\hat{D}_{n+1}^{(k)}(x) = x \hat{D}_n^{(k)}(x-1) + k \sum_{j=0}^n \frac{\binom{n}{j}}{\binom{n+k}{j+k}} \frac{S_1(n+k, j+k)}{j+1} ((x+k-1)^{j+1} - B_{j+1}(x+k-1)).$$

From (10) and (18), we note that

$$\frac{d}{dx} \hat{D}_n^{(k)}(x) = n! \sum_{l=0}^{n-1} \frac{(-1)^{n-l-1}}{l!(n-l)} \hat{D}_l^{(k)}(x). \tag{29}$$

By (6) and (18), we see that

$$\begin{aligned}
 \hat{D}_n^{(k)}(y) &= \left\langle \sum_{l=0}^{\infty} \hat{D}_l^{(k)}(y) \frac{t^l}{l!} \middle| x^n \right\rangle \quad (n \geq 1) \\
 &= \left\langle \left(\frac{(1+t) \log(1+t)}{t} \right)^k (1+t)^y \middle| x^n \right\rangle \\
 &= \left\langle \partial_t \left(\left(\frac{(1+t) \log(1+t)}{t} \right)^k (1+t)^y \right) \middle| x^{n-1} \right\rangle \\
 &= \left\langle \left(\partial_t \left(\frac{(1+t) \log(1+t)}{t} \right)^k \right) (1+t)^y \middle| x^{n-1} \right\rangle \\
 &\quad + y \left\langle \left(\frac{(1+t) \log(1+t)}{t} \right)^k (1+t)^{y-1} \middle| x^{n-1} \right\rangle \\
 &= y \hat{D}_{n-1}^{(k)}(y-1) \\
 &\quad + k \left\langle \left(\frac{(1+t) \log(1+t)}{t} \right)^{k-1} (1+t)^y \middle| \left(\log(1+t) + 1 - \frac{(1+t) \log(1+t)}{t} \right) \frac{x^n}{n} \right\rangle \\
 &= y \hat{D}_{n-1}^{(k)}(y-1) + \frac{k}{n} \left\langle \left(\frac{(1+t) \log(1+t)}{t} \right)^{k-1} (1+t)^y \middle| \log(1+t) x^n \right\rangle \\
 &\quad + \frac{k}{n} \left\langle \left(\frac{(1+t) \log(1+t)}{t} \right)^{k-1} (1+t)^y \middle| x^n \right\rangle \\
 &\quad - \frac{k}{n} \left\langle \left(\frac{(1+t) \log(1+t)}{t} \right)^k (1+t)^y \middle| x^n \right\rangle \\
 &= y \hat{D}_{n-1}^{(k)}(y-1) + \frac{k}{n} \hat{D}_n^{(k-1)}(y) - \frac{k}{n} \hat{D}_n^{(k)}(y) \\
 &\quad + \frac{k}{n} \sum_{1 \leq l \leq n} \frac{(-1)^{l-1} (n)_l}{l} \left\langle \left(\frac{(1+t) \log(1+t)}{t} \right)^{k-1} (1+t)^y \middle| x^{n-l} \right\rangle. \tag{30}
 \end{aligned}$$

Thus, by (30), we get

$$\begin{aligned} \hat{D}_n^{(k)}(x) &= \frac{n}{n+k}x\hat{D}_{n-1}^{(k)}(x-1) + \frac{k}{n+k}\hat{D}_n^{(k-1)}(x) \\ &\quad + \frac{k}{n+k} \sum_{1 \leq l \leq n} (-1)^{l-1} \binom{n}{l} (l-1)! \hat{D}_{n-l}^{(k-1)}(x). \end{aligned} \tag{31}$$

Therefore, by (31), we obtain the following theorem.

Theorem 4 For $n \geq 0, k \geq 1$, we have

$$\begin{aligned} \hat{D}_n^{(k)}(x) &= \frac{n}{n+k}x\hat{D}_{n-1}^{(k)}(x-1) + \frac{k}{n+k}\hat{D}_n^{(k-1)}(x) \\ &\quad + \frac{k}{n+k} \sum_{1 \leq l \leq n} (-1)^{l-1} \binom{n}{l} (l-1)! \hat{D}_{n-l}^{(k-1)}(x). \end{aligned}$$

Now, we compute $\langle (\frac{(1+t)\log(1+t)}{t})^k (\log(1+t))^m | x^n \rangle$ in two different ways:

$$\begin{aligned} &\left\langle \left(\frac{(1+t)\log(1+t)}{t} \right)^k (\log(1+t))^m | x^n \right\rangle \\ &= \left\langle \left(\frac{(1+t)\log(1+t)}{t} \right)^k | (\log(1+t))^m x^n \right\rangle \\ &= \sum_{0 \leq l \leq n-m} \frac{m!}{(l+m)!} S_1(l+m, m)(n)_{l+m} \left\langle \left(\frac{(1+t)\log(1+t)}{t} \right)^k | x^{n-l-m} \right\rangle \\ &= \sum_{0 \leq l \leq n-m} m! \binom{n}{l+m} S_1(l+m, m) \hat{D}_{n-l-m}^{(k)} \\ &= \sum_{0 \leq l \leq n-m} m! \binom{n}{l} S_1(n-l, m) \hat{D}_l^{(k)}. \end{aligned} \tag{32}$$

On the other hand,

$$\begin{aligned} &\left\langle \left(\frac{(1+t)\log(1+t)}{t} \right)^k (\log(1+t))^m | x^n \right\rangle \\ &= \left\langle \partial_t \left(\left(\frac{(1+t)\log(1+t)}{t} \right)^k (\log(1+t))^m \right) | x^{n-1} \right\rangle \\ &= k \left\langle \left(\frac{(1+t)\log(1+t)}{t} \right)^{k-1} \left(\frac{\log(1+t) + 1 - \frac{(1+t)\log(1+t)}{t}}{t} \right) (\log(1+t))^m | x^{n-1} \right\rangle \\ &\quad + m \left\langle \left(\frac{(1+t)\log(1+t)}{t} \right)^k (1+t)^{-1} (\log(1+t))^{m-1} | x^{n-1} \right\rangle \\ &= \frac{k}{n} \left\langle \left(\frac{(1+t)\log(1+t)}{t} \right)^{k-1} (\log(1+t))^{m+1} | x^n \right\rangle \\ &\quad + \frac{k}{n} \left\langle \left(\frac{(1+t)\log(1+t)}{t} \right)^{k-1} (\log(1+t))^m | x^n \right\rangle \end{aligned}$$

$$\begin{aligned}
 & - \frac{k}{n} \left\langle \left(\frac{(1+t)\log(1+t)}{t} \right)^k (\log(1+t))^m \middle| x^n \right\rangle \\
 & + m \left\langle \left(\frac{(1+t)\log(1+t)}{t} \right)^k (1+t)^{-1} (\log(1+t))^{m-1} \middle| x^{n-1} \right\rangle.
 \end{aligned} \tag{33}$$

Thus, by (33), we get

$$\begin{aligned}
 & \frac{n+k}{n} \left\langle \left(\frac{(1+t)\log(1+t)}{t} \right)^k (\log(1+t))^m \middle| x^n \right\rangle \\
 & = \frac{k}{n} \left\langle \left(\frac{(1+t)\log(1+t)}{t} \right)^{k-1} (\log(1+t))^{m+1} \middle| x^n \right\rangle \\
 & \quad + \frac{k}{n} \left\langle \left(\frac{(1+t)\log(1+t)}{t} \right)^{k-1} (\log(1+t))^m \middle| x^n \right\rangle \\
 & \quad + m \left\langle \left(\frac{(1+t)\log(1+t)}{t} \right)^k (1+t)^{-1} (\log(1+t))^{m-1} \middle| x^{n-1} \right\rangle.
 \end{aligned} \tag{34}$$

From (34), we derive the following equation:

$$\begin{aligned}
 & \frac{n+k}{k} \sum_{0 \leq l \leq n-m} m! \binom{n}{l} S_1(n-l, m) \hat{D}_l^{(k)} \\
 & = \frac{k}{n} \sum_{0 \leq l \leq n-m-1} (m+1)! \binom{n}{l} S_1(n-l, m+1) \hat{D}_l^{(k-1)} \\
 & \quad + \frac{k}{n} \sum_{0 \leq l \leq n-m} m! \binom{n}{l} S_1(n-l, m) \hat{D}_l^{(k-1)} \\
 & \quad + m \sum_{0 \leq l \leq n-m} (m-1)! \binom{n-1}{l} S_1(n-l-1, m-1) \hat{D}_l^{(k)}(-1).
 \end{aligned} \tag{35}$$

Therefore, by (35), we obtain the following theorem.

Theorem 5 For $n-1 \geq m \geq 1$, we have

$$\begin{aligned}
 & \sum_{l=0}^{n-m} \binom{n}{l} S_1(n-l, m) \hat{D}_l^{(k)} \\
 & = \frac{k(m+1)}{n+k} \sum_{0 \leq l \leq n-m-1} \binom{n}{l} S_1(n-l, m+1) \hat{D}_l^{(k-1)} \\
 & \quad + \frac{k}{n+k} \sum_{0 \leq l \leq n-m} \binom{n}{l} S_1(n-l, m) \hat{D}_l^{(k-1)} \\
 & \quad + \frac{n}{n+k} \sum_{0 \leq l \leq n-m} \binom{n-1}{l} S_1(n-l-1, m-1) \hat{D}_l^{(k)}(-1).
 \end{aligned}$$

For $\hat{D}_n^{(k)}(x) \sim ((\frac{e^t-1}{te^t})^k, e^t - 1)$, and $(x)_n \sim (1, e^t - 1)$, let us assume that

$$\hat{D}_n^{(k)}(x) = \sum_{m=0}^n C_{n,m}(x) m. \tag{36}$$

Then, by (16) and (17), we get

$$\begin{aligned}
 C_{n,m} &= \frac{1}{m!} \left\langle \left(\frac{(1+t)\log(1+t)}{t} \right)^k \middle| t^m x^n \right\rangle \\
 &= \binom{n}{m} \left\langle \left(\frac{(1+t)\log(1+t)}{t} \right)^k \middle| x^{n-m} \right\rangle \\
 &= \binom{n}{m} \hat{D}_{n-m}^{(k)}.
 \end{aligned} \tag{37}$$

Therefore, by (36) and (37), we obtain the following theorem.

Theorem 6 For $n \geq 0$, we have

$$\begin{aligned}
 \hat{D}_n^{(k)}(x) &= \sum_{0 \leq m \leq n} \binom{n}{m} \hat{D}_{n-m}^{(k)}(x)_m \\
 &= \sum_{0 \leq m \leq n} m! \binom{n}{m} \hat{D}_{n-m}^{(k)} \left(\frac{x}{m} \right).
 \end{aligned}$$

Now, we consider the following two Sheffer sequences:

$$\hat{D}_n^{(k)}(x) \sim \left(\left(\frac{e^t - 1}{te^t} \right)^k, e^t - 1 \right) \tag{38}$$

and

$$H_n^{(s)}(x|\lambda) \sim \left(\left(\frac{e^t - \lambda}{1 - \lambda} \right)^s, t \right), \quad s \in \mathbb{N}, \lambda \in \mathbb{C} \text{ with } \lambda \neq 1.$$

Let

$$\hat{D}_n^{(k)}(x) = \sum_{m=0}^n C_{n,m} H_m^{(s)}(x|\lambda). \tag{39}$$

Here

$$\begin{aligned}
 C_{n,m} &= \frac{1}{m!(1-\lambda)^s} \left\langle \left(\frac{(1+t)\log(1+t)}{t} \right)^k (\log(1+t))^m (1-\lambda+t)^s \middle| x^n \right\rangle \\
 &= \frac{1}{m!(1-\lambda)^s} \sum_{j=0}^n \binom{s}{j} (1-\lambda)^{s-j} (n)_j \\
 &\quad \times \left\langle \left(\frac{(1+t)\log(1+t)}{t} \right)^k (\log(1+t))^m \middle| x^{n-j} \right\rangle \\
 &= \sum_{j=0}^{n-m} \binom{s}{j} (1-\lambda)^{-j} (n)_j \sum_{l=0}^{n-m-j} \binom{n-j}{l+m} S_1(l+m, m) \hat{D}_{n-j-l-m}^{(k)} \\
 &= \sum_{j=0}^{n-m} \sum_{l=0}^{n-m-j} \binom{s}{j} \binom{n-j}{l} (n)_j (1-\lambda)^{-j} S_1(n-j-l, m) \hat{D}_l^{(k)}.
 \end{aligned} \tag{40}$$

Therefore, by (39) and (40), we obtain the following theorem.

Theorem 7 For $n \geq 0, k \geq 1$ and $\lambda \in \mathbb{C}$ with $\lambda \neq 1$, we have

$$\hat{D}_n^{(k)}(x) = \sum_{m=0}^n \left\{ \sum_{j=0}^{n-m} \sum_{l=0}^{n-m-j} \binom{s}{j} \binom{n-j}{l} (n)_j \right. \\ \left. \times (1-\lambda)^{-j} S_1(n-j-l, m) \hat{D}_l^{(k)} \right\} H_m^{(s)}(x|\lambda).$$

We consider the following two Sheffer sequences:

$$\hat{D}_n^{(k)}(x) \sim \left(\left(\frac{e^t - 1}{te^t} \right)^k, e^t - 1 \right), \quad B_n^{(s)}(x) \sim \left(\left(\frac{e^t - 1}{t} \right)^s, t \right).$$

Let

$$\hat{D}_n^{(k)}(x) = \sum_{m=0}^n C_{n,m} B_m^{(s)}(x). \tag{41}$$

Here

$$C_{n,m} = \frac{1}{m!} \left\langle \left(\frac{t}{\log(1+t)} \right)^s (\log(1+t))^m \middle| x^n \right\rangle \\ = \frac{1}{m!} \left\langle (1+t)^s \left(\frac{t}{(1+t)\log(1+t)} \right)^s (\log(1+t))^m \middle| x^n \right\rangle. \tag{42}$$

Case 1. For $s > k$, we have

$$C_{n,m} = \frac{1}{m!} \left\langle \left(\frac{t}{(1+t)\log(1+t)} \right)^{s-k} (\log(1+t))^m \middle| (1+t)^s x^n \right\rangle \\ = \frac{1}{m!} \sum_{0 \leq j \leq n} \binom{s}{j} (n)_j \left\langle \left(\frac{t}{(1+t)\log(1+t)} \right)^{s-k} \middle| (\log(1+t))^m x^{n-j} \right\rangle \\ = \sum_{0 \leq j \leq n-m} \binom{s}{j} (n)_j \sum_{m \leq l \leq n-j} S_1(l, m) \\ \times \binom{n-j}{l} \left\langle \left(\frac{t}{(1+t)\log(1+t)} \right)^{s-k} \middle| x^{n-j-l} \right\rangle \\ = \sum_{0 \leq j \leq n-m} \sum_{m \leq l \leq n-j} \binom{s}{j} \binom{n-j}{l} (n)_j S_1(l, m) \hat{C}_{n-j-l}^{(s-k)}, \tag{43}$$

where $\hat{C}_i^{(s-k)}$ is the i th Cauchy number of the second kind of order $s - k$ (see [14]).

Case 2. For $s = k$, we have

$$C_{n,m} = \frac{1}{m!} \left\langle (\log(1+t))^m \middle| (1+t)^s x^n \right\rangle \\ = \frac{1}{m!} \left\langle (\log(1+t))^m \middle| \sum_{j=0}^s \binom{s}{j} t^j x^n \right\rangle$$

$$\begin{aligned}
 &= \sum_{0 \leq j \leq n-m} \binom{s}{j} (n)_j \sum_{m \leq l < \infty} \frac{S_1(l, m)}{l!} \langle t^l | x^{n-j} \rangle \\
 &= \sum_{0 \leq j \leq n-m} \binom{s}{j} (n)_j S_1(n-j, m).
 \end{aligned} \tag{44}$$

Case 3. For $s < k$, we have

$$\begin{aligned}
 C_{n,m} &= \frac{1}{m!} \left\langle \left(\frac{(1+t) \log(1+t)}{t} \right)^{k-s} (\log(1+t))^m \middle| (1+t)^s x^n \right\rangle \\
 &= \sum_{0 \leq j \leq n-m} \sum_{m \leq l \leq n-j} \binom{s}{j} \binom{n-j}{l} (n)_j S_1(l, m) \hat{D}_{n-j-l}^{(k-s)}.
 \end{aligned} \tag{45}$$

Therefore, by (41), (42), (43), (44), and (45), we obtain the following theorem.

Theorem 8 *Let $n \geq 0$, we have:*

(I) *For $s > k$, we have*

$$\begin{aligned}
 \hat{D}_n^{(k)}(x) &= \sum_{0 \leq m \leq n} \left\{ \sum_{0 \leq j \leq n-m} \sum_{m \leq l \leq n-j} \binom{s}{j} \binom{n-j}{l} \right. \\
 &\quad \left. \times (n)_j S_1(l, m) \hat{C}_{n-j-l}^{(s-k)} \right\} B_m^{(s)}(x).
 \end{aligned}$$

(II) *For $s = k$, we have*

$$\hat{D}_n^{(k)}(x) = \sum_{0 \leq m \leq n} \left\{ \sum_{0 \leq j \leq n-m} \binom{s}{j} (n)_j S_1(n-j, m) \right\} B_m^{(s)}(x).$$

(III) *For $s < k$, we have*

$$\begin{aligned}
 \hat{D}_n^{(k)}(x) &= \sum_{0 \leq m \leq n} \left\{ \sum_{0 \leq j \leq n-m} \sum_{m \leq l \leq n-j} \binom{s}{j} \binom{n-j}{l} \right. \\
 &\quad \left. \times (n)_j S_1(l, m) \hat{D}_{n-j-l}^{(k-s)} \right\} B_m^{(s)}(x).
 \end{aligned}$$

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to this work. All authors read and approved the final manuscript.

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