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A Note on q-Analogue of Lambda-Daehee Polynomials

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Abstract

In this paper, we consider the q-analogue of lambda-Daehee polynomials and we give some new identities of these polynomials which are derived from p-adic invariant integral on $\mathbb{Z}_p$

Keywords: p-adic integral on $\mathbb{Z}_p$, lambda-Daehee polynomials, stirling numbers

1. Introduction

As is well known, the lambda-Daehee polynomials are defined by the generating function to be

$$\sum_{n=0}^{\infty} D_{n.\lambda}(x)\frac{t^n}{n!} = \frac{\lambda \log(1+t)}{(1+t)^{\lambda}-1}(1+t)^x, \text{ (see [7]).} \tag{1}$$

Let p be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic rational integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. The p-adic norm $|\cdot|$ is normalized as $|p|_p = 1/p$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$, the p-adic invariant integral on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the p-adic invariant integral on $\mathbb{Z}_p$ is defined to be

$$\begin{aligned} I_0(f) = \int_{\mathbb{Z}_p} f(x)d\mu_0(x) &= \lim_{N\to\infty}\sum_{x=0}^{p^N-1} f(x)\mu_0(x+p^N\mathbb{Z}_p) \\ &= \lim_{N\to\infty}\frac{1}{p^N}\sum_{x=0}^{p^N-1} f(x), \text{ (see [1-18]).} \end{aligned} \tag{2}$$

By (2), we easily get

$$I_0(f_1) - I_0(f) = f\prime(0), \text{ (see [8, 10, 11])} \tag{3}$$

where $f_1(x) = f(x+1)$.

From (3), we have

$$\int_{\mathbb{Z}_p} e^{xt}d\mu_0(x) = \frac{t}{e^t-1} = \sum_{n=0}^{\infty} B_n\frac{t^n}{n!}, \text{ (see [1-8]),} \tag{4}$$

where B_n are called the Bernoulli numbers.

In particular, the Bernoulli polynomials are given by

$$\int_{\mathbb{Z}_p} e^{(x+y)t}d\mu_0(y) = \frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}. \tag{5}$$

By (4) and (5), we get

$$B_n(x) = \sum_{\ell=1}^{n}\binom{n}{\ell}B_\ell x^{n-\ell}, \text{ (see [10-18]).} \tag{6}$$

The Stirling number of the first kind is defined by the falling factorial sequence to be

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{\ell=1}^{n} S_1(n,\ell)x^\ell,\ (n \in \mathbb{Z}_{\geq 0}). \tag{7}$$

As is known, the Stirling number of the second kind is given by

$$(e^t - a)^n = n! \sum_{\ell=n}^{\infty} S_2(\ell,n)\frac{t^\ell}{\ell!},\ \text{(see [8, 16])}. \tag{8}$$

In viewpoint of (1), we consider the q-analogue of lambda-Daehee polynomials and investigate some properties of those polynomials which are derived from the p-adic invariant integral on $\mathbb{Z}_p$.

2. Some identities for the higher-order q-Bernoulli polynomials of the second kind

In this section, we assume that $q, t \in \mathbb{C}_p$ with $|t|_p < |\frac{p^{\frac{1}{p-1}}}{q}|_p$ and $\lambda \in \mathbb{Z}_p$ with $\lambda \neq 0$. For $f(x) = (1+qt)^{\lambda x}$, by (3), we get

$$\int_{\mathbb{Z}_p} (1+qt)^{x+\lambda y} d\mu_0(y) = \frac{\lambda \log(1+qt)}{(1+qt)^\lambda}(1+qt)^x. \tag{9}$$

In viewpoint of (1), we define the q-analogue lambda-Daehee polynomials as follows:

$$\frac{\lambda \log(1+qt)}{(1+qt)^\lambda}(1+qt)^x = \sum_{n=0}^{\infty} BD_{n,q}(x|\lambda)\frac{t^n}{n!}. \tag{10}$$

When $x = 0, BD_{n,q}(\lambda) = BD_{n,q}(0|\lambda)$ are called the q-analogue of lambda-Daehee numbers.

Remark. Note that $\lim_{q\to 1} BD_{n,q}(x|\lambda) = D_{n,\lambda}(x)$.

From (9) and (10), we have

$$\begin{aligned}\sum_{n=0}^{\infty} BD_{n,q}(x|\lambda)\frac{t^n}{n!} &= \int_{\mathbb{Z}_p} (1+qt)^{\lambda y + x} d\mu_0(y) \\ &= \sum_{n=0}^{\infty} q^n \int_{\mathbb{Z}_p} (\lambda y + x)_n d\mu_0(y)\frac{t^n}{n!}.\end{aligned} \tag{11}$$

Therefore, by (11), we obtain the following theorem.

Theorem 2.1. *For $n \geq 0$, we have*

$$q^n \int_{\mathbb{Z}_p} (x+\lambda y)_n d\mu_0(dy) = BD_{n,q}(x|\lambda).$$

By replacing qt by e^t-1 in (10), we get

$$\begin{aligned}\sum_{n=0}^{\infty} q^{-n} BD_{n,q}(x|\lambda)\frac{(e^t-1)^n}{n!} &= \frac{\lambda t}{e^{\lambda t}-1}e^{tx}\\ &= \sum_{n=0}^{\infty} B_n\left(\frac{x}{\lambda}\right)\lambda^n\frac{t^n}{n!}.\end{aligned} \tag{12}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} q^{-n} BD_{n,q}(x|\lambda)\frac{(e^t-1)^n}{n!} &= \sum_{n=0}^{\infty} q^{-n} BD_{n,q}(x|\lambda)\sum_{m=n}^{\infty} S_2(m,n)\frac{t^m}{m!}\\ &= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m} q^{-n} BD_{n,q}(x|\lambda)S_2(m,n)\right)\frac{t^m}{m!}.\end{aligned} \tag{13}$$

Therefore, by (12) and (13), we obtain the following theorem.

Theorem 2.2. *For $m \geq 0$, we have*

$$\sum_{n=0}^{m} q^{-n} BD_{n,q}(x|\lambda)S_2(m,n) = \lambda^m B_m\left(\frac{x}{\lambda}\right).$$

From Theorem 1, we have

$$\begin{aligned}q^{-n} BD_{n,q}(x|\lambda) &= \sum_{\ell=0}^{n} S_1(n,\ell)\int_{\mathbb{Z}_p}(x+y\lambda)^{\ell} d\mu_0(y)\\ &= \sum_{\ell=0}^{n} S_1(n,\ell)\lambda^{\ell}\int_{\mathbb{Z}_p}\left(\frac{x}{\lambda}+y\right)^{\ell} d\mu_0(y)\\ &= \sum_{\ell=0}^{n} S_1(n,\ell)\lambda^{\ell}B_{\ell}\left(\frac{x}{\lambda}\right).\end{aligned} \tag{14}$$

Theorem 2.3. *For $n \geq 0$, we have*

$$q^{-n} BD_{n,q}(x|\lambda) = \sum_{\ell=0}^{n} S_1(n,\ell)\lambda^{\ell}B_{\ell}\left(\frac{x}{\lambda}\right).$$

Let us consider the q-analogue of lambda-Daehee polynomals of order $k \in \mathbb{N}$ as follows:

$$q^{-n} BD_{n,q}^{(k)}(x|\lambda) = \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left(\lambda\sum_{i=1}^{k} x_i + x\right)_n d\mu_0(x_1)\cdots d\mu_0(x_k). \tag{15}$$

Thus, by (15), we get

$$q^{-n} BD_{n,q}^{(k)}(x|\lambda) = \sum_{\ell=1}^{n} S_1(n,\ell)\lambda^{\ell}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left(\sum_{i=1}^{k} x_i + \frac{x}{\lambda}\right)^{\ell} d\mu_0(x_1)\cdots d\mu_0 y x_k). \tag{16}$$

Now, we observe that

$$\begin{aligned}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} e^{\left(\sum_{i=1}^{k} x_i+x\right)} d\mu_0(x_1)\cdots d\mu_0(x_k) &= \left(\frac{t}{e^t+1}\right)^k e^{xt} \\ &= \sum_{n=0}^{\infty} B_n^{(k)}(x)\frac{t^n}{n!}.\end{aligned} \tag{17}$$

where $B_n^{(k)}(x)$ are called Bernoulli polynomials of order k.

By (17), we get

$$B_n^{(k)}(x) = \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left(\sum_{i=1}^{k} x_i + x\right)^n d\mu_0\cdots d\mu_0(x_k). \tag{18}$$

From (16) and (18), we have

$$q^{-n}BC_{n,q}^{(k)}(x|\lambda) = \sum_{\ell=1}^{n} S_1(n,\ell)\lambda^{\ell} B_{\ell}^{(k)}\left(\frac{x}{\lambda}\right). \tag{19}$$

From (15), we can derive the generating function of $BD_{n,q}^{(k)}(x|\lambda)$ as follows:

$$\begin{aligned}\sum_{n=0}^{\infty} BD_{n,q}^{(k)}(x|\lambda)\frac{t^n}{n!} &= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+qt)^{\lambda\sum_{i=1}^{k} x_i+x} d\mu_0(x_1)\cdots d\mu_0(x_k) \\ &= \left(\frac{\lambda\log(1+qt)}{(1+qt)^{\lambda}-1}\right)^k (1+qt)^x.\end{aligned} \tag{20}$$

by replacing qt by e^t-1, we get

$$\begin{aligned}\sum_{n=0}^{\infty} q^{-n}BD_{n,q}^{(k)}(x|\lambda)\frac{1}{n!}(e^t-1)^n &= \left(\frac{\lambda t}{e^{\lambda t}-1}\right)^k e^{xt} \\ &= \sum_{n=0}^{\infty} B_n^{(k)}\left(\frac{x}{\lambda}\right)\lambda^n\frac{t^n}{n!}\end{aligned} \tag{21}$$

and

$$\sum_{n=0}^{\infty} q^{-n}BD_{n,q}^{(k)}(x|\lambda)\frac{1}{n!}(e^t-1)^n = \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m} q^{-n}BD_{n,q}^{(k)}(x|\lambda)S_2(m,n)\right)\frac{t^m}{m!}. \tag{22}$$

Therefore, by (21) and (22), we obtain the following theorem.

Theorem 2.4. *For $m \geq 0$, we have*

$$\sum_{n=0}^{\infty} BD_{n,q}^{(k)}(x|\lambda)S_2(m,n)q^{-n} = \lambda^m B_m^{(k)}\left(\frac{x}{\lambda}\right).$$

For $n \geq 0$, the rising factorial sequence is defined by

$$\begin{aligned} x^{\underline{n}} =& x(x-1)\cdots(x-n+1) = (-1)^n(-x)_n \\ =& \sum_{\ell=0}^{n} |S_1(n,l)| x^{\ell}, \end{aligned} \tag{23}$$

where $|S_1(n,l)| = (-1)^{n-l} S_1(n,\ell)$.

We consider the q-analogue of lambda-Daehee polynomials of the second kind as follows:

$$\widehat{BD}_{n,q}(x|\lambda) = q^n \int_{\mathbb{Z}_p} (-\lambda y + x)_n d\mu_0(y), \ (n \geq 0). \tag{24}$$

From (24), we have

$$\begin{aligned} q^n \widehat{BD}_{n,q}(x|\lambda) =& \sum_{\ell=0}^{n} S_1(n,\ell)(-1)^{\ell}\lambda^{\ell} \int_{\mathbb{Z}_p} \left(-\frac{x}{\lambda} + y\right)^{\ell} d\mu_0(y) \\ =& \sum_{\ell=0}^{n} S_1(n,\ell)(-1)^{\ell}\lambda^{\ell} B_{\ell}\left(-\frac{x}{\lambda}\right). \end{aligned} \tag{25}$$

When $x = 0, \widehat{BD}_{n,q}(\lambda) = \widehat{BD}_{n,q}(0|\lambda)$ are called the q-analogue of lambda-Daehee numbers of the second kind. The generating function of $\widehat{BD}_{n,q}(x|\lambda)$ is given by

$$\begin{aligned} \sum_{n=0}^{\infty} \widehat{BD}_{n,q}(x|\lambda)\frac{t^n}{n!} =& \int_{\mathbb{Z}_p} (1+qt)^{-\lambda y + x} d\mu_0(y) \\ =& \frac{\lambda \log(1+qt)}{(1+qt)^{\lambda} - 1}(1+qt)^{\lambda + x}. \end{aligned} \tag{26}$$

By replacing qt by $e^t - 1$, we get

$$\sum_{n=0}^{\infty} q^{-n} \widehat{BD}_{n,q}(x|\lambda)\frac{1}{n!}(e^t-1)^n = \sum_{m=0}^{\infty} \lambda^m B_m\left(\frac{\lambda + x}{\lambda}\right)\frac{t^m}{m!} \tag{27}$$

and

$$\sum_{n=0}^{\infty} \widehat{BD}_{n,q}(x|\lambda)\frac{q^{-n}}{n!}(e^t-1)^n = \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m} \widehat{BD}_{n,q}(x|\lambda) S_2(m,n) q^{-n}\right)\frac{t^m}{m!}. \tag{28}$$

Therefore, by (27) and (28), we obtain the following theorem.

Theorem 2.5. *For $m \geq 0$, we have*

$$q^{-m}\widehat{BD}_{m,q}(x|\lambda) = \sum_{\ell=0}^{m} S_1(m,\ell)(-1)^{\ell}\lambda^{\ell} B_{\ell}\left(-\frac{x}{\lambda}\right)$$

and

$$\lambda^m B_m\left(\frac{\lambda+x}{\lambda}\right)=\sum_{n=0}^{m}\widehat{BD}_{n,q}(x|\lambda)S_2(m,n)q^{-n}.$$

For $k\in\mathbb{N}$, let us consider the q-analogue of lambda-Daehee polynomials of the second kind with order k as follwos:

$$\widehat{BD}^{(k)}_{n,q}(x|\lambda)=q^n\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left(-\lambda\sum_{i=1}^{k}x_i+x\right)_n d\mu_0(x_1)\cdots d\mu_0(x_k), \tag{29}$$

where $n\geq 0$.

From (29), we have

$$q^{-n}\widehat{BD}^{(k)}_{n,q}(x|\lambda)=\sum_{\ell=0}^{n}S_1(n,\ell)(-1)^{\ell}B^{(k)}_{\ell}\left(-\frac{x}{\lambda}\right)\lambda^{\ell}. \tag{30}$$

The generating function of $\widehat{BD}^{(k)}_{n,q}(x|\lambda)$ is given by

$$\begin{aligned}\sum_{n=0}^{\infty}\widehat{BD}^{(k)}_{n,q}(x|\lambda)\frac{t^n}{n!}&=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+qt)^{-\lambda\sum_{i=1}^{k}x_i+x}d\mu_0(x_1)\cdots d\mu_0(x_k)\\&=\left(\frac{\lambda\log(1+qt)}{(1+qt)^{\lambda}-1}\right)^k(1+qt)^{\lambda k+x}.\end{aligned} \tag{31}$$

By replacing qt by e^t-1 in (31), we get

$$\begin{aligned}\sum_{n=0}^{\infty}q^{-n}\widehat{BD}^{(k)}_{n,q}(x|\lambda)\frac{1}{n!}(e^t-1)^n&=\left(\frac{\lambda t}{e^{\lambda t}-1}\right)^k e^{(\lambda k+x)t}\\&=\sum_{m=0}^{\infty}\lambda^m B^{(k)}_m\left(k+\frac{x}{\lambda}\right)\frac{t^m}{m!}\end{aligned} \tag{32}$$

and

$$\sum_{n=0}^{\infty}q^{-n}\widehat{BD}^{(k)}_{n,q}(x|\lambda)\frac{1}{n!}(e^t-1)^n=\sum_{m=0}^{\infty}\left(\sum_{n=0}^{m}\widehat{BD}^{(k)}_{n,q}(x|\lambda)S_2(m,n)q^{-n}\right)\frac{t^m}{m!}. \tag{33}$$

Therefore, by (32) and (33), we obtain the following theorem.

Theorem 2.6. *Form ≥ 0, we have*

$$(-q)^m\widehat{BD}^{(k)}_{m,q}(x|\lambda)=\sum_{\ell=0}^{m}|S_1(m,l)|\lambda^{\ell}B^{(k)}_{\ell}\left(-\frac{x}{\lambda}\right) \tag{34}$$

and

$$\lambda^m B^{(k)}_m\left(k+\frac{x}{\lambda}\right)=\sum_{n=0}^{m}\widehat{BD}^{(k)}_{n,q}(x|\lambda)S_2(m,n)q^{-n}. \tag{35}$$

Now, we observe that

$$\begin{aligned} q^{-n}(-1)^n \frac{BD_{n,q}(x|\lambda)}{n!} &= (-1)^n \int_{\mathbb{Z}_p} \binom{x+\lambda y}{n} d\mu_0(y) \\ &= \int_{\mathbb{Z}_p} \binom{-\lambda y - x + n - 1}{n} d\mu_0(y) \\ &= \sum_{m=0}^{n} \binom{n-1}{m-1} \int_{\mathbb{Z}_p} \binom{-y\lambda - x}{m} d\mu_0(y) \\ &= \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{q^{-m}\widehat{BD}_{m,q}(-x|\lambda)}{m!} \end{aligned} \tag{36}$$

and

$$(-1)^n q^{-n} \frac{\widehat{BD}_{n,q}(x|\lambda)}{n!} = \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{BD_{m,q}(-x|\lambda)}{m!} q^{-m}. \tag{37}$$

Therefore, by (36) and (37), we obtain the following theorem.

Theorem 2.7. *For*$n \geq 1$, *we have*

$$q^{-n}(-1)^n \frac{BD_{n,q}(x|\lambda)}{n!} = \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{\widehat{BD}_{m,q}(-x|\lambda)}{m!} q^{-m}, \tag{38}$$

and

$$q^{-n}(-1)^n \frac{\widehat{BD}_{n,q}(x|\lambda)}{n!} = \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{\widehat{BD}_{m,q}(-x|\lambda)}{m!} q^{-m}. \tag{39}$$

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