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Chebyshev-type quadrature and zeros of Faber polynomials

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Abstract

With any probability measure μ on $[-1,1]$ we associate a sequence of polynomials $F_n(z)$ which are Faber polynomials of a univalent function $F(z)$ on $|z|>1$. If the zeros of $F_n(z)$ are in the open unit disk then there exists a Chebyshev-type quadrature formula for μ with n nodes which is exact for all polynomials $f(t)$ up to degree $n-1$.

For the normalized Jacobi measures $\mathrm{d}\mu(t) = C_\lambda(1-t)^{-1/2-\lambda}(1+t)^{-1/2}\,\mathrm{d}t$ with $\lambda < 1/2$ the function $F(z)$ can be expressed in terms of hypergeometric functions. Using this expression it is proved that the zeros of the associated Faber polynomials are in the open unit disk in case $\lambda \in (0,\lambda_0]$ for some $\lambda_0 > 0$. This result solves to a large extent a problem of Förster.

Keywords: Chebyshev-type quadrature; Ultraspherical and Jacobi measures; Faber polynomials; Hypergeometric functions

1. Introduction and statement of results

A Chebyshev-type quadrature formula is a numerical integration formula in which all weights are equal. Given a probability measure μ on $[-1,1]$, this is a formula of the type

$$\int_{-1}^{1} \phi(t)\,\mathrm{d}\mu(t) \approx \frac{1}{n}\sum_{j=1}^{n}\phi(x_j), \tag{1.1}$$

with (not necessarily distinct) nodes $x_1,\ldots,x_n \in [-1,1]$. We call n the size of (1.1). The degree is the maximal number p such that equality holds for every polynomial ϕ of degree $\leqslant p$. See [3,4,7] for surveys on Chebyshev-type quadrature and related topics.

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The central question concerning Chebyshev-type quadrature is: with μ and n fixed, what is the maximal degree that can be achieved by a suitable choice of the nodes $x_1, \ldots, x_n$?

For the arcsine measure $\mathrm{d}\mu(t) = \pi^{-1}(1-t^2)^{-1/2}\,\mathrm{d}t$, the Gauss formulas have equal weights. Hence for every n, there is a Chebyshev-type formula of size n and degree $2n-1$. Various modifications of the arcsine measure admit Chebyshev-type quadrature of size n and degree $\geqslant n$ for every n, see [3] and the references given there. On the basis of numerical computations, Förster [2] concluded that it might be possible that also certain ultraspherical measures

$$\mathrm{d}\mu(t) = w_\lambda^{(0)}(t)\,\mathrm{d}t := \frac{4^\lambda \Gamma(1-2\lambda)}{\Gamma(\frac{1}{2}-\lambda)^2}(1-t^2)^{-1/2-\lambda}\,\mathrm{d}t, \quad 0 < \lambda < \tfrac{1}{2} \tag{1.2}$$

have this property. In particular, he found that for $0 \leqslant \lambda \leqslant 0.18$, Chebyshev-type formulas of size n and degree $\geqslant n$ exist for every n up to 55.

Instead of (1.2) we will consider the Jacobi measures

$$\mathrm{d}\mu(t) = w_\lambda(t)\,\mathrm{d}t := \frac{2^\lambda \Gamma(1-\lambda)}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}-\lambda)}(1+t)^{-1/2}(1-t)^{-1/2-\lambda}\,\mathrm{d}t, \quad 0 < \lambda < \tfrac{1}{2}, \tag{1.3}$$

which are related to the ultraspherical measures (1.2) by a quadratic transformation. The main result of this paper is the following.

Theorem 1.1. *There is a $\lambda_0 > 0$ such that for every $\lambda \in (0, \lambda_0]$, the measure $w_\lambda(t)\,\mathrm{d}t$ of (1.3) admits Chebyshev-type quadrature of size n and degree $\geqslant n-1$ for every n.*

From Theorem 1.1 easily follows:

Corollary 1.2. *There is a $\lambda_0 > 0$ such that for every $\lambda \in (0, \lambda_0]$, the measure $w_\lambda^{(0)}(t)\,\mathrm{d}t$ of (1.2) admits Chebyshev-type quadrature of size $2n$ and degree $\geqslant 2n-1$ for every n.*

To prove Theorem 1.1 we use the connection between Chebyshev-type quadrature and Faber polynomials which has also been used by Peherstorfer [12,13]. With any probability measure μ on $[-1,1]$ one associates a mapping $F(z)$,

$$F(z) = z + a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots, \quad |z| > 1. \tag{1.4}$$

The Faber polynomial $F_n(z)$ of degree n is the polynomial part of $F(z)^n$. The property which relates Faber polynomials and Chebyshev-type quadrature is the fact that if the zeros of $F_n(z)$ are in the open unit disk then μ admits Chebyshev-type quadrature of size n and degree $\geqslant n-1$, see Theorem 2.1.

Let $\Phi(w)$ be the inverse of $F(z)$,

$$\Phi(w) = w + b_0 + b_1 w^{-1} + b_2 w^{-2} + \cdots, \tag{1.5}$$

and let ρ be the smallest number such that $\Phi(w)$ is univalent in $|w| > \rho$. We denote by K the complement of $\Phi(|w| > \rho)$. Then it is known, see Theorem 2.5, that limit points of the zeros of $F_n(z)$ are in K, and under additional assumptions one can prove that all zeros are in the interior of K. Thus if K is contained in $\{|z| \leqslant 1\}$ one has a chance to prove that the zeros of $F_n(z)$ are in $|z| < 1$.

We will prove the following results for the set $K = K(\lambda)$ associated with the measure $w_\lambda(t)\,dt$.

Theorem 1.3. *There exists a $\lambda^* > 0$ such that the set $K(\lambda)$ is contained in $\{|z| \leqslant 1\}$ if and only if $0 < \lambda \leqslant \lambda^*$.*

Furthermore, for $0 < \lambda \leqslant \lambda^*$, $K(\lambda)$ is starlike with respect to the origin.

Theorem 1.4. *There exists a $\lambda_0 > 0$, ($\lambda_0 \leqslant \lambda^*$) such that for every $0 < \lambda \leqslant \lambda_0$ all zeros of $F_n(z)$ are in the interior of $K(\lambda)$ for every n.*

Numerical experiments indicate that we have $\lambda^* \approx 0.1768\ldots$, which is pretty close to the value 0.18 found by Förster [2]. From the proof of Theorem 1.4 it is not clear how to obtain estimates for λ_0. It might be true that λ^* is the optimal value for λ_0.

Theorem 1.1 is an immediate consequence of Theorems 1.3, 1.4, and the relation between Chebyshev-type quadrature and the zeros of Faber polynomials.

Finally, we present a result about the asymptotic zero distribution of the Faber polynomials $F_n(z)$ associated with the measures $w_\lambda(t)\,dt$. More results on distributions of zeros of Faber polynomials can be found in [10].

Theorem 1.5. *Let $0 < \lambda < \lambda^*$, where λ^* is as in Theorem 1.3. Let $\zeta_{j,n}$, $j = 1, \ldots, n$ denote the zeros of $F_n(z)$ and let ν_n be the normalized zero distribution of $F_n(z)$, i.e.,*

$$\nu_n = \frac{1}{n}\sum_{j=1}^{n} \delta_{\zeta_{j,n}}. \tag{1.6}$$

Then in weak-star sense

$$\lim_{n\to\infty} \nu_n = \nu_{K(\lambda)}, \tag{1.7}$$

where $\nu_{K(\lambda)}$ is the equilibrium distribution of $K(\lambda)$.

2. Faber polynomials associated with Chebyshev-type quadrature

Let μ be a probability measure on $[-1,1]$. We define the function $G(z) = G(z;\mu)$ by

$$G(z) := -\int_{-1}^{1} \log(1 - 2tz + z^2)\,d\mu(t). \tag{2.1}$$

Because of the relation (see [17])

$$-\log(1 - 2tz + z^2) = \sum_{k=1}^{\infty} \frac{2}{k} T_k(t) z^k,$$

where $T_k(t)$ is the Chebyshev polynomial of the first kind of degree k, $G(z)$ has the power series expansion

$$G(z) = \sum_{k=1}^{\infty} \frac{c_k}{k} z^k, \quad c_k = 2\int_{-1}^{1} T_k(t)\,d\mu(t). \tag{2.2}$$

From either (2.1) or (2.2) it is easily seen that $G(z)$ is analytic in the open unit disk $|z| < 1$.

For every nonnegative integer n, we introduce a polynomial $P_n(z) = P_n(z;\mu)$ of degree $\leqslant n$ as follows: $P_n(z)$ agrees with the power series expansion of $\exp(-nG(z))$ up to degree n. Thus,

$$P_n(z) := \exp(-nG(z)) + \mathcal{O}(z^{n+1}), \quad (z \to 0). \tag{2.3}$$

We also need the reversed polynomials which we denote by $F_n(z)$:

$$F_n(z) := z^n P_n(1/z). \tag{2.4}$$

$F_n(z)$ is a monic polynomial of precise degree n.

The importance of these polynomials for Chebyshev-type quadrature is given in the following theorem, due to Geronimus [5, Theorem 1] and Peherstorfer [12, Theorem 1], see also [9].

Theorem 2.1. *If all zeros of $P_n(z)$ have absolute value > 1 (or equivalently, all zeros of $F_n(z)$ have absolute value < 1), then μ admits Chebyshev-type quadrature of size n and degree $\geqslant n - 1$.*

We will also make frequent use of the following two functions:

$$f(z) := z \exp G(z), \quad |z| < 1, \tag{2.5}$$

$$F(z) := 1/f(1/z) = z \exp(-G(1/z)), \quad |z| > 1. \tag{2.6}$$

Recall from the theory of univalent functions that a univalent mapping on $|z| < 1$ is called *starlike* if the image domain is starlike with respect to the origin. We denote by S_R^* the class of univalent functions $g(z)$ on $|z| < 1$, normalized by $g(0) = 0$, $g'(0) = 1$, which are starlike and have real coefficients.

Lemma 2.2. *The function $f(z)$ of (2.5) belongs to S_R^* and conversely, every function in S_R^* equals $z \exp G(z;\mu)$ for some probability measure μ on $[-1,1]$.*

Proof. By [14, Theorem 2.6] a function $f(z) = z + a_2 z^2 + \cdots$ is starlike in the open unit disk if and only if $f(z) = z \exp G(z)$ with

$$G(z) = -2 \int_{-\pi}^{\pi} \log(1 - \mathrm{e}^{-\mathrm{i}t} z)\, \mathrm{d}\gamma(t)$$

for some probability measure γ on $[-\pi,\pi]$. The coefficients of $f(z)$ are real if and only if γ is symmetric with respect to $t = 0$. Simple transformations then show that $G(z)$ has the form (2.1) for some probability measure μ on $[-1,1]$. □

Corollary 2.3. *For $|z| < 1$, $\Re(1 + zG'(z)) > 0$.*

Proof. By [14, Theorem 2.5] $f(z)$ is starlike if and only if $\Re(zf'(z)/f(z)) > 0$ for $|z| < 1$. Then the corollary follows from (2.5) and Lemma 2.2. □

From Lemma 2.2 we also conclude that $F(z)$ is univalent in $|z| > 1$ and it maps $|z| > 1$ onto the complement of a compact set which is starlike and symmetric with respect to the real axis. The inverse mapping of $w = F(z)$ will be denoted by $\Phi(w)$ as in (1.5).

Lemma 2.4. *The polynomial $F_n(z)$ defined in* (2.4) *satisfies*

$$F_n(z) = F(z)^n + \mathcal{O}(1/z), \quad (z \to \infty). \tag{2.7}$$

Proof. Combination of (2.4) and (2.3) yields

$$F_n(z) = z^n \left[\exp(-nG(1/z)) + \mathcal{O}(z^{-(n+1)})\right] = [z\exp(-G(1/z))]^n + \mathcal{O}(1/z), \quad (z \to \infty),$$

which in view of (2.6) gives (2.7). □

Hence the polynomials $F_n(z)$ are the *Faber polynomials* of $F(z)$, see e.g. [16], and as such many properties of the polynomials $F_n(z)$ are known. We have, for example, the generating function

$$\frac{w\Phi'(w)}{\Phi(w) - z} = \sum_{n=0}^{\infty} F_n(z) w^{-n}, \tag{2.8}$$

which holds for $|w|$ sufficiently large (depending on z). The relation (2.8) is often taken as the definition of Faber polynomials, see [16, p. 130].

Note however, that usually the Faber polynomials arise as polynomials satisfying (2.7) where $F(z)$ is a univalent mapping from the complement of a compact set onto a domain $|w| > \rho$. In our case the function $F(z)$ is defined on $|z| > 1$ and it maps $|z| > 1$ onto the complement of a compact set.

In view of Theorem 2.1 we are interested in the zeros of $F_n(z)$. To state what is known we need to consider a domain which is mapped by $F(z)$, or by an *analytic extension* of $F(z)$, univalently onto $|w| > \rho$ for some ρ. In that case $\Phi(w)$ is univalent on $|w| > \rho$. Let ρ be the smallest number such that $\Phi(w)$ (or an analytic extension of $\Phi(w)$) is defined and univalent on $|w| > \rho$ and let K be the complement of $\Phi(|w| > \rho)$ in the z-plane. Thus ρ is the logarithmic capacity (transfinite diameter) of K. Then the following hold.

Theorem 2.5. (a) *All limit points of the zeros of $F_n(z)$ are in K.*

(b) *If K is convex then the zeros of $F_n(z)$ are in K for every n. If in addition K is not a line segment then the zeros of $F_n(z)$ are in the interior of K for every n. If K is a line segment then no zeros of $F_n(z)$ are at the end points of K.*

Proof. (a) See [16, p. 137]. (b) See [8]. □

It was conjectured that in general the convex hull of K would contain the zeros of all $F_n(z)$. However, Goodman [6] gave an example of a compact set K for which this fails. Goodman's example is not starlike, so it might be that the conjecture holds for starlike K.

Combining Theorems 2.1 and 2.5 we have the following sufficient condition for the existence of Chebyshev-type quadrature of size n and degree $\geqslant n - 1$.

Corollary 2.6. *Suppose $F(z)$ has an analytic extension (also denoted by $F(z)$) to the complement of a compact convex set K contained in the closed unit disk which maps $\mathbb{C}\setminus K$ univalently onto $\{|w| > \rho\}$ for some $\rho \geqslant 0$. Then the measure μ admits Chebyshev-type quadrature of size n and degree $\geqslant n-1$ for every n.*

The assumptions of Corollary 2.6 are satisfied if and only if $f(z)$ has an analytic extension to a domain $D \supset \{|z| < 1\}$ for which $K = \{z \mid 1/z \notin D\}$ is convex, and this extension maps D one–one onto a disk $|w| < \rho^{-1}$.

Corollary 2.6 applies to a large collection of measures μ. Indeed, one can take any compact convex set $K \subset \{|z| \leqslant 1\}$ which is symmetric with respect to the real axis and then consider the map $F(z) = z + a_0 + a_1 z^{-1} + \cdots$ which maps $\mathbb{C}\setminus K$ univalently onto $\{|w| > \rho\}$, where ρ is the logarithmic capacity of K. Then $1/F(1/z)$ is a function in S_R^* (it can be proved that it is starlike) and by Lemma 2.2 it is equal to $z \exp G(z;\mu)$ for some probability measure μ on $[-1,1]$. By Corollary 2.6 μ admits Chebyshev-type quadrature of size n and degree $\geqslant n-1$ for every n.

The simplest example of this kind is to take $K = \{b\}$ with $-1 < b < 1$. Then $F(z) = z - b$ and the associated measure μ is given by

$$\mathrm{d}\mu(t) = \frac{1 - bt}{1 - 2bt + b^2} \frac{\mathrm{d}t}{\pi\sqrt{1-t^2}}.$$

These measures were considered by Ullman [18].

3. The measures $w_\lambda(t)\,\mathrm{d}t$ and the proof of Theorem 1.3

In the rest of the paper we apply the results of Section 2 to the Jacobi measures $w_\lambda(t)\,\mathrm{d}t$ defined in (1.3). These measures are related to the ultraspherical measures $w_\lambda^{(0)}(t)\,\mathrm{d}t$ of (1.2) by a quadratic transformation:

$$2s w_\lambda(2s^2 - 1) = w_\lambda^{(0)}(s), \quad 0 < s < 1. \tag{3.1}$$

Lemma 3.1. *Let $\lambda < \frac{1}{2}$. If $w_\lambda(t)\,\mathrm{d}t$ admits a Chebyshev-type formula of size n and degree p then $w_\lambda^{(0)}(t)\,\mathrm{d}t$ admits a Chebyshev-type quadrature formula of size $2n$ and degree $2p+1$.*

Proof. Suppose $x_1, \ldots, x_n \in [-1,1]$ are such that

$$\int_{-1}^{1} \phi(t) w_\lambda(t)\,\mathrm{d}t = \frac{1}{n} \sum_{j=1}^{n} \phi(x_j) \tag{3.2}$$

holds for all polynomials ϕ of degree $\leqslant p$. Define $y_j := \sqrt{(1+x_j)/2}$ and consider the Chebyshev-type formula of size $2n$

$$\int_{-1}^{1} g(s) w_\lambda^{(0)}(s)\,\mathrm{d}s \approx \frac{1}{2n} \sum_{j=1}^{n} [g(y_j) + g(-y_j)]\,. \tag{3.3}$$

For odd polynomials g both sides of (3.3) are equal to zero. Let g be an even polynomial of degree $\leqslant 2p$. Then there is a polynomial ϕ of degree $\leqslant p$ such that $g(s)=\phi(s^2)$ and we find using (3.1)

$$\int_{-1}^{1} g(s)w_\lambda^{(0)}(s)\,\mathrm{d}s = 2\int_0^1 \phi(s^2)w_\lambda^{(0)}(s)\,\mathrm{d}s = 2\int_0^1 \phi(s^2)2sw_\lambda(2s^2-1)\,\mathrm{d}s.$$

Putting $2s^2 - 1 = t$ and applying (3.2) we obtain

$$\int_{-1}^{1} g(s)w_\lambda^{(0)}(s)\,\mathrm{d}s = \int_{-1}^{1} \phi((1+t)/2)w_\lambda(t)\,\mathrm{d}t = \frac{1}{n}\sum_{j=1}^{n}\phi((1+x_j)/2).$$

Since $g(y_j)=g(-y_j)=\phi((1+x_j)/2)$, it follows that equality holds in (3.3) and the lemma follows. □

From Lemma 3.1 it is clear that Corollary 1.2 is an immediate consequence of Theorem 1.1.

From now on we take $\lambda\in(0,\frac{1}{2})$ and we use the notations of Section 2 applied to the Jacobi measure $w_\lambda(t)\,\mathrm{d}t$. If we want to emphasize the dependence on λ, we write $G(z;\lambda)$, $F_n(z;\lambda)$, $K(\lambda)$ and so on. We first compute the coefficients $c_k = c_k(\lambda)$ of (2.2). We use the standard notation $(a)_k := a(a+1)\cdots(a+k-1)$.

Lemma 3.2. *For* $k \geqslant 1$,

$$c_k = 2\frac{(\lambda)_k}{(1-\lambda)_k}. \tag{3.4}$$

Proof. We use Rodrigues' formula for $T_k(t)$:

$$T_k(t) = \frac{(-1)^k}{2^k(\frac{1}{2})_k}(1-t^2)^{1/2}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^k(1-t^2)^{k-1/2}. \tag{3.5}$$

Inserting (1.3) and (3.5) into (2.2), one can evaluate c_k by integrating by parts k times.

$$\begin{aligned} c_k &= 2\frac{2^{\lambda-k}\Gamma(1-\lambda)(-1)^k}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}-\lambda)(\frac{1}{2})_k}\int_{-1}^{1}\left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^k(1-t^2)^{k-1/2}\right](1-t)^{-\lambda}\,\mathrm{d}t \\ &= 2\frac{2^{\lambda-k}\Gamma(1-\lambda)}{\Gamma(k+\frac{1}{2})\Gamma(\frac{1}{2}-\lambda)}\int_{-1}^{1}(1-t^2)^{k-1/2}\left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^k(1-t)^{-\lambda}\right]\mathrm{d}t \\ &= 2\frac{2^{\lambda-k}\Gamma(1-\lambda)(\lambda)_k}{\Gamma(k+\frac{1}{2})\Gamma(\frac{1}{2}-\lambda)}\int_{-1}^{1}(1+t)^{k-1/2}(1-t)^{-\lambda-1/2}\,\mathrm{d}t. \end{aligned}$$

The last integral is a beta integral which has the value $2^{k-\lambda}\Gamma(k+\frac{1}{2})\Gamma(\frac{1}{2}-\lambda)/\Gamma(k+1-\lambda)$, so that

$$c_k = 2\frac{\Gamma(1-\lambda)(\lambda)_k}{\Gamma(k+1-\lambda)} = 2\frac{(\lambda)_k}{(1-\lambda)_k}. \quad \square$$

By (3.4) the function $G(z)=G(z;\lambda)$ of (2.2) is

$$G(z)=2\sum_{k=1}^{\infty}\frac{(\lambda)_k}{(1-\lambda)_k}\frac{z^k}{k}. \tag{3.6}$$

It follows that $G(z)$ can be expressed in terms of hypergeometric functions. We use the notation $F(a,b;c;z)$ for hypergeometric functions as in [1, Ch. II]. Then we have

$$G'(z)=2\sum_{k=1}^{\infty}\frac{(\lambda)_k}{(1-\lambda)_k}z^{k-1}=2\frac{\lambda}{1-\lambda}F(1,1+\lambda;2-\lambda;z).$$

Thus $G'(z)$ has an analytic continuation to the complex plane with a cut along $[1,\infty)$ given by Euler's integral representation (see [1, Section 2.1.3])

$$G'(z)=2\frac{\Gamma(1-\lambda)}{\Gamma(\lambda)\Gamma(1-2\lambda)}\int_0^1 t^{\lambda}(1-t)^{-2\lambda}(1-tz)^{-1}\,\mathrm{d}t. \tag{3.7}$$

After integration we see that $G(z)$ has an analytic extension to $\mathbb{C}\setminus[1,\infty)$ given by

$$G(z)=2\frac{\Gamma(1-\lambda)}{\Gamma(\lambda)\Gamma(1-2\lambda)}\int_0^1 t^{\lambda-1}(1-t)^{-2\lambda}\log\frac{1}{1-tz}\,\mathrm{d}t,\quad |\arg(1-z)|<\pi. \tag{3.8}$$

Here log denotes the principal branch of the logarithm. Hence also $f(z)=z\exp G(z)$ has an analytic extension to $\mathbb{C}\setminus[1,\infty)$.

Before we come to the proof of Theorem 1.3 we need several lemmas. The first two describe the behavior of $|f(z)|$ along circles $|z|=r$ and along rays $\arg z=\theta$, respectively.

Lemma 3.3. *Let $r>0$ be fixed. The function*

$$\theta\mapsto|f(re^{i\theta})|,\quad 0<\theta<\pi$$

is strictly decreasing.

Proof. Clearly, $|f(re^{i\theta})|=r\exp(\Re G(re^{i\theta}))$, and by (3.8),

$$\Re G(re^{i\theta})=-2\frac{\Gamma(1-\lambda)}{\Gamma(\lambda)\Gamma(1-2\lambda)}\int_0^1 t^{\lambda-1}(1-t)^{-2\lambda}\log|1-tre^{i\theta}|\,\mathrm{d}t.$$

For every $t\in(0,1)$, $\theta\mapsto|1-tre^{i\theta}|$ is strictly increasing for $0<\theta<\pi$. Hence $\Re G(re^{i\theta})$ is strictly decreasing and the lemma follows. □

Lemma 3.4. (a) *Let $0<\lambda<\frac{1}{2}$. For every $\theta\in[\frac{1}{2}\pi,\frac{3}{2}\pi]$, there is a unique $R(\theta)\geqslant 1$ such that*

$$r\mapsto|f(re^{i\theta})|,\quad r>0$$

is increasing for $0<r<R(\theta)$ and decreasing for $r>R(\theta)$.

(b) *Let $0<\lambda\leqslant\frac{1}{4}$. For every $\theta\in(0,2\pi)$, there is a unique $R(\theta)\geqslant 1$ such that*

$$r\mapsto|f(re^{i\theta})|,\quad r>0$$

is increasing for $0<r<R(\theta)$ and decreasing for $r>R(\theta)$.

Proof. It suffices to consider $0 < \theta \leqslant \pi$. Fix θ and write $z(r) = re^{i\theta}$. We have

$$\frac{d}{dr}\log|f(z(r))| = \Re\left[\frac{d}{dr}\log f(z(r))\right] = \frac{1}{r}\Re(1 + zG'(z)).$$

Thus we have to prove that in the two cases (a) and (b) there is a unique $R(\theta) \geqslant 1$ such that

$$\Re(1 + zG'(z)) \begin{cases} > 0 & \text{for } z = re^{i\theta}, \quad 0 < r < R(\theta), \\ < 0 & \text{for } z = re^{i\theta}, \quad r > R(\theta). \end{cases} \tag{3.9}$$

By Corollary 2.3 we need only consider $r > 1$.

We distinguish the two cases $\frac{1}{2}\pi \leqslant \theta \leqslant \pi$, $0 < \lambda < \frac{1}{2}$, and $0 < \theta < \frac{1}{2}\pi$, $0 < \lambda \leqslant \frac{1}{4}$. These two cases cover the cases (a) and (b) of the lemma.

Case 1. $\frac{1}{2}\pi \leqslant \theta \leqslant \pi$ and $0 < \lambda < \frac{1}{2}$.

From (3.7) it follows that

$$1 + zG'(z) = \frac{\Gamma(1-\lambda)}{\Gamma(\lambda)\Gamma(1-2\lambda)}\int_0^1 t^{\lambda-1}(1-t)^{-2\lambda}\frac{1+tz}{1-tz}\,dt. \tag{3.10}$$

For every $t \in (0,1)$, $z \mapsto (1+tz)/(1-tz)$ is a Möbius transformation which maps $\arg z = \theta$ onto a circular arc from $+1$ to -1. Using $\frac{1}{2}\pi \leqslant \theta \leqslant \pi$, we can easily see that $\Re((1+tz)/(1-tz))$, $z = re^{i\theta}$ is a strictly decreasing function of $r > 0$. Thus $\Re(1 + zG'(z))$, $z = re^{i\theta}$ is strictly decreasing. Since it is positive for $r < 1$ and has limit -1 for $r \to \infty$, (3.9) follows in Case 1.

Case 2. $0 < \theta < \frac{1}{2}\pi$ and $0 < \lambda \leqslant \frac{1}{4}$.

We have from (3.6),

$$1 + zG'(z) = -1 + 2F(1, \lambda; 1-\lambda; z).$$

By a transformation for hypergeometric functions, see [1, Section 2.10, formula(1)], and some manipulations we arrive at

$$1 + zG'(z) = -1 + F(1, \lambda; 1+2\lambda; 1-z) + \frac{\Gamma(1-\lambda)\Gamma(1+2\lambda)}{\Gamma(1+\lambda)}z^{\lambda}(1-z)^{-2\lambda}. \tag{3.11}$$

We show that the real parts of the terms on the right-hand side of (3.11) are decreasing for $z = re^{i\theta}$, $r > 1$.

We have for $z = re^{i\theta}$,

$$\Re\left[z^{\lambda}(1-z)^{-2\lambda}\right] = r^{\lambda}(1 - 2r\cos\theta + r^2)^{-\lambda}\cos\left(2\lambda\pi + \lambda\theta - 2\lambda\,\operatorname{arccot}\frac{r\cos\theta - 1}{r\sin\theta}\right).$$

Since $\operatorname{arccot}((r\cos\theta - 1)/(r\sin\theta))$ is decreasing for $r > 0$, we obtain

$$\lambda\pi < \cos\left(2\lambda\pi + \lambda\theta - 2\lambda\operatorname{arccot}\frac{r\cos\theta - 1}{r\sin\theta}\right) < 2\lambda\pi - \lambda\theta < \frac{\pi}{2}.$$

The last inequality holds because we have assumed that $0 < \theta$ and $0 < \lambda \leqslant \frac{1}{4}$. Hence it follows that $\cos(2\lambda\pi + \lambda\theta - 2\lambda\operatorname{arccot}(r\cos\theta - 1)/(r\sin\theta))$ is positive and strictly decreasing for $r > 0$. Next since

$$\frac{\partial}{\partial r}\left[\frac{r}{1-2r\cos\theta+r^2}\right]=\frac{1-r^2}{(1-2r\cos\theta+r^2)^2},$$

it is also clear that $r^\lambda(1-2r\cos\theta+r^2)^{-\lambda}$ is positive and strictly decreasing for $r>1$. Hence the real part of the last term in (3.11) is strictly decreasing for $z=re^{i\theta}$, $r>1$.

We turn to the term $F(1,\lambda;1+2\lambda;1-z)$ in (3.11), for which we have the integral representation, see [1, Section 2.1.3],

$$F(1,\lambda;1+2\lambda;1-z)=\frac{\Gamma(1+2\lambda)}{\Gamma(\lambda)\Gamma(1+\lambda)}\int_0^1 t^{\lambda-1}(1-t)^\lambda(1-t(1-z))^{-1}\,dt.$$

An argument similar to the one given in Case 1 shows that for every $t\in(0,1)$, the real part of $(1-t(1-z))^{-1}$, $z=re^{i\theta}$ is strictly decreasing for $r>0$. Here it is important that $0<\theta<\frac{1}{2}\pi$. Hence $r\mapsto\Re(F(1,\lambda;1+2\lambda;1-z))$, $z=re^{i\theta}$ is strictly decreasing for $r>0$.

Now (3.9) follows just as in Case 1. □

We will also need the following lemma.

Lemma 3.5. *If $h(t)$ is a decreasing function on* $[0,1]$, *then the function*

$$\lambda\mapsto\frac{\Gamma(1-\lambda)}{\Gamma(\lambda)\Gamma(1-2\lambda)}\int_0^1 t^{\lambda-1}(1-t)^{-2\lambda}h(t)\,dt,\quad 0<\lambda<\tfrac{1}{2}$$

is decreasing.

Proof. Write

$$p_\lambda(t)=\frac{\Gamma(1-\lambda)}{\Gamma(\lambda)\Gamma(1-2\lambda)}t^{\lambda-1}(1-t)^{-2\lambda}.$$

Let $0<\lambda_1<\lambda_2<\frac{1}{2}$. It is easily seen that there is a $t_0\in(0,1)$ such that $p_{\lambda_1}(t)>p_{\lambda_2}(t)$ for $0<t<t_0$ and $p_{\lambda_1}(t)<p_{\lambda_2}(t)$ for $t_0<t<1$. Using the fact that $\int_0^1 p_{\lambda_1}(t)\,dt=\int_0^1 p_{\lambda_2}(t)\,dt$, we find

$$\int_0^1(p_{\lambda_1}(t)-p_{\lambda_2}(t))h(t)\,dt=\int_0^1(p_{\lambda_1}(t)-p_{\lambda_2}(t))(h(t)-h(t_0))\,dt\geqslant 0$$

since the integrand in the last integral is nonnegative on $[0,1]$. □

Proof of Theorem 1.3. Write

$$D=D(\lambda):=\{z\mid 1/z\notin K(\lambda)\}.$$

Clearly $K(\lambda)\subset\{|z|\leqslant 1\}$ if and only if $D(\lambda)\supset\{|z|<1\}$. The function $f(z)$ maps D one–one onto the disk $\{|w|<\rho^{-1}\}$. Since $z=1$ is a branch point of $f(z)$ it cannot belong to D. Hence $\{|z|<1\}\subset D$ if and only if $z=1$ belongs to ∂D and in that case $f(z)$ maps D onto $\{|w|<f(1)\}$, i.e. $\rho=f(1)^{-1}$.

For numerical computations it is helpful that $f(1)$ can be evaluated explicitly. Since [1, Section 1.7.4, formula (30)]

$$G(1) = 2\sum_{k=1}^{\infty} \frac{(\lambda)_k}{(1-\lambda)_k}\frac{1}{k} = 2(\psi(1-\lambda) - \psi(1-2\lambda)), \tag{3.12}$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$, we have

$$f(1) = \exp\left[2(\psi(1-\lambda) - \psi(1-2\lambda))\right]. \tag{3.13}$$

Consider $f(z)$ on the negative real axis. Since $f(z) = z\exp G(z)$ and $G(z)$ is real for $\arg z = \pi$, see (3.8), it is clear that $f(z)$ maps the negative real axis into itself. By Lemma 3.4(a) there is a unique $R_0 = R_0(\lambda) \geqslant 1$ such that $f(z)$ is increasing for $-R_0 < z < 0$ and decreasing for $z < -R_0$. Thus the negative real axis is mapped onto the interval $[f(-R_0), 0)$. This gives a *necessary condition* for $\{|z| < 1\} \subset D$, namely

$$|f(-R_0)| \geqslant f(1). \tag{3.14}$$

This is a condition on λ. Note that both $f(z)$ and R_0 depend on λ; we write $f(z;\lambda)$ and $R_0(\lambda)$. The right-hand side of (3.14) is strictly increasing for $0 < \lambda < \frac{1}{2}$. This is obvious from (3.13) and (3.12). We prove that the left-hand side is decreasing. By (3.8) we have for a fixed $x > 0$,

$$\log|f(-x;\lambda)| = G(-x;\lambda) + \log x = \frac{\Gamma(1-\lambda)}{\Gamma(\lambda)\Gamma(1-2\lambda)}\int_0^1 t^{\lambda-1}(1-t)^{-2\lambda}\log\frac{x}{(1+xt)^2}\,dt. \tag{3.15}$$

Since $\log(x/(1+xt)^2)$ is decreasing for $t \in [0,1]$, Lemma 3.5 gives that $\log|f(-x;\lambda)|$ is decreasing in λ. Then if $0 < \lambda_1 < \lambda_2 < \frac{1}{2}$,

$$|f(-R_0(\lambda_2);\lambda_2)| \leqslant |f(-R_0(\lambda_2);\lambda_1)| \leqslant |f(-R_0(\lambda_1);\lambda_1)|,$$

where the second inequality holds because $|f(-x;\lambda_1)|$, $x > 0$ is maximal for $x = R_0(\lambda_1)$. This proves that the left-hand side of (3.14) is decreasing for $0 < \lambda < \frac{1}{2}$.

Since $\lim_{\lambda\to 0}|f(-R_0)| = \infty$, $\lim_{\lambda\to 0} f(1) = 0$ and $\lim_{\lambda\to 1/2} f(1) = \infty$, it now follows that there is a unique $\lambda^* > 0$ such that (3.14) holds if and only if $0 < \lambda \leqslant \lambda^*$. Numerical computations give the approximation

$$\lambda^* \approx 0.1768\ldots . \tag{3.16}$$

Next we prove that the necessary condition (3.14) (or equivalently $0 < \lambda \leqslant \lambda^*$), is also a *sufficient condition* for $\{|z| < 1\} \subset D$.

Thus suppose (3.14) holds and let $R_0 = R_0(\lambda)$ be as before. By Lemma 3.3 and (3.14) we have for every θ, $0 < \theta \leqslant \pi$

$$|f(R_0 e^{i\theta})| \geqslant f(1).$$

Let $R(\theta)$ be as in Lemma 3.4(b) (which we may apply since $\lambda^* < \frac{1}{4}$, see also remark (2) below). It follows that $|f(R(\theta)e^{i\theta})| \geqslant f(1)$ and that there is a unique $r(\theta)$, $1 \leqslant r(\theta) \leqslant R(\theta)$ such that

$$|f(r(\theta)e^{i\theta})| = f(1), \quad 0 < \theta \leqslant \pi. \tag{3.17}$$

The uniqueness of $r(\theta)$ implies that $r(\theta)$ depends continuously on θ. We define $r(0) = 1$, and $r(2\pi - \theta) = r(\theta)$. Then (3.17) holds for $0 \leqslant \theta \leqslant 2\pi$. The closed curve $\theta \mapsto r(\theta)e^{i\theta}$, $0 \leqslant \theta \leqslant 2\pi$ encloses a domain D_0,

$$D_0 := \{re^{i\theta} \mid 0 \leqslant \theta < 2\pi, 0 \leqslant r < r(\theta)\}, \tag{3.18}$$

which contains $\{|z| < 1\}$. We can now easily see (using Rouché's theorem and the fact that $f(z)$ has precisely one zero in D_0) that $f(z)$ maps D_0 one–one onto $\{|w| < f(1)\}$. Thus $D_0 = D(\lambda)$ and we have shown that (3.14) is also a sufficient condition for $\{|z| < 1\} \subset D$.

Finally, it is clear from (3.18) that $D(\lambda)$ (and thus also $K(\lambda)$) is starlike with respect to the origin. □

Corollary 3.6. *For* $0 < \lambda \leqslant \lambda^*$, *we have*

$$\Re(1 + zG'(z)) > 0, \quad z \in D(\lambda). \tag{3.19}$$

Proof. Since $D(\lambda)$ is starlike (by Theorem 1.3), we obtain from [14, Theorem 2.5] that $\Re(w\phi'(w)/\phi(w)) > 0$, $|w| < \rho^{-1}$, where $\phi(w)$ is the inverse of $f(z)$. This readily implies (3.19). □

Note that Corollary 3.6 is an extension of Corollary 2.3.

Remarks. (1) The proof of Theorem 1.3 also gives a simple method to compute λ^* numerically. For given λ, compute the unique maximum of $|f(-x;\lambda)|$ for $x > 0$, with the aid of formula (3.15). If this maximum is larger than $f(1;\lambda)$ then $\lambda < \lambda^*$; if not then $\lambda \geqslant \lambda^*$. The value of $f(1;\lambda)$ is evaluated explicity in (3.13). In this way we obtained the approximation (3.16).

(2) In the proof of Theorem 1.3 we used the fact that $\lambda^* < \frac{1}{4}$. This follows of course from the numerical estimate (3.16), but can also be shown directly as follows.

If $\lambda = \frac{1}{4}$ and $x > 1$, then it can be seen from (3.11) that

$$\Re(1 + xG'(x)) = -1 + F(1, \lambda; 1 + 2\lambda; 1 - x).$$

Hence $\lim_{x \downarrow 1} \Re(1 + xG'(x)) = 0$, and it follows as in the proof of Lemma 3.4 that $|f(x)|$ is decreasing for $x > 1$. Thus $|f(R_0)| \leqslant f(1)$. By Lemma 3.3 we have $|f(-R_0)| < |f(R_0)|$ and it follows that $|f(-R_0)| < f(1)$. Hence (3.14) does not hold if $\lambda = \frac{1}{4}$ and therefore $\lambda^* < \frac{1}{4}$.

4. The proof of Theorem 1.4: starlikeness with respect to $z = 1$ and positive coefficients

4.1. A result on the zeros of $F_n(z)$

In view of Theorem 2.5 it would be sufficient for our purposes to prove that for λ sufficiently small, $K(\lambda)$ is a convex set. Unfortunately, we were not able to prove this, although computer experiments indicate that it is very likely. Instead of Theorem 2.5 we will use the following result to conclude that the zeros of the Faber polynomials are in the interior of $K(\lambda)$.

Proposition 4.1. *Let $0 < \lambda \leqslant \lambda^*$ with λ^* as in Theorem 1.3. Let $\Phi(w)$ be the inverse of $F(z)$ and write*

$$\Phi(w) = w + \sum_{k=0}^{\infty} b_k w^{-k}.$$

Suppose that the following properties are satisfied:
(A) *$K(\lambda)$ is starlike with respect to $z = 1$.*
(B) *The coefficients b_k are positive for every k.*
Then the zeros of every Faber polynomial $F_n(z)$ are located in the interior of $K(\lambda)$.

Proof. Let as before $\rho = F(1) = \text{cap}(K)$. Then $\theta \mapsto \Phi(\rho e^{i\theta})$, $\theta \in [0, 2\pi)$ is a parametrization of ∂K. Property (A) implies that $\arg(\Phi(\rho e^{i\theta}) - 1)$, $0 < \theta < 2\pi$ is nondecreasing. Thus,

$$0 \leqslant \frac{\partial}{\partial\theta} \arg(\Phi(\rho e^{i\theta}) - 1) = \Im\left[\frac{\partial}{\partial\theta}\log(\Phi(\rho e^{i\theta}) - 1)\right] = \Re\left[\frac{\rho e^{i\theta}\Phi'(\rho e^{i\theta})}{\Phi(\rho e^{i\theta}) - 1}\right], \quad 0 < \theta < 2\pi,$$

or

$$\Re\left[\frac{w\Phi'(w)}{\Phi(w) - 1}\right] \geqslant 0, \quad |w| = \rho, w \neq \rho.$$

Since $\Re((w + \rho)/(w - \rho)) = 0$ for $|w| = \rho$, $w \neq \rho$, it follows that

$$\Re[H(w)] \geqslant 0, \quad |w| = \rho, w \neq \rho, \tag{4.1}$$

where

$$H(w) := \frac{w\Phi'(w)}{\Phi(w) - 1} - \frac{1}{2}\frac{w + \rho}{w - \rho}, \quad |w| \geqslant \rho, w \neq \rho. \tag{4.2}$$

To study the behavior of $H(w)$ near $w = \rho$ we put $w = 1/f(z)$, where $f(z) = z \exp G(z)$. Then (4.2) gives

$$H\left(\frac{1}{f(z)}\right) = \frac{1}{(1 - z)(1 + zG'(z))} - \frac{1}{2}\frac{1 + \rho f(z)}{1 - \rho f(z)}, \quad z \in \overline{D}, z \neq 1. \tag{4.3}$$

Here $D = D(\lambda)$ is as in the proof of Theorem 1.3. Since by (3.11)

$$1 + zG'(z) = \frac{\Gamma(1 - \lambda)\Gamma(1 + 2\lambda)}{\Gamma(1 + \lambda)}(1 - z)^{-2\lambda} + \mathcal{O}((1 - z)^{1-2\lambda}), \quad (z \in D, z \to 1),$$

we have

$$\frac{1}{(1 - z)(1 + zG'(z))} = \frac{\Gamma(1 + \lambda)}{\Gamma(1 - \lambda)\Gamma(1 + 2\lambda)}(1 - z)^{-1+2\lambda} + \mathcal{O}((1 - z)^{2\lambda}), \quad (z \in D, z \to 1).$$

Further, from

$$G(1) - G(z) = \frac{\Gamma(1 - \lambda)\Gamma(1 + 2\lambda)}{(1 - 2\lambda)\Gamma(1 + \lambda)}(1 - z)^{1-2\lambda} + \mathcal{O}(1 - z), \quad (z \in D, z \to 1),$$

see (4.9)–(4.11) below, it follows that

$$1-\rho f(z)=\frac{\Gamma(1-\lambda)\Gamma(1+2\lambda)}{(1-2\lambda)\Gamma(1+\lambda)}(1-z)^{1-2\lambda}+\mathcal{O}(1-z),\quad (z\in D, z\to 1),$$

and

$$\frac{1}{2}\frac{1+\rho f(z)}{1-\rho f(z)}=\frac{(1-2\lambda)\Gamma(1+\lambda)}{\Gamma(1-\lambda)\Gamma(1+2\lambda)}(1-z)^{-1+2\lambda}+\mathcal{O}((1-z)^{-1+4\lambda}),\quad (z\in D, z\to 1).$$

So from (4.3)

$$H(1/f(z))=2\lambda\frac{\Gamma(1+\lambda)}{\Gamma(1-\lambda)\Gamma(1+2\lambda)}(1-z)^{-1+2\lambda}+\mathcal{O}((1-z)^{-1+4\lambda}),\quad (z\in D, z\to 1),$$

which implies that $H(1/f(z))>0$ for $z\in\mathbb{R}$, $z<1$ close to $z=1$. Consequently $H(w)>0$ for $w\in\mathbb{R}$, $w>\rho$ close to $w=\rho$. Together with (4.1) this implies by the maximum principle for harmonic functions that $\Re(H(w))>0$ for $|w|>\rho$.

We have the series expansion, see (2.8) and (4.2),

$$H(w)=\frac{1}{2}+\sum_{n=1}^{\infty}(F_n(1)-\rho^n)w^{-n}$$

which converges for $|w|>\rho$, since $H(w)$ is analytic for $|w|>\rho$. Hence the function

$$w\mapsto 2H(\rho/w)=1+2\sum_{n=1}^{\infty}(F_n(1)-\rho^n)\frac{w^n}{\rho^n}$$

has positive real part for $|w|<1$. Then Carathéodory's coefficient estimate, see [14, Corollary 2.3], gives

$$|F_n(1)-\rho^n|<\rho^n,\quad n=1,2,\ldots. \tag{4.4}$$

Next we write

$$F_n(\Phi(w))-w^n=\sum_{k=1}^{\infty}\beta_{nk}w^{-k},\quad |w|\geqslant\rho.$$

The coefficients β_{nk} are called the Grunsky coefficients. Schur [15] proved that each β_{nk} can be expressed as a polynomial with nonnegative integer coefficients in the coefficients b_k of Φ. Therefore, Property (B) of the proposition implies that all Grunsky coefficients are positive. Hence for $|w|\geqslant\rho$,

$$|F_n(\Phi(w))-w^n|\leqslant\sum_{k=1}^{\infty}\beta_{nk}\rho^{-k}=F_n(1)-\rho^n<\rho^n\leqslant|w|^n, \tag{4.5}$$

where we used the inequality (4.4). From (4.5) it is clear that $|F_n(\Phi(w))|\neq 0$ for $|w|\geqslant\rho$. That is, all zeros of $F_n(z)$ are in the interior of $K(\lambda)$. □

In the rest of this section it will be proved that for λ sufficiently small, the Properties (A) and (B) of Proposition 4.1 are satisfied.

4.2. Some estimates

First we need estimates on $1+zG'(z)$ and $G(z)+\log z$. We write as in (3.11),

$$1+zG'(z)=A(z)+B(z) \tag{4.6}$$

with

$$A(z)=-1+F(1,\lambda;1+2\lambda;1-z)=\frac{\lambda}{1+2\lambda}(1-z)F(1,1+\lambda;2+2\lambda;1-z), \tag{4.7}$$

$$B(z)=\frac{\Gamma(1-\lambda)\Gamma(1+2\lambda)}{\Gamma(1+\lambda)}z^{\lambda}(1-z)^{-2\lambda}. \tag{4.8}$$

After integration we obtain

$$G(z)+\log z=G(1)+A_0(z)+B_0(z) \tag{4.9}$$

with

$$A_0(z)=\int_1^z\frac{A(t)}{t}\,dt,\quad B_0(z)=\int_1^z\frac{B(t)}{t}\,dt. \tag{4.10}$$

For $B_0(z)$ we have the following expressions:

$$B_0(z)=\frac{\Gamma(1-\lambda)\Gamma(1+2\lambda)}{\Gamma(1+\lambda)}\int_1^z t^{\lambda-1}(1-t)^{-2\lambda}\,dt \tag{4.11}$$

$$=-\frac{\Gamma(1-\lambda)\Gamma(1+2\lambda)}{(1-2\lambda)\Gamma(1+\lambda)}(1-z)^{1-2\lambda}z^{\lambda-1}F(1,1-\lambda;2-2\lambda;1-1/z). \tag{4.12}$$

To obtain (4.12) from (4.11) we have applied [1, Section 2.5.3, p.87] to express the incomplete beta integral as a hypergeometric function followed by the transformation [1, Section 2.10, formula (6)].

Lemma 4.2. *For $\Re z\geqslant 1$ with $|z-1|=r$ the following estimates hold:*

$$|A(z)|\leqslant\lambda r, \tag{4.13}$$

$$|A_0(z)|\leqslant\lambda r^2(r+1)^{-1}, \tag{4.14}$$

$$|B(z)|\geqslant r^{-2\lambda}, \tag{4.15}$$

$$|B_0(z)|\geqslant r^{1-2\lambda}(r+1)^{\lambda-1}. \tag{4.16}$$

Proof. The Euler integral formula gives

$$F(1,1+\lambda;2+2\lambda;1-z)=\frac{\Gamma(2+2\lambda)}{\Gamma(1+\lambda)^2}\int_0^1 t^{\lambda}(1-t)^{\lambda}\frac{1}{1-t(1-z)}\,dt.$$

For $\Re z\geqslant 1$ and $t\in[0,1]$ we have $|1-t(1-z)|\geqslant 1$. Hence,

$$|F(1,1+\lambda;2+2\lambda;1-z)|\leqslant 1,\quad \Re z\geqslant 1.$$

In view of (4.7) this gives (4.13).

For $z = 1 + re^{i\psi}$, $|\psi| \leqslant \frac{1}{2}\pi$, we have by (4.10)

$$A_0(z) = \int_0^r \frac{A(1+se^{i\psi})}{1+se^{i\psi}} e^{i\psi}\, ds.$$

By (4.13) we have $|A(1+se^{i\psi})| \leqslant \lambda s$ for every $s > 0$. Also $|1+se^{i\psi}| \geqslant \sqrt{1+s^2}$. Hence,

$$|A_0(z)| \leqslant \lambda \int_0^r \frac{s}{\sqrt{1+s^2}}\, ds = \lambda(\sqrt{1+r^2} - 1) \leqslant \lambda \frac{r^2}{r+1},$$

which is (4.14).

The estimate (4.15) is obvious from (4.8), since $\Gamma(1-\lambda)\Gamma(1+2\lambda) > \Gamma(1+\lambda)$.

To estimate $|B_0(z)|$ we use (4.12). From the Euler integral

$$F(1, 1-\lambda; 2-2\lambda; 1-1/z) = \frac{\Gamma(2-2\lambda)}{\Gamma(1-\lambda)^2} \int_0^1 t^{-\lambda}(1-t)^{-\lambda} \frac{1}{1-t(1-1/z)}\, dt \tag{4.17}$$

and the fact that $\Re(1/(1-t(1-1/z))) \geqslant 1$ for $\Re z \geqslant 1$, $t \in [0,1]$, we get

$$|F(1, 1-\lambda; 2-2\lambda; 1-1/z)| \geqslant 1, \quad \Re z \geqslant 1.$$

Now (4.16) follows from this and (4.12). □

4.3. *Starlikeness with respect to $z = 1$*

We will now concentrate on Property (A) and prove that $K(\lambda)$ is starlike with respect to $z = 1$ in case λ is sufficiently small.

Lemma 4.3. *Let $0 < \lambda \leqslant \lambda^*$. Then $K(\lambda)$ is starlike with respect to $z = 1$ if and only if*

$$\Re[(1-z)(1+zG'(z))] \geqslant 0, \quad z \in D(\lambda). \tag{4.18}$$

Proof. Proof as in [14, Theorem 2.9]. See also the proof of Proposition 4.1. □

To prove that (4.18) holds we divide $D(\lambda)$ into two parts,

$$D_1 = D_1(\lambda) := D(\lambda) \cap \{\Re z \leqslant 1\}, \quad D_2 = D_2(\lambda) := D(\lambda) \cap \{\Re z \geqslant 1\} \tag{4.19}$$

and we consider (4.18) in the two parts D_1, D_2 separately.

Proposition 4.4. *Let $0 < \lambda \leqslant \lambda^*$. For $z \in D_1(\lambda)$ the inequality* (4.18) *holds.*

Proof. Let $z \in D_1(\lambda)$. Without loss of generality we may assume $\Im z \geqslant 0$. From (3.10) it is easily seen that $\Im(1+zG'(z)) \geqslant 0$. By Corollary 3.6 we also have $\Re(1+zG'(z)) \geqslant 0$. Hence $0 \leqslant \arg(1+zG'(z)) \leqslant \frac{1}{2}\pi$. Since $-\frac{1}{2}\pi \leqslant \arg(1-z) \leqslant 0$ for $z \in D_1$, it follows that

$$-\tfrac{1}{2}\pi \leqslant \arg((1-z)(1+zG'(z))) \leqslant \tfrac{1}{2}\pi, \quad z \in D_1,$$

which gives (4.18). □

The part $D_2(\lambda)$ will be more difficult to handle. We need two lemmas.

Lemma 4.5. *For $0 < \lambda \leqslant \lambda^*$, let $y(\lambda)$ be defined by*

$$y(\lambda) := \sup\{y > 0 \mid 1 + iy \in D(\lambda)\}. \tag{4.20}$$

There is a constant $C_1 > 0$ such that for λ small enough,

$$y(\lambda) \leqslant C_1\sqrt{\lambda}. \tag{4.21}$$

Proof. Clearly $\lim_{\lambda\to 0} y(\lambda) = 0$. Let λ be such that $y(\lambda) < 1$. If $z = 1 + iy(\lambda)$ then $\log(1/|1 - tz|) \geqslant 0$ for every $t \in [0, 1]$. Hence by (3.8) $\Re G(z) \geqslant 0$ and then $f(1) = |f(z)| = |z \exp G(z)| \geqslant |z|$. By (3.13) this gives

$$y(\lambda)^2 \leqslant -1 + \exp(4(\psi(1-\lambda) - \psi(1-2\lambda)),$$

and (4.21) follows. □

Lemma 4.6. *Let $0 < \lambda \leqslant \lambda^*$. The boundary of $D(\lambda)$ makes in $z = 1$ an angle*

$$\theta = \theta(\lambda) := \frac{1}{2}\pi - \frac{\lambda}{1-2\lambda}\pi \tag{4.22}$$

with the interval $[1, \infty)$. Furthermore, for λ sufficiently small, the half-line $z = 1 + re^{i\theta(\lambda)}, r > 0$ is completely outside $D(\lambda)$.

Proof. From (4.9)–(4.11) it follows that

$$G(z) + \log z = G(1) - \frac{\Gamma(1-\lambda)\Gamma(1+2\lambda)}{(1-2\lambda)\Gamma(1+\lambda)}(1-z)^{1-2\lambda} + \mathcal{O}(1-z), \quad (z \to 1).$$

Since the boundary of $D(\lambda)$ is the curve $\Re(G(z) + \log z) = G(1)$, it follows that the angle with $[1, \infty)$ is given by (4.22).

Next we write $z(r) = 1 + re^{i\theta}$, $r > 0$ with $\theta = \theta(\lambda)$ defined by (4.22) and we prove that for λ small, $r \mapsto |f(z(r))|$ is increasing for $0 < r < 1$. An easy calculation shows that this amounts to proving that

$$\Re\left[\frac{z'}{z}(1 + zG'(z))\right] > 0, \quad z = z(r), 0 < r < 1. \tag{4.23}$$

From (4.8) we have for $z = z(r)$,

$$\arg B(z) = 2\lambda\theta + \lambda \arg z = \tfrac{1}{2}\pi - \theta + \lambda \arg z.$$

From (4.13) and (4.15) it follows that

$$|\arg(1 + A(z)/B(z))| \leqslant |A(z)/B(z)| \leqslant \lambda r^{1+2\lambda}.$$

It now readily follows that

$$\arg\left[\frac{z'}{z}(1 + zG'(z))\right] = \tfrac{1}{2}\pi - (1-\lambda)\arg z + \delta(z;\lambda), \quad \text{where } |\delta(z;\lambda)| \leqslant \lambda r^{1+2\lambda}.$$

Since $\arg z \geqslant Cr$ for some constant $C > 0$ (independent of λ), we see that for λ sufficiently small, (4.23) holds.

Now (4.23) implies that $z = z(r)$, $0 < r < 1$ is outside of $D(\lambda)$ for λ small. It is easy to see that $z = z(r)$, $r > 1$ is outside of $D(\lambda)$ for λ sufficiently small, and therefore the lemma is proved. □

Proposition 4.7. *For $\lambda > 0$ sufficiently small, the inequality* (4.18) *holds for every $z \in D_2(\lambda)$.*

Proof. Let λ be such that the half-line $z = 1 + re^{i\theta(\lambda)}$, $r > 0$ is outside of $D(\lambda)$ (see Lemma 4.6) and such that $y(\lambda) \leqslant C_1\sqrt{\lambda}$ (see Lemma 4.5). Let $z \in D_2(\lambda)$, $|z-1| = r$. We may assume $\Im z \geqslant 0$. From Lemma 4.6 it follows that

$$-\pi + \theta \leqslant \arg(1-z) \leqslant -\tfrac{1}{2}\pi. \tag{4.24}$$

From (4.8) we have

$$\arg((1-z)B(z)) = (1-2\lambda)\arg(1-z) + \lambda \arg z. \tag{4.25}$$

Using (4.22), (4.24) and (4.25) we get

$$-\tfrac{1}{2}\pi + \lambda \arg z \leqslant \arg((1-z)B(z)) \leqslant 0.$$

Since by (4.15) $|(1-z)B(z)| \geqslant r^{1-2\lambda}$ we obtain from this

$$\Re((1-z)B(z)) \geqslant r^{1-2\lambda}\sin(\lambda \arg z). \tag{4.26}$$

Next from (4.13) we have $|(1-z)A(z)| \leqslant \lambda r^2$ and it follows that

$$\Re[(1-z)(1+zG'(z))] \geqslant r^{1-2\lambda}\sin(\lambda\arg z) - \lambda r^2. \tag{4.27}$$

Hence to prove (4.18) it suffices to show that the right-hand side of (4.27) is positive.

From Lemma 4.6 we have

$$\arg z \geqslant \arg(1 + re^{i\theta}) = \arctan\frac{r\sin\theta}{1 + r\cos\theta}$$

and from Lemma 3.3 and the definition of $y(\lambda)$ it follows that $r \leqslant y(\lambda) \leqslant C_1\sqrt{\lambda}$. Thus we are left to prove

$$\sin\left(\lambda \arctan\frac{r\sin\theta}{1+r\cos\theta}\right) \geqslant \lambda r^{1+2\lambda}, \quad r \leqslant C_1\sqrt{\lambda}, \tag{4.28}$$

for λ sufficiently small.

Using (4.22), it is rather straightforward to show that for $r \leqslant C_1\lambda$,

$$\sin\left(\lambda \arctan\frac{r\sin\theta}{1+r\cos\theta}\right) = \lambda r(1 + \mathcal{O}(\lambda)), \quad (\lambda \to 0).$$

For the right-hand side of (4.28) we have for $r \leqslant C_1\sqrt{\lambda}$,

$$\lambda r^{1+2\lambda} \leqslant \lambda r(C_1\sqrt{\lambda})^{2\lambda} = \lambda r(1 + \lambda\log\lambda + \mathcal{O}(\lambda)), \quad (\lambda \to 0).$$

Now (4.28) easily follows and the proposition is proved. □

Combining Lemma 4.3 with Propositions 4.4 and 4.7 we have proved the following result.

Theorem 4.8. *There is $\lambda_1 \in (0, \lambda^*]$ such that for every $0 < \lambda \leqslant \lambda_1$, the set $K(\lambda)$ is starlike with respect to $z = 1$.*

4.4. Positive coefficients

We turn to the proof of Property (B) of Proposition 4.1. Let

$$\Phi(w) = w + b_0 + \frac{b_1}{w} + \frac{b_2}{w^2} + \cdots \tag{4.29}$$

be the inverse of $F(z)$. Of course, the coefficients b_k depend on λ. If necessary we write $b_k = b_k(\lambda)$.

Lemma 4.9. *For $k \geqslant 1$,*

$$b_k = -\frac{1}{2\pi i k} \int_\gamma f(z)^{-k} z^{-2} \,\mathrm{d}z, \tag{4.30}$$

where γ is a closed contour which encloses $z = 0$ once in the positive direction.

Proof. From (4.29) and the Cauchy formula we get

$$b_k = \frac{1}{2\pi i} \int_C \Phi(w) w^{k-1} \,\mathrm{d}w.$$

Set in the integral $w = 1/f(z)$ to obtain

$$b_k = \frac{1}{2\pi i} \int_\gamma f(z)^{-k-1} z^{-1} f'(z) \,\mathrm{d}z$$

and integrate by parts

$$b_k = \frac{1}{2\pi i} \int_\gamma f(z)^{-k} z^{-2} \,\mathrm{d}z + (k+1) b_k.$$

This gives (4.30). □

To obtain the positivity of b_k from the integral in (4.30), we will consider the path of steepest descent from $z = 1$. This is the curve for which $G(z) + \log z$ is real and $> G(1)$. To be precise, we define $z(s) = z(s; \lambda)$ as the solution of

$$G(z) + \log z = G(1) + s, \quad z(0) = 1$$

which lies on the sheet of the Riemann surface of $G(z) + \log z$ where

$$-\tfrac{3}{2}\pi \leqslant \arg(1 - z) \leqslant -\pi, \quad -\tfrac{1}{2}\pi \leqslant \arg z \leqslant 0.$$

By (4.9) we have

$$A_0(z) + B_0(z) = s, \quad z = z(s). \tag{4.31}$$

We need several estimates on $z(s)$.

Lemma 4.10. (1) *For* $s > 0$, $\arg z(s) < 0$.

(2) *There is a constant* $C_2 > 0$ *such that for all* λ *sufficiently small, we have* $\arg(1-z(s)) > -\frac{5}{4}\pi$, *as long as* $|z(s)-1| \leqslant C_2\lambda^{-1/3}$.

(3) *If* $|z(s)-1| < 1$ *then* $\arg(1-z(s)) < -\pi - \lambda\pi$.

(4) *There is a constant* $C_3 > 0$, *independent of* λ, *such that for* $|z(s)-1| < 1$, *we have* $|z(s)-1| \geqslant C_3 s^{1/(1-2\lambda)}$.

Proof. (1) For $\arg z = 0$, $z > 1$, $A_0(z)$ is real and by (4.12) $\arg B_0(z) = (1-2\lambda)(-\pi) + \pi = 2\lambda\pi$. Hence $A_0(z) + B_0(z)$ is not real for such z, and it follows that the curve $z(s)$ does not cross the real axis. This proves (1).

(2) Let $\arg(1-z) = -\frac{5}{4}\pi$ and $r = |z-1|$. It is easy to see from the integral representation (4.17) that $\arg F(1, 1-\lambda; 2-2\lambda; 1-1/z) < 0$. Thus it follows from (4.12) that

$$\arg B_0(z) < \pi - (1-2\lambda)\tfrac{5}{4}\pi + (\lambda - 1)\arg z = -\tfrac{1}{4}\pi + \tfrac{5}{2}\lambda\pi + (1-\lambda)|\arg z|.$$

Since

$$|\arg z| = \arctan\left(1 - \frac{\sqrt{2}}{\sqrt{2}+r}\right) \leqslant \tfrac{1}{4}\pi - \frac{\sqrt{2}}{2\sqrt{2}+2r},$$

we find

$$\arg B_0(z) \leqslant \tfrac{9}{4}\lambda\pi - (1-\lambda)\frac{\sqrt{2}}{2\sqrt{2}+2r}.$$

Further from (4.14) and (4.16) it follows that $|A_0(z)/B_0(z)| \leqslant \lambda r^{1+\lambda}$, so that

$$\arg(A_0(z) + B_0(z)) \leqslant \tfrac{9}{4}\lambda\pi - (1-\lambda)\frac{\sqrt{2}}{2\sqrt{2}+2r} + \lambda r^{1+\lambda}.$$

It is easy to see from this that there exists $C_2 > 0$, independent of λ, such that for $r < C_2\lambda^{-1/3}$ and λ sufficiently small, we have $\arg(A_0(z) + B_0(z)) < 0$. This implies that $z(s)$ does not cross the line $\arg(1-z) = -\frac{5}{4}\pi$ as long as $|z(s)-1| < C_2\lambda^{-1/3}$. This proves (2).

(3) Let $z = 1 + re^{-i\lambda\pi}, 0 < r < 1$. From (4.11) we get

$$\begin{aligned} B_0(z) &= e^{2\lambda\pi i}\frac{\Gamma(1-\lambda)\Gamma(1+2\lambda)}{\Gamma(1+\lambda)}\int_1^z (t-1)^{-2\lambda}t^{\lambda-1}\,dt \\ &= e^{2\lambda\pi i}e^{-i\lambda\pi}e^{2\lambda^2\pi i}\frac{\Gamma(1-\lambda)\Gamma(1+2\lambda)}{\Gamma(1+\lambda)}\int_0^r \rho^{-2\lambda}\frac{1}{(1+\rho e^{-i\lambda\pi})^{1-\lambda}}\,d\rho, \end{aligned}$$

where we put $t = 1 + \rho e^{-i\lambda\pi}$. It follows that

$$\arg B_0(z) > 2\lambda\pi - \lambda\pi + 2\lambda^2\pi > \lambda\pi.$$

Since from (4.14) and (4.16) we have for $0 < r < 1$, $|A_0(z)/B_0(z)| \leqslant \lambda r^{1+\lambda} < \lambda$, we see that

$$\arg(A_0(z) + B_0(z)) > \lambda\pi - \lambda > 0$$

and (3) follows.

(4) From (4.12) we easily get that

$$|B_0(z)| \leqslant Kr^{1-2\lambda}, \quad \Re z \geqslant 1, r = |z-1| < 1.$$

Here $K > 1$ is a constant independent of λ. Combined with (4.14) this gives

$$|A_0(z) + B_0(z)| \leqslant Kr^{1-2\lambda}, \quad \Re z \geqslant 1, r = |z-1| < 1,$$

with a (possibly larger) positive constant K, independent of λ. So if $|z(s) - 1| = r < 1$, then

$$s = |A_0(z(s)) + B_0(z(s))| \leqslant Kr^{1-2\lambda}.$$

This gives (4) with e.g., $C_3 = K^{-2}$. □

Theorem 4.11. *There is $\lambda_2 \in (0, \lambda^*]$ such that for every $0 < \lambda \leqslant \lambda_2$, all coefficients $b_k(\lambda)$ are positive.*

Proof. In formula (4.30) we take the following contour γ. Let $R_1(\lambda) := C_2\lambda^{-1/3}$, where C_2 is as in Lemma 4.10 part (2), and let γ be the closed contour starting in $z = 1$, following the upper side of the cut $[1, \infty)$ up to the point $R_1(\lambda)$, followed by the circle $|z| = R_1(\lambda)$, and then on the lower side of the cut back to $z = 1$. Let λ be so small that if $|z(s) - 1| < R_1(\lambda)$ then $\arg(1 - z(s)) > -\frac{5}{4}\pi$, see Lemma 4.10 part (2). [In the course of the proof we might take a smaller λ.]

Since γ is symmetric in the real axis, (4.30) gives

$$b_k = -\frac{1}{\pi k}\Im \int_{\gamma^+} f(z)^{-k} z^{-2}\, dz, \tag{4.32}$$

where γ^+ is the part of γ in the upper half-plane.

Let $s_0 = s_0(\lambda)$ be such that $|z(s_0(\lambda); \lambda)| = R_1(\lambda)$, where $z(s) = z(s; \lambda)$ is defined as in (4.31). We deform γ^+ into a contour γ^* as follows. Starting at $z = 1$ we follow the curve $z(s)$ from $s = 0$ to $s = s_0$. This part will be denoted by γ_1^*. Then we follow the circle $|z| = R_1(\lambda)$ from $z(s_0)$ to $-R_1(\lambda)$. This part will be denoted by γ_2^*.

The integral in (4.32) can be taken over γ^* instead of γ^+ and we get $b_k = b_{k,1} + b_{k,2}$, where

$$b_{k,j} = -\frac{1}{\pi k}\Im \int_{\gamma_j^*} f(z)^{-k} z^{-2}\, dz, \quad j = 1, 2.$$

We are going to prove that $b_{k,1} > |b_{k,2}|$ for all k in case λ is sufficiently small.

Using the substitution $G(z) + \log z = G(1) + s$, $0 < s < s_0$, we get

$$b_{k,1} = -\frac{1}{\pi k} e^{-kG(1)} \Im \int_0^{s_0} e^{-ks} \frac{z'(s)}{z^2(s)}\, ds.$$

We integrate by parts to obtain

$$b_{k,1} = \frac{1}{\pi k} e^{-kG(1)} \Im \left[e^{-ks_0} \frac{1}{z(s_0)} - 1 + k \int_0^{s_0} e^{-ks} \frac{1}{z(s)}\, ds \right].$$

By Lemma 4.10, parts (1) and (2) and by the definition of s_0, $1/z(s)$ has positive imaginary part for $0 < s \leqslant s_0$. Hence

$$b_{k,1} \geqslant \frac{1}{\pi} e^{-kG(1)} \int_0^{s_0} e^{-ks} \Im\left(\frac{1}{z(s)}\right) ds. \tag{4.33}$$

Next, we let $s_1 = s_1(\lambda)$ be such that $|z(s_1(\lambda);\lambda) - 1| = 1$. Since $z(s_1(\lambda);\lambda) \to 2$ for $\lambda \to 0$, we get $s_1(\lambda) \to \log 2$, and there is a constant $\sigma > 0$ such that $s_1(\lambda) > \sigma$ for all λ. For $s < \sigma$ we have

$$\Im(1/z(s)) = -\frac{\Im z(s)}{|z(s)|^2} \geqslant \tfrac{1}{4}|\Im z(s)| = \tfrac{1}{4}|\Im(z(s)-1)| = \tfrac{1}{4}|z(s)-1||\sin\arg(1-z(s))|.$$

Using Lemma 4.10 parts (3) and (4) we obtain from this

$$\Im\left(\frac{1}{z(s)}\right) \geqslant K_1 s^{1/(1-2\lambda)}\lambda \geqslant K_1 s^{1/2}\lambda, \tag{4.34}$$

with a constant $K_1 > 0$, independent of λ. Thus from (4.33) and (4.34) and the positivity of $\Im(1/z(s))$,

$$\begin{aligned} b_{k,1} &\geqslant \frac{1}{\pi}e^{-kG(1)}\int_0^\sigma e^{-ks}\Im\left(\frac{1}{z(s)}\right)ds \geqslant \frac{1}{\pi}e^{-kG(1)}K_1\lambda\int_0^\sigma e^{-ks}s^{1/2}\,ds \\ &\geqslant K_2\lambda k^{-3/2}e^{-kG(1)} \end{aligned} \tag{4.35}$$

with another positive constant K_2.

Next we estimate $b_{k,2}$. On the circle $|z| = R_1(\lambda)$ we have $|f(z)| \geqslant |f(-R_1(\lambda))|$, see Lemma 3.3. Recall that $f(z) = z\exp G(z)$ and by (3.8)

$$G(-R_1) = -2\frac{\Gamma(1-\lambda)}{\Gamma(\lambda)\Gamma(1-2\lambda)}\int_0^1 t^{\lambda-1}(1-t)^{-2\lambda}\log(1+tR_1)\,dt.$$

We use $\log(1+tR_1) \leqslant tR_1$ to obtain

$$G(-R_1) \geqslant -2R_1\frac{\Gamma(1-\lambda)}{\Gamma(\lambda)\Gamma(1-2\lambda)}\int_0^1 t^{\lambda}(1-t)^{-2\lambda}\,dt = -2R_1\frac{\lambda}{1-\lambda}.$$

Now $R_1 = R_1(\lambda) = C_2\lambda^{-1/3}$, so that $G(-R_1) \geqslant -2C_2\lambda^{2/3}/(1-\lambda)$ and

$$|f(-R_1)| = R_1 e^{G(-R_1)} \geqslant C_2\lambda^{-1/3}\exp\left(-2C_2\frac{\lambda^{2/3}}{1-\lambda}\right) \geqslant K_3\lambda^{-1/3}$$

for a positive constant K_3. Hence

$$|b_{k,2}| \leqslant \frac{1}{\pi k}\int_{\gamma_2^*}|f(z)|^{-k}|z|^{-2}|dz| \leqslant K_4\frac{1}{k}\lambda^{k/3}\lambda^{1/3}K_3^{-k}. \tag{4.36}$$

Now $b_k(\lambda)$ will be positive if $b_{k,1} > |b_{k,2}|$ which by (4.35), (4.36) is the case if

$$K_4\frac{1}{k}\lambda^{k/3}\lambda^{1/3}K_3^{-k} < K_2\lambda k^{-3/2}e^{-kG(1)},$$

or written otherwise,

$$k^{1/2}\exp\left[-k\left(\tfrac{1}{3}\log\frac{1}{\lambda} + \log K_3 - G(1)\right)\right] \leqslant K_5\lambda^{2/3}. \tag{4.37}$$

Note that for λ sufficiently small,

$$\tfrac{1}{3}\log\frac{1}{\lambda} + \log K_3 - G(1) > 2.$$

[Actually, $G(1)$ also depends on λ, but it remains bounded near $\lambda = 0$, see (3.12).] For such λ, the left-hand side of (4.37) is decreasing in k. Thus for such λ and $k \geqslant 3$,

$$k^{1/2} \exp\left[-k\left(\tfrac{1}{3}\log\frac{1}{\lambda} - G(1)\right)\right] \leqslant 3^{1/2}\lambda \exp(3G(1))K_3^{-3}, \tag{4.38}$$

and for possibly even smaller λ the right-hand side of (4.38) is less than $K_5\lambda^{2/3}$. Hence for λ sufficiently small the coefficients $b_k(\lambda)$ are positive for $k \geqslant 3$. It is easily checked that $b_0(\lambda)$, $b_1(\lambda)$ and $b_2(\lambda)$ are also positive for small λ. The theorem now follows. □

4.5. Conclusion

Theorem 1.4 is an immediate consequence of Proposition 4.1 and Theorems 4.8 and 4.11.

5. Asymptotic distribution of the zeros: the proof of Theorem 1.5

In this section we fix $\lambda < \lambda^*$ and use the notation as before. Thus we have by Theorem 1.3 a starlike compact set $K = K(\lambda)$ such that $F(z)$ is the conformal mapping from $\mathbb{C}\backslash K$ onto $\{|w| > \rho\}$ and $\operatorname{cap}(K) = \rho$.

Let $\nu_n = \nu(F_n)$ be the normalized zero distribution of the Faber polynomials $F_n(z)$. That is, if $\zeta_{1,n}, \zeta_{2,n}, \ldots, \zeta_{n,n}$ are the zeros of $F_n(z)$, then

$$\nu_n := \frac{1}{n}\sum_{j=1}^{n} \delta_{\zeta_{j,n}}, \tag{5.1}$$

where δ_ζ is the point distribution with total mass 1 at the point ζ. Let ν_K be the equilibrium measure of K. This is the unique probability measure on K such that

$$-\int_K \log|z - t|\,\mathrm{d}\nu_K(t) = -\log\operatorname{cap}(K) = -\log\rho, \quad z \in K. \tag{5.2}$$

For every measure ν on K we denote by $U^\nu(z) = -\int_K \log|z - t|\,\mathrm{d}\nu(t)$ the potential of ν. For the normalized zero distributions ν_n we have

$$U^{\nu_n}(z) = -\frac{1}{n}\log|F_n(z)|. \tag{5.3}$$

Lemma 5.1.

$$\lim_{n\to\infty} |F_n(0)|^{1/n} = \rho. \tag{5.4}$$

Proof. From (2.3) and (2.4) we see that $F_n(0)$ is the coefficient of z^n in the Taylor expansion of $\exp(-nG(z))$. Thus by the Cauchy formula

$$F_n(0) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z^{n+1}\exp(nG(z))}\,\mathrm{d}z = \frac{1}{2\pi i}\int_\gamma f(z)^{-n}z^{-1}\,\mathrm{d}z, \tag{5.5}$$

where γ is a closed contour which encloses $z = 0$ once in the positive direction.

For large n, we apply asymptotic analysis to the integral in (5.5). Since $0 < \lambda < \lambda^*$, we can choose γ such that γ passes through $z = 1$ and such that $|f(z)| > f(1)$ for all $z \in \gamma$, $z \neq 1$. To be specific, we take γ to be the segment $[1, R_0(\lambda)]$ on the upper side of the cut $[1, \infty)$, followed by the circle $|z| = R_0(\lambda)$ and on the lower side of the cut back to $z = 1$. Here $R_0(\lambda)$ is as in the proof of Theorem 1.3. Using the symmetry with respect to the real axis one gets from (5.5)

$$F_n(0) = \frac{1}{\pi}\Im \int_{\gamma^+} f(z)^{-n} z^{-1}\, dz,$$

where γ^+ is the part of γ in the upper half-plane. Since for $z = x$ real on the upper side of the cut, one has by (4.9), (4.12),

$$G(x) + \log x = G(1) + \frac{\Gamma(1-\lambda)\Gamma(1+2\lambda)}{(1-2\lambda)\Gamma(1+\lambda)} e^{2\lambda\pi i}(x-1)^{1-2\lambda} + \mathcal{O}(x-1), \quad (x \searrow 1),$$

asymptotic analysis as in Wong [19, II 5] yields

$$F_n(0) \sim -C e^{-nG(1)} n^{-1/(1-2\lambda)},$$

where C is a positive constant. Here $c_n \sim d_n$ means $\lim_{n\to\infty} c_n/d_n = 1$.

Now (5.4) follows, since $\exp G(1) = 1/\rho$. □

Proof of Theorem 1.5. In view of Theorem 2.5(a) every weak-star limit of the sequence (ν_n) is a probability measure on K and it suffices to prove that ν_K is the only possible limit.

Let ν be any weak-star limit of the sequence (ν_n). Since

$$\lim_{n\to\infty} |F_n(z)|^{1/n} = |F(z)|, \qquad \text{uniformly on compact subsets of } \mathbb{C}\backslash K,$$

cf. [16, p. 135], we find that

$$U^\nu(z) = -\log|F(z)|, \quad z \in \mathbb{C}\backslash K. \tag{5.6}$$

Because $U^\nu(z)$ is superharmonic (5.6) gives

$$U^\nu(z) \geqslant -\log\rho, \qquad z \in K. \tag{5.7}$$

Next from (5.3), (5.4), the Principle of descent (see [11, Theorem 1.3]) and (5.7) it follows that

$$U^\nu(0) = -\log\rho. \tag{5.8}$$

From (5.7) and (5.8) it follows by the minumum principle for superharmonic functions that $U^\nu(z) = -\log\rho$ for every $z \in K$. Hence ν satisfies the property (5.2) which characterizes the equilibrium measure ν_K and Theorem 1.5 follows. □

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