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FOUR COMPOSITION IDENTITIES FOR CHEBYSHEV POLYNOMIALS

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1. INTRODUCTION

Let $\{t_n(x)\}_{n=0}$ be the sequence of Chebyshev polynomials defined by

$$t_0(x) = 1,\ t_1(x) = x,\ t_n(x) = 2xt_{n-1}(x) - t_{n-2}(x) \text{ for } n \geq 2.$$

These are often called *Chebyshev polynomials of the first kind* to distinguish them from *Chebyshev polynomials of the second kind*, which are defined by

$$u_0(x) = 1,\ u_1(x) = 2x,\ u_n(x) = 2xu_{n-1}(x) - u_{n-2}(x) \text{ for } n \geq 2.$$

It is well known that any two Chebyshev polynomials of the first kind commute under composition. Explicitly,

$$t_m(t_n(x)) = t_n(t_m(x)) = t_{mn}(x) \text{ for nonnegative } m \text{ and } n.$$

Similar identities involving Chebyshev polynomials of the second kind are not well known. This paper offers three such identities, one for each of the expressions $\overline{u}_m(\overline{u}_n(x))$, $t_m(\overline{u}_n(x))$, and $\overline{u}_m(t_n(x))$, where $\overline{u}_m(x) = u_m(x)\sqrt{1 - x^2}$.

Literature relating to the identity $t_m(t_n) = t_n(t_m)$ shows that this commutativity, also called *permutability*, is, among polynomials with coefficients in a field of characteristic 0, a distinctive property of Chebyshev polynomials of the first kind. For example, Bertram [1] shows that if p is a polynomial of degree $m > 1$ which is permutable with some t_n for $n \geq 2$, then $p = \pm t_m$. Another theorem (e.g., Kuczma [5, pp. 215-218] and Rivlin [6, pp. 160-164]) characterizes the sequence $\{t_n\}$ as the only nontrivial *semipermutable chain* (up to equivalence, as described below). Sections 3 and 4 of this paper deal with analogous results for the functions $\overline{u}_n$.

We deal with the Chebyshev polynomials in slightly altered form. Assume throughout that all numbers, including coefficients of all polynomials, lie in a field of characteristic 0. With this in mind, the nonmonic polynomials t_n and u_n are altered as follows: define

$$T_0(x, y) = 2,\ T_1(x, y) = x,\ T_n(x, y) = xT_{n-1}(x, y) - yT_{n-2}(x, y) \text{ for } n \geq 2;$$

$$U_0(x, y) = 0,\ U_1(x, y) = 1,\ U_n(x, y) = xU_{n-1}(x, y) - yU_{n-2}(x, y) \text{ for } n \geq 2.$$

In the sequel, the polynomials T_n are regarded as Chebyshev polynomials of the first kind, and the polynomials U_n are regarded as Chebyshev polynomials of the second kind. The connections with the polynomials t_n and u_n are simply

$$T_n(x, 1) = 2t_n(x/2) \text{ for } n \geq 0 \quad \text{and} \quad U_n(x, 1) = u_{n-1}(x/2) \text{ for } n \geq 1.$$

All the results obtained below for $\{T_n\}$ and $\{U_n\}$ carry over, as in Corollary 1, to $\{t_n\}$ and $\{u_n\}$. We also wish to carry over some results to certain polynomials of number-theoretic interest, namely the *generalized Lucas polynomials* $L_n(x, y)$ and *generalized Fibonacci polynomials* $F_n(x, y)$, discussed in [4] and elsewhere. For these, we have

$$T_n(x, y) = L_n(x, -y) \quad \text{and} \quad U_n(x, y) = F_n(x, -y).$$

2. THE FOUR IDENTITIES

Consistent with the modification $\overline{u}_n(x)$ of $u_n(x)$ already mentioned, we introduce a modification of $U_n(x, y)$:

$$\overline{U}_n(x, y) = U_n(x, y)\sqrt{4y - x^2} \text{ for } n \geq 0.$$

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Although $\overline{U}_n$ is not a polynomial for $n \geq 1$, it is convenient to say that $\overline{U}_n(x, y)$ has degree n in x. [The polynomial $U_n(x, y)$ has degree $n - 1$ in x.] Generally, a function $P(x, y)\sqrt{S(x, y)}$, where $P(x, y)$ and $S(x, y)$ are polynomials of degrees n and $2k$, respectively, in x, is regarded as a *function of degree* $n + k$ in x.

Definition: Suppose $P(x, y)$ and $Q(x, y)$ are functions of degrees m and n, respectively, in x. The composite function $P \circ Q$ is defined by

$$P \circ Q(x, y) = P[Q(x, y), y^n].$$

Theorem 1: Suppose m and n are nonnegative integers. Then

$$(1) \qquad T_m \circ T_n(x, y) = T_{mn}(x, y)$$

$$(2) \qquad \overline{U}_m \circ T_n(x, y) = \overline{U}_{mn}(x, y)$$

$$(3) \qquad T_m \circ \overline{U}_n(x, y) = \begin{cases} (-1)^{m/2}T_{mn}(x, y) & \text{for even } m \\ (-1)^{(m-1)/2}\overline{U}_{mn}(x, y) & \text{for odd } m \end{cases}$$

$$(4) \qquad \overline{U}_m \circ \overline{U}_n(x, y) = \begin{cases} (-1)^{(m-2)/2}\overline{U}_{mn}(x, y) & \text{for even } m \\ (-1)^{(m-1)/2}T_{mn}(x, y) & \text{for odd } m. \end{cases}$$

Proof: It is easy to establish (as in [4]) that

$$T_m(x, y) = 2y^{m/2}\cos(m \cos^{-1}x/2\sqrt{y})$$

and

$$U_m(x, y) = (4y - x^2)^{-1/2}\, 2y^{m/2}\sin(m \cos^{-1}x/2\sqrt{y}),$$

so that

$$\overline{U}_m(x, y) = 2y^{m/2}\sin(m \cos^{-1}x/2\sqrt{y}).$$

Then

$$T_m \circ T_n(x, y) = 2y^{mn/2}\cos\left[m \cos^{-1} \frac{2y^{n/2}\cos(n \cos^{-1}x/2\sqrt{y})}{2y^{n/2}}\right] = T_{mn}(x, y).$$

Similarly,

$$\overline{U}_m \circ T_n(x, y) = 2y^{mn/2}\sin\left[m \cos^{-1} \frac{2y^{n/2}\cos(n \cos^{-1}x/2\sqrt{y})}{2y^{n/2}}\right] = \overline{U}_{mn}(x, y).$$

Next,

$$\begin{aligned} T_m \circ \overline{U}_n(x, y) &= 2y^{mn/2}\cos[m \cos^{-1} \sin(n \cos^{-1}x/2\sqrt{y})] \\ &= 2y^{mn/2}\cos[m(\pi/2 - n \cos^{-1}x/2\sqrt{y})] \\ &= 2y^{mn/2}[\cos m\pi/2 \cos(mn \cos^{-1}x/2\sqrt{y}) \\ &\quad + \sin m\pi/2 \sin(mn \cos^{-1}x/2\sqrt{y})], \end{aligned}$$

and from this, (3) clearly follows. Finally,

$$\begin{aligned} \overline{U}_m \circ \overline{U}_n(x, y) &= 2y^{mn/2}\sin[m \cos^{-1} \sin(n \cos^{-1}x/2\sqrt{y})] \\ &= 2y^{mn/2}\sin[m(\pi/2 - n \cos^{-1}x/2\sqrt{y})] \\ &= 2y^{mn/2}[\sin m\pi/2 \cos(mn \cos^{-1}x/2\sqrt{y}) \\ &\quad - \cos m\pi/2 \sin(mn \cos^{-1}x/2\sqrt{y})], \end{aligned}$$

and this proves (4).

Corollary 1: Let $\{t_n\}_{n=0}$ and $\{u_n\}_{n=0}$ be the sequences of (unaltered) Chebyshev polynomials of the first and second kinds, respectively. Put $\overline{u}_{-1}(x) \equiv 0$ and $\overline{u}_n(x) = u_n(x)\sqrt{1 - x^2}$ for $n \geq 0$. Then for nonnegative m and n, (1′) $t_m(t_n(x)) = t_{mn}(x)$

(2′) $\overline{u}_m(t_n(x)) = \overline{u}_{mn+n-1}(x)$

(3′) $t_m(\overline{u}_n(x)) = \begin{cases} (-1)^{m/2} t_{mn+m}(x) & \text{for even } m \\ (-1)^{(m-1)/2}\overline{u}_{mn+m-1}(x) & \text{for odd } m \end{cases}$

(4′) $\overline{u}_m(\overline{u}_n(x)) = \begin{cases} (-1)^{m/2} t_{(m+1)(n+1)}(x) & \text{for even } m \\ (-1)^{(m-1)/2}\overline{u}_{mn+m+n}(x) & \text{for odd } m. \end{cases}$

Proof: These identities come directly from Theorem 1 via the transformations

$$t_n(x) = \frac{1}{2}T_n(2x,\ 1) \quad \text{and} \quad \overline{u}_n(x) = \frac{1}{2}\overline{U}_{n+1}(2x,\ 1) \text{ for } n \geq 0.$$

We turn now to the problem of expressing (1)-(4) in terms of generalized Lucas and Fibonacci polynomials. Corresponding to the functions $\overline{U}_n(x,\ y)$ we define

$$\overline{F}_n(x,\ y) = F_n(x,\ y)\sqrt{x^2 + 4y} \text{ for } n \geq 0,$$

noting that this equals $i\overline{U}_n(x,\ -y)$. Two lemmas are helpful.

Lemma 2a: For $0 \leq m \leq n$,

(5) $L_m(x,\ y)L_n(x,\ y) - \overline{F}_m(x,\ y)\overline{F}_n(x,\ y) = 2(-y)^m L_{n-m}(x,\ y).$

Proof: It is well known and easily shown by induction that

$$L_n(x,\ y) = \alpha^n + \beta^n \quad \text{and} \quad \overline{F}_n(x,\ y) = \alpha^n - \beta^n,$$

where $\alpha + \beta = x$ and $\alpha\beta = -y$. The desired identity now follows immediately.

Lemma 2b: For $m \geq 0$,

$$L_m(ix,\ -y) = i^m L_m(x,\ y) \quad \text{and} \quad \overline{F}_m(ix,\ -y) = i^m\overline{F}_m(x,\ y).$$

Proof: This is easily seen by induction, using the recurrence relation

$$H_m(x,\ y) = xH_{m-1}(x,\ y) + yH_{m-2}(x,\ y)$$

satisfied by both $\{L_m\}$ and $\{\overline{F}_m\}$ for $m \geq 2$.

From (1) and the relation $T_n(x,\ -y) = L_n(x,\ y)$ comes

$$T_m[L_n(x,\ y),\ (-1)^n y^n] = L_{mn}(x,\ y),$$

so that

(1a) $L_m \circ L_n(x,\ y) = L_{mn}(x,\ y)$ for odd n.

But, for even n,

(6) $L_n^m - a_{m-2}L_n^{m-2}y^n + a_{m-4}L_n^{m-4}y^{2n} - \cdots + (-1)^{\left[\frac{m}{2}\right]}a_\ell L_n^\ell y^{\left[\frac{m}{2}\right]n} = L_{mn}(x,\ y),$

where the a_k's are coefficients in the polynomial

$$T_m(x,\ y) = x^m - a_{m-2}x^{m-2}y + a_{m-4}x^{m-4}y - \cdots + (-1)^{\left[\frac{m}{2}\right]}a_\ell x^\ell y^{\left[\frac{m}{2}\right]};$$

here, $\ell = 0$ if m is even and $\ell = 1$ if m is odd (see Lemma 2e). Adding

$$2a_{m-2}L_n^{m-2}y^n + 2a_{m-6}L_n^{m-6}y^{3n} + \cdots$$

to both sides of (6) gives

(1b) $L_m \circ L_n(x,\ y) = L_{mn}(x,\ y) + 2(a_{m-2}L_n^{m-2}y + a_{m-6}L_n^{m-6}y^{3n} + \cdots + a_s L_n^s y^{tn})$

for even n, where

$$s = \begin{cases} 0 \text{ if } m \equiv 2 \bmod 4 \\ 1 \text{ if } m \equiv 3 \bmod 4 \\ 2 \text{ if } m \equiv 0 \bmod 4 \\ 3 \text{ if } m \equiv 1 \bmod 4 \end{cases} \quad \text{and} \quad t = 1\left[\frac{m-2}{4}\right] + 1.$$

Now from (2) and the relation $\overline{U}_n(x, -y) = -i\overline{F}_n(x, y)$ comes

$$i\overline{U}_m[L_n(x, y), (-1)^n y^n] = \overline{F}_{mn}(x, y),$$

so that

(2a) $$\overline{F}_m \circ L_n(x, y) = \overline{F}_{mn}(x, y) \text{ for odd } n.$$

But for even n,

$$\begin{aligned} \overline{F}_{mn}(x, y) &= i\overline{U}_m[L_n(x, y), y^n] \\ &= \overline{F}_m[L_n(x, y), -y^n] \\ &= \sqrt{L_n^2 - 4y^n} F_m[L_n(x, y), -y^n] \\ &= \overline{F}_n(x, y) F_m[L_n(x, y), -y^n], \end{aligned}$$

by Lemma 2a. Thus,

(7) $$\overline{F}_{mn}(x, y) = \overline{F}_n(x, y)\left\{L_n^{m-1} - b_{m-3}L_n^{m-3}y + b_{m-5}L_n^{m-5}y^{2n} - \cdots + (-1)^{\left[\frac{m-1}{2}\right]} b_\ell L_n^\ell y^{\left[\frac{m-1}{2}\right]n}\right\},$$

where the b_k's are the coefficients of the polynomial

$$F_m(x, y) = x^{m-1} + b_{m-3}x^{m-3}y + b_{m-5}x^{m-5}y^2 + \cdots + b_\ell x^\ell y^{\left[\frac{m-1}{2}\right]};$$

here, $\ell = 0$ if m is even and $\ell = 1$ if m is odd (see Lemma 2e). Adding

$$2\overline{F}_n(x, y)(b_{m-3}L_n^{m-3}y^n + b_{m-7}L_n^{m-7}y^{3n} + \cdots)$$

to both sides of (7) gives

$$\overline{F}_n(x, y)F_m \circ L_n(x, y) = \overline{F}_{mn}(x, y) + 2\overline{F}_n(x, y)(b_{m-3}L_n^{m-3}y^n + b_{m-7}L_n^{m-7}y^{3n} + \cdots).$$

For $n > 0$, we divide both sides by $\overline{F}_n(x, y)$ and have

(2b) $$F_m \circ L_n(x, y) = \frac{F_{mn}(x, y)}{F_n(x, y)} + 2(b_{m-3}L_n^{m-3}y^n + b_{m-7}L_n^{m-7}y^{3n} + \cdots + b_s L_n^s y^{tn}) \text{ for even } n > 0,$$

where

$$s = \begin{cases} 0 \text{ if } m \equiv 3 \bmod 4 \\ 1 \text{ if } m \equiv 0 \bmod 4 \\ 2 \text{ if } m \equiv 1 \bmod 4 \\ 3 \text{ if } m \equiv 2 \bmod 4 \end{cases} \quad \text{and} \quad t = 2\left[\frac{m-3}{4}\right] + 1.$$

Identity (3) leads to

(8) $$T_m[-i\overline{F}_n(x, y), (-1)^n y^n] = \begin{cases} (-1)^{\frac{m}{2}} L_{mn}(x, y) & \text{for even } m \\ (-1)^{\frac{m+1}{2}} i\overline{F}_{mn}(x, y) & \text{for odd } m. \end{cases}$$

For even $n \geq 0$, we apply Lemma 2b to find, without difficulty, that

(3a) $$L_m \circ \overline{F}_n = \begin{cases} L_{mn} \text{ for even } n \text{ and even } m \\ \overline{F}_{mn} \text{ for even } n \text{ and odd } m. \end{cases}$$

For odd n, suppose first that m is odd also. Then (8) with Lemma 2a gives

$$L_m[\overline{F}_n(x, y), -y^n] = \overline{F}_{mn}(x, y).$$

As in the derivation of (1b), we add

$$2(a_{m-2}\overline{F}_n^{m-2}y^n + a_{m-6}\overline{F}_n^{m-6}y^{3n} + \cdots)$$

to both sides. This gives

(3b) $$L_m \circ \overline{F}_n(x, y) = \overline{F}_{mn}(x, y) + 2(a_{m-2}\overline{F}_n^{m-2}y^n + a_{m-6}\overline{F}_n^{m-6}y^{3n} + \cdots + a_s\overline{F}_n^s y^{tn}) \text{ for odd } n \text{ and odd } m,$$

where the a_k's, s, and t are the same as for (1b).

Continuing with odd n, suppose now that m is even. Using (8) and Lemma 2a, we find

(3c) $$L_m \circ F_n(x, y) = L_{mn}(x, y) + 2(a_{m-2}\overline{F}_n^{m-2}y^n + a_{m-6}\overline{F}_n^{m-6}y^{3n} + \cdots + a_s\overline{F}_n^s y^{tn}) \text{ for odd } n \text{ and even } m,$$

where the a_k's, s, and t are the same as for (1b).

Identity (4) leads to

(9) $$\overline{U}_m[-i\overline{F}_n(x, y), (-1)^n y^n] = \begin{cases} (-1)^{\frac{m}{2}} i\overline{F}_{mn}(x, y) & \text{for even } m \\ (-1)^{\frac{m-1}{2}} L_{mn}(x, y) & \text{for odd } m. \end{cases}$$

whence,

(4a) $$\overline{F}_m \circ \overline{F}_n = \begin{cases} \overline{F}_{mn} & \text{for even } n \text{ and even } m \\ L_{mn} & \text{for even } n \text{ and odd } m. \end{cases}$$

For odd n, suppose first that m is odd also. Then (9) and Lemmas 2a and 2b apply, and we find

$$L_{mn}(x, y) = \overline{F}_m[\overline{F}_n(x, y), -y^n] = \sqrt{\overline{F}_n^2 - 4y^n}F_m[\overline{F}_n(x, y), -y^n]$$
$$= L_n(x, y)(\overline{F}_n^{m-1} - b_{m-3}\overline{F}_n^{m-3}y^n + b_{m-5}\overline{F}_n^{m-5}y^{2n} - \cdots).$$

At this point, we add $2L_n(x, y)(b_{m-3}\overline{F}_n^{m-3}y^n + b_{m-7}\overline{F}_n^{m-7}y^{3n} + \cdots)$ to both sides and then divide both sides by $L_n(x, y)$, getting

(4b) $$F_m \circ \overline{F}_n(x, y) = \frac{L_{mn}(x, y)}{L_n(x, y)} + 2(b_{m-3}\overline{F}_n^{m-3}y^n + b_{m-7}\overline{F}_n^{m-7}y^{3n} + \cdots + b_s\overline{F}^s y^{tn}) \text{ for odd } n \text{ and odd } m,$$

where the b_k's, s, and t are the same as for (2b).

Continuing with odd n, suppose now that m is even. With the method which is now familiar, we find

(4c) $$F_m \circ F_n(x, y) = \frac{F_{mn}(x, y)}{F_n(x, y)} + 2(b_{m-3}\overline{F}_n^{m-3}y^n + b_{m-7}\overline{F}_n^{m-7}y^{3n} + \cdots + b_s\overline{F}_n^s y^{tn}) \text{ for odd } m \text{ and even } m,$$

where the b_k's, s, and t are the same as for (2b).

Table 1. Examples of Composites Involving Generalized Lucas and Fibonacci Polynomials

From (1b) and (2b), for even $n > 0$:

$L_2 \circ L_n = L_{2n} + 4y^n$	$F_2 \circ L_n = F_{2n}/F_n$
$L_3 \circ L_n = L_{3n} + 6L_n y^n$	$F_3 \circ L_n = F_{3n}/F_n + 2y^n$
$L_4 \circ L_n = L_{4n} + 8L_n^2 y^n$	$F_4 \circ L_n = F_{4n}/F_n + 4L_n y^n$
$L_5 \circ L_n = L_{5n} + 10L_n^3 y^n$	$F_5 \circ L_n = F_{5n}/F_n + 6L_n^2 y^n$
$L_6 \circ L_n = L_{6n} + 12L_n^4 y^n + 4y^{3n}$	$F_6 \circ L_n = F_{6n}/F_n + 8L_n^3 y^n$
$L_7 \circ L_n = L_{7n} + 14L_n^5 y^n + 14L_n y^{3n}$	$F_7 \circ L_n = F_{7n}/F_n + 10L_n^4 y^n + 2y^{3n}$

Table 1—continued

From (3b) and (3c), for odd $n \geq 1$:

$L_3 \circ \overline{F}_n = \overline{F}_{3n} + 6\overline{F}_n y^n$ $\qquad$ $L_2 \circ \overline{F}_n = L_{2n} + 4y^n$

$L_5 \circ \overline{F}_n = \overline{F}_{5n} + 10\overline{F}_n^3 y^n$ $\qquad$ $L_4 \circ \overline{F}_n = L_{4n} + 8\overline{F}_n^2 y^n$

$L_7 \circ \overline{F}_n = \overline{F}_{7n} + 14\overline{F}_n^5 y^n + 14\overline{F}_n^3 y^n$ $\qquad$ $L_6 \circ \overline{F}_n = L_{6n} + 12\overline{F}_n^4 y^n + 4y^{3n}$

$L_9 \circ \overline{F}_n = \overline{F}_{9n} + 18\overline{F}_n^7 y^n + 60\overline{F}_n^3 y^n$ $\qquad$ $L_8 \circ \overline{F}_n = L_{8n} + 16\overline{F}_n^6 y^n + 32\overline{F}_n^2 y^{3n}$

From (4b) and (4c), for odd $n \geq 1$:

$F_1 \circ \overline{F}_n = 1$ $\qquad$ $F_2 \circ \overline{F}_n = \overline{F}_{2n}/L_n$

$F_3 \circ \overline{F}_n = L_{3n}/L_n + 2y^n$ $\qquad$ $F_4 \circ \overline{F}_n = \overline{F}_{4n}/L_n + 4\overline{F}_n y^n$

$F_5 \circ \overline{F}_n = L_{5n}/L_n + 6\overline{F}_n^2 y^n$ $\qquad$ $F_6 \circ \overline{F}_n = \overline{F}_{6n}/L_n + 8\overline{F}_n^3 y^n$

$F_7 \circ \overline{F}_n = L_{7n}/L_n + 10\overline{F}_n^4 y^n + 2y^{3n}$ $\qquad$ $F_8 \circ \overline{F}_n = \overline{F}_{8n}/L_n + 12\overline{F}_n^5 y^n + 8\overline{F}_n y^{3n}$

$F_9 \circ \overline{F}_n = L_{9n}/L_n + 14\overline{F}_n^6 y^n + 20\overline{F}_n^2 y^{3n}$ $\qquad$ $F_{10} \circ \overline{F}_n = \overline{F}_{10n}/L_n + 16\overline{F}_n^7 y^n + 40\overline{F}_n y^{3n}$

For $m \geq 0$, define

$$V_m(x, y) = \binom{m}{0}x^m + \binom{m}{1}x^{m-2}y + \cdots + \binom{m}{[m/2]}x^{\ell}y^{\left[\frac{m}{2}\right]n}$$

and

$$W_m(x, y) = V_m(x, -y),$$

where $\ell = 0$ for even m and $\ell = 1$ for odd m.

<u>*Lemma 2c*</u>: Suppose m and n are nonnegative integers. Then

$$L_n^m(x, y) = \begin{cases} V_m \circ L_1^n(x, y) & \text{for even } n \\ W_m \circ L_1^n(x, y) & \text{for odd } n, \end{cases}$$

$$\overline{F}_n^m(x, y) = \begin{cases} W_m \circ L_1^n(x, y) & \text{for even } m \text{ and even } n \\ V_m \circ L_1^n(x, y) & \text{for even } m \text{ and odd } n, \end{cases}$$

$$\overline{F}_n^m(x, y) = \begin{cases} W_m \circ \overline{F}_1^n(x, y) & \text{for odd } m \text{ and even } n \\ V_m \circ \overline{F}_1^n(x, y) & \text{for odd } m \text{ and odd } n; \end{cases}$$

in these formulas, after expansions on the right sides, each symbol of the form L_1^j (or $\overline{F}_1^j$) is to be changes to L_j (or $\overline{F}_j$). (This "symbolic substitution" is discussed in Hoggatt and Lind [3].)

<u>*Proof*</u>: These are direct results of writing

$$L_n(x, y) = \alpha^n + \beta^n \quad \text{and} \quad \overline{F}_n(x, y) = \alpha^n - \beta^n$$

and applying the binomial formula, where $\alpha + \beta = x$ and $\alpha\beta = -y$.

<u>*Lemma 2d*</u>: Suppose m and n are nonnegative integers. Then

$$L_{mn}(x, y) = \begin{cases} T_m \circ L_n(x, y) & \text{for even } n \\ L_m \circ L_n(x, y) & \text{for odd } n \end{cases}$$

and

$$\frac{F_{mn}(x, y)}{F_n(x, y)} = \begin{cases} U_m \circ L_n(x, y) & \text{for even } n > 0 \\ F_m \circ L_n(x, y) & \text{for odd } n. \end{cases}$$

Proof: Near (1a) and (2a) these two are already proved. (They are restated here for later convenience and as inverse formulas for the formulas in Lemma 2c. Tables of coefficients for these two formulas are found in Brousseau [2, pp. 145-150].)

Lemma 2e: For $m \geq 0$,

$$L_m(x,\ y) = \sum_{j=0}^{p} \frac{m}{m-2j}\binom{m-j-1}{j}x^{m-2j}y^j \text{ with } p = \begin{cases} m/2 & \text{for even } m \\ (m-1)/2 & \text{for odd } m \end{cases}$$

where the summand on the right equals $2y^p$, by definition, in case $j = p = m/2$. Also

$$F_m(x,\ y) = \sum_{j=0}^{q}\binom{m-j-1}{j}x^{m-2j-1}y^j \text{ with } q = \begin{cases} (m-2)/2 & \text{for even } m \\ (m-1)/2 & \text{for odd } m. \end{cases}$$

Proof: These well-known formulas are easily proved by induction.

The composite functions in Table 1 can also be expressed as linear combinations of terms of the form $L_{jn}y^k$ or $\overline{F}_{jn}y^k$. To obtain such expressions, one may use Table 1 with substitutions from Lemma 2c, or one may use Binet forms (e.g., $\overline{F}_n = \alpha^n - \beta^n$) and binomial expansions. These methods give the following results.

For even n, the coefficients c_{m-2j} in the expression

$$L_m \circ L_n = c_m L_{mn} + c_{m-2}L_{(m-2)n}y^n + \cdots + c_{m-2p}L_{(m-2p)n}y^{pn},$$

where p is as in Lemma 2e and for temporary convenience $L_0 \equiv 1$ (instead of 2):

Table 2

	c_m	c_{m-2}	c_{m-4}	c_{m-6}	c_{m-8}	c_{m-10}
$m = 2$	1	4				
3	1	6				
4	1	8	16			
5	1	10	30			
6	1	12	48	76		
7	1	14	70	154		
8	1	16	96	272	384	
9	1	18	126	438	810	
10	1	20	160	660	1520	2004

Formula: $c_{m-2j} = \sum_{k=0}^{j}\frac{m}{m-2k}\binom{m-k-1}{k}\binom{m-2k}{j-k}$ for $0 \leq j \leq p$, where the summand on the right = 2, by definition, in case $k = m/2$ (which occurs in c_{m-2p} for even m).

For even n, the coefficients c_{m-2j-1} in the expression

$$F_m \circ L_n = c_{m-1}L_{(m-1)n} + c_{m-3}L_{(m-3)n}y^n + \cdots + c_{m-2q-1}L_{(m-2q-1)n}y^{qn},$$

where q is as in Lemma 2e and for temporary convenience $L_0 \equiv 1$ (instead of 2):

Table 3

	c_{m-1}	c_{m-3}	c_{m-5}	c_{m-7}	c_{m-9}
$m=2$	1				
3	1	3			
4	1	5			
5	1	7	13		
6	1	9	25		
7	1	11	41	63	
8	1	13	61	129	
9	1	15	85	231	321
10	1	17	113	377	681

Formula: $c_{m-2j-1} = \sum_{k=0}^{j} \binom{m-k-1}{k}\binom{m-2k-1}{j-k}$ for $0 \le j \le q$.

For odd $n \ge 1$, the coefficients c_{m-2j} in the expression

$$L_m \circ \overline{F}_n = \begin{cases} c_m L_{mn} + c_{m-2}L_{(m-2)n}y^n + \cdots + c_{m-2p}L_{(m-2p)n}y^{pn} & \text{for even } m \ge 0 \\ c_m \overline{F}_{mn} + c_{m-2}\overline{F}_{(m-2)n}y^n + \cdots + c_{m-2p}\overline{F}_{(m-2p)n}y^{pn} & \text{for odd } m \ge 1 \end{cases}$$

are precisely the same as in Table 2. Similarly, for odd $n \ge 1$, the coefficients c_{m-2j-1} in the expression

$$F_m \circ \overline{F}_n = \begin{cases} c_{m-1}\overline{F}_{(m-1)n} + c_{m-3}\overline{F}_{(m-3)n}y^n + \cdots + c_{m-2q-1}\overline{F}_{(m-2q-1)n}y^{qn} & \text{for even } m \ge 2 \\ c_{m-1}L_{(m-1)n} + c_{m-3}L_{(m-3)n}y^n + \cdots + c_{m-2q-1}L_{(m-2q-1)n}y^{qn} & \text{for odd } m \ge 1. \end{cases}$$

are precisely the same as in Table 3.

Now let us recall (1a), (2a), (3a), and (4a): For odd $n \ge 1$,

$$\overline{F}_m \circ L_n = \overline{F}_{mn} \quad \text{and} \quad L_m \circ L_n = L_{mn};$$

for even $n \ge 0$,

$$L_m \circ \overline{F}_n = \begin{cases} L_{mn} & \text{for even } m \ge 0 \\ \overline{F}_{mn} & \text{for odd } m \ge 1 \end{cases} \quad \text{and} \quad \overline{F}_m \circ \overline{F}_n = \begin{cases} \overline{F}_{mn} & \text{for even } m \ge 0 \\ L_{mn} & \text{for odd } m \ge 1. \end{cases}$$

These four identities lead to identities for products of composites. For example, suppose s and σ are odd positive integers and t and τ are even nonnegative integers. Then

$$\overline{F}_s \circ \overline{F}_t = L_{st} \quad \text{and} \quad \overline{F}_\sigma \circ \overline{F}_\tau = L_{\sigma\tau}.$$

By identity (5) in [4], $L_{st}L_{\sigma\tau} = L_{st+\sigma\tau} + L_{st-\sigma\tau}$. Therefore,

$$(\overline{F}_s \circ \overline{F}_t)(\overline{F}_\sigma \circ \overline{F}_\tau) = L_{st+\sigma\tau} + L_{st-\sigma\tau}.$$

Ten identities are obtainable in this way. To facilitate listing them, we make certain abbreviations. The identity just derived appears below in (10) as

$$(\overline{F}_s \circ \overline{F}_t)(\overline{F}_\sigma \circ \overline{F}_\tau) = L_\sharp + L_\flat, \text{ oeoe},$$

where the designation "oeoe" means "for odd s, even t, odd σ, even τ."

Table 4. Product-Composition Identities

Notation: s, t, σ, τ are nonnegative integers and $st \ge \sigma\tau$. Also, $\sharp = st + \sigma\tau$ and $\flat = st - \sigma\tau$ as in the example above.

(10)
$$(\overline{F}_s \circ \overline{F}_t)(\overline{F}_\sigma \circ \overline{F}_\tau) = \begin{cases} L_\sharp + L_\flat, & \text{oeoe} \\ \overline{F}_\sharp - \overline{F}_\flat, & \text{oeee} \\ \overline{F}_\sharp + \overline{F}_\flat, & \text{eeoe} \\ L_\sharp - L_\flat, & \text{eeee} \end{cases}$$

(11)
$$(\overline{F}_s \circ \overline{F}_t)(\overline{F}_\sigma \circ L_\tau) = \begin{cases} \overline{F}_\sharp + \overline{F}_\flat, & \text{oeoo} \\ \overline{F}_\sharp - \overline{F}_\flat, & \text{oeeo} \\ L_\sharp + L_\flat, & \text{eeoo} \\ L_\sharp - L_\flat, & \text{eeeo} \end{cases}$$

Table 4.—continued

(12)
$$(\overline{F}_s \circ \overline{F}_t)(L_\sigma \circ \overline{F}_\tau) = \begin{cases} \overline{F}_\sharp - \overline{F}_\flat, & \text{oeoe} \\ L_\sharp + L_\flat, & \text{oeee} \\ L_\sharp - L_\flat, & \text{eeoe} \\ \overline{F}_\sharp + \overline{F}_\flat, & \text{eeee} \end{cases}$$

(13)
$$(\overline{F}_s \circ \overline{F}_t)(L_\sigma \circ L_\tau) = \begin{cases} L_\sharp - L_\flat, & \text{oeoo} \\ L_\sharp + L_\flat, & \text{oeeo} \\ \overline{F}_\sharp - \overline{F}_\flat, & \text{eeoo} \\ \overline{F}_\sharp + \overline{F}_\flat, & \text{eeeo} \end{cases}$$

(14)
$$(L_s \circ \overline{F}_t)(L_\sigma \circ \overline{F}_\tau) = \begin{cases} L_\sharp - L_\flat, & \text{oeoe} \\ \overline{F}_\sharp + \overline{F}_\flat, & \text{oeee} \\ \overline{F}_\sharp - \overline{F}_\flat, & \text{eeoe} \\ L_\sharp + L_\flat, & \text{eeee} \end{cases}$$

(15)
$$(L_s \circ \overline{F}_t)(L_\sigma \circ L_\tau) = \begin{cases} \overline{F}_\sharp - \overline{F}_\flat, & \text{oeoo} \\ \overline{F}_\sharp + \overline{F}_\flat, & \text{oeeo} \\ L_\sharp - L_\flat, & \text{eeoo} \\ L_\sharp + L_\flat, & \text{eeeo} \end{cases}$$

(16)
$$(\overline{F}_s \circ L_t)(\overline{F}_\sigma \circ L_\tau) = \begin{cases} L_\sharp + L_\flat, & \text{oooo and eooo} \\ L_\sharp - L_\flat, & \text{ooeo and eoeo} \end{cases}$$

(17)
$$(\overline{F}_s \circ L_t)(L_\sigma \circ \overline{F}_\tau) = \begin{cases} L_\sharp - L_\flat, & \text{oooe and eooe} \\ \overline{F}_\sharp + \overline{F}_\flat, & \text{ooee and eoee} \end{cases}$$

(18)
$$(\overline{F}_s \circ L_t)(L_\sigma \circ L_\tau) = \begin{cases} \overline{F}_\sharp - \overline{F}_\flat, & \text{oooo and eooo} \\ \overline{F}_\sharp + \overline{F}_\flat, & \text{ooeo and eoeo} \end{cases}$$

(19)
$$(L_s \circ L_t)(L_\sigma \circ L_\tau) = \begin{cases} L_\sharp - L_\flat, & \text{oooo and eooo} \\ L_\sharp + L_\flat, & \text{ooeo and eoeo} \end{cases}$$

3. FUNCTIONS COMMUTING WITH $\overline{U}$ (x)

Bertram [1] proves that, except for a possible factor -1, the only nonconstant polynomials that are permutable (i.e., commute) with nonlinear Chebyshev polynomials (of the first kind) are Chebyshev polynomials (of the first kind). Here we obtain analogous results for Chebyshev polynomials of the second kind. The same arguments give further analogous results for composites involving one Chebyshev polynomial of each kind.

There is no real loss in disregarding the symbol y in $T_n(x, y)$, $U_n(x, y)$, and $\overline{U}_n(x, y)$ in this section. Accordingly, we write $T_n(x)$ for $T_n(x, 1)$, $U_n(x)$ for $U_n(x, 1)$, and $\overline{U}_n(x)$ for $\overline{U}_n(x, 1)$. Following the notation and arguments in Bertram, if P and Q are functions, the substitution of $Q(x)$ for x in $P(x)$ is denoted either by $P(Q(x))$ or $P(Q)$. Ordinary multiplication of functions is given by juxtaposition, as in $\sqrt{4 - x^2}U_n(x)$, or by brackets, as in $A[\overline{P}']^j$ and $(4 - x^2)[U_n'(x)]$, in order to avoid confusion with the composition (i.e., substitution) operation.

Proofs in this section are abbreviated or omitted, but the interested reader with [1] at hand should have no trouble writing out the proofs in full. One must of course bear in mind the transformations already given between T_n, $\overline{U}_n$, and t_n, $\overline{u}_n$.

Lemma 3a: Suppose $\overline{P}(x)$ satisfies the following differential equation for some positive integer n:

(20)
$$(4 - x^2)[\overline{P}'(x)]^2 = n^2[4 - \overline{P}^2(x)].$$

If $\overline{P}(x)$ is of the form $\sqrt{4 - x^2}P(x)$, where $P(x)$ is a polynomial, then

$$P(x) = \pm U_n(x). \quad [\text{That is, } \overline{P}(x) = \pm\overline{U}_n(x).]$$

Lemma 3b: Suppose $A(x)$, a polynomial of degree $j \geq 0$, and $\overline{Q}(x) = \sqrt{4 - x^2}Q(x)$, where $Q(x)$ is a polynomial of degree $n - 1 \geq 1$, satisfy the differential equation

(21)
$$\{A(x)[\overline{Q}'(x)]^j\}^2 = [n^j A(\overline{Q}(x))]^2.$$

If $\overline{P}(x) = \sqrt{4 - x^2}P(x)$, where $P(x)$ is a polynomial of degree $m - 1 \geq 0$, is permutable with $\overline{Q}(x)$, then $\overline{P}(x)$ satisfies the same differential equation with n replaced by m.

Proof: Let

$$G = \{A[\overline{P}']^j\}^2 - [m^j A(\overline{P})]^2,$$

and suppose $G \not\equiv 0$. The highest degree term of both $\{A[\overline{P}']^j\}^2$ and $[m^j A(\overline{P})]^2$ is

$$(-1)^j m^{2j} a_j^2 p_{m-1}^{2j} x^{2mj},$$

so that the degree d of G is strictly less than $2jm$. We next prove that $G \not\equiv 0$ also implies $d = 2jm$. Using (21), the commutativity, and the chain rule,

$$\begin{aligned} n^{2j} G(\overline{Q}) &= n^{2j}\{A(\overline{Q})[\overline{P}'(\overline{Q})]^j\}^2 - m^{2j} n^{2j} [A(\overline{P}(\overline{Q}))]^2 \\ &= A^2[\overline{Q}']^{2j}[\overline{P}'(\overline{Q})]^{2j} - m^{2j}[A(\overline{P})]^2[\overline{Q}'(\overline{P})]^{2j} \\ &= A^2[\overline{P}']^{2j}[\overline{Q}'(\overline{P})]^{2j} - m^{2j}[A(\overline{P})]^2[\overline{Q}'(\overline{P})]^{2j} \\ &= [\overline{Q}'(\overline{P})]^{2j}\{A^2[\overline{P}']^{2j} - m^{2j}[A(\overline{P})]^2\} = [\overline{Q}'(\overline{P})]^{2j} G. \end{aligned}$$

Equating degrees gives $nd = d + 2j(n-1)m$, so that $d = 2jm$ since $n \neq 1$. This contradiction shows that $G \equiv 0$, as desired.

Theorem 3: Let $\{U_n\}_{n=0}$ be the sequence of (altered) Chebyshev polynomials of the second kind. Suppose P is a polynomial of degree $m - 1 \geq 0$ such that the functions

$$\overline{U}_n(x) = \sqrt{4 - x^2}\,U_n(x) \quad \text{and} \quad \overline{P}(x) = \sqrt{4 - x^2}\,P(x)$$

are permutable for some positive integer n. Then $P = U_m$ if n is odd, and $P = \pm U_m$ if n is even.

Proof: First suppose $n = 1$. If $m = 1$ also, then the desired result is easily obtained. If $m > 1$, then the method of proof of Theorem 6 below shows that $P = \pm U_m$. Now suppose $n > 1$. By Lemma 3a, $\pm U_n$ are the only polynomials Y of degree $n - 1 \geq 1$ which satisfy the differential equation

$$A^2[\overline{Y}']^4 = n^4[A(\overline{Y})]^2,$$

where $\overline{Y}(x) = \sqrt{4 - x^2}\,Y(x)$ and $A(x) = 4 - x^2$. But the hypothesis that $\overline{U}_n(\overline{P}) = \overline{P}(\overline{U}_n)$ for $n \geq 1$, together with Lemma 3b implies that $\overline{P}$ satisfies this differential equation with n replaced by m. Thus, taking square roots,

$$(4 - x^2)[\overline{P}'(x)]^2 = n^2[4 - \overline{P}^2(x)] \quad \text{or} \quad -n^2[4 - \overline{P}^2(x)].$$

The latter leads to $m^2 + n^2 = 0$, which is impossible. Therefore, Lemma 3a applies, and $P = \pm U_m$. If n is odd, then U_n is an even function, and $P = U_m$; if n is even, then U_n is an odd function, and $P = \pm U_m$.

Identities (2) and (3) show that $\overline{U}_m$ and T_n sometimes commute. Theorems 4 and 5 below tell precisely when this happens and also answer the following questions: What polynomials Q commute with a given $\overline{U}_m$? What functions of the form $\sqrt{4 - x^2}P(x)$ commute with a given T_n for $n \geq 2$? The proofs, which are omitted, follow closely the arguments already used in this section.

Theorem 4: Suppose $Q(x)$ is a polynomial of degree $m \geq 2$ and $Q(x)$ commutes with $\overline{U}_n(x)$ for some $n \geq 1$. Then $m \equiv 1 \bmod 4$ and $Q(x) = T_m(x)$. Moreover, if

$$Q(\overline{U}_n(x)) \equiv -\overline{U}_n(Q(x)) \text{ for some } n \geq 1,$$

then $m \equiv 3 \bmod 4$ and $P(x) = T_m(x)$.

Theorem 5: Suppose $P(x)$ is a polynomial of degree $m - 1 \geq 0$ and

$$\overline{P}(x) = \sqrt{4 - x^2}\,P(x).$$

If $\overline{P}(x)$ commutes with $T_n(x)$ for some $n \geq 2$, then $m \equiv 1 \bmod 4$ and $P(x) = U_m(x)$. Moreover, if $\overline{P}(T_n(x)) \equiv -T_n(\overline{P}(x))$ for some $n \geq 2$, then $m \equiv 3 \bmod 4$ and $P(x) = U_m(x)$.

4. SEMIPERMUTABLE CHAINS

Two functions $f(x)$ and $g(x)$ are defined in Kuczma [5, p. 215] to be *semi-permutable* if there exists a function

$$\Phi(x) = \frac{Kx + L}{Mx + N}$$

such that

$$f(g(x)) = \Phi[g(f(x))]. \tag{22}$$

Two functions $f(x)$ and $v(x)$ are *equivalent* if there exists a function

$$\phi(x) = rx + s, \text{ where } r \neq 0, \tag{23}$$

such that

$$\phi^{-1}[f(\phi(x))] = v(x).$$

Lemma 6a: Suppose $\phi(x)$ and $\Phi(x)$ are as just described and that (22) holds. Then the functions

$$F(x) = \phi^{-1}[f(\phi(x))] \quad \text{and} \quad G(x) = \phi^{-1}[g(\phi(x))]$$

are semipermutable.

Proof: For $\Psi(x) = \dfrac{Ax + B}{Cx + D}$, where $A = K - sM$, $B = L - sN$, $C = rM$, and $D = rN$, we have

$$\begin{aligned} F(G(x)) &= \phi^{-1} \circ f \circ g \circ \phi(x) = \phi^{-1} \circ \Phi \circ g \circ f \circ \phi(x) \\ &= \Psi \circ \phi^{-1} \circ g \circ f \circ \phi(x) = \Psi[G(F(x))], \end{aligned}$$

where the symbol $\circ$ indicates composition.

Suppose Γ is a sequence of positive integers and

$$P = \{p_n(x)\} \quad \text{and} \quad D = \{d_n(x)\}$$

are sequences of functions indexed by Γ. We define P to be an *SP chain under* D if every pair of functions in the set

$$\{p_n(x)d_n(x) : n \in \Gamma\}$$

are semipermutable. This definition generalizes that for SP chains given in [5], which is obtainable from the present definition in the case $d_n(x) \equiv 1$ for all positive integers n.

If $P = \{p_n(x)\}_{n \in \Gamma}$ is an SP chain under $D = \{d_n(x)\}_{n \in \Gamma}$ and $Q = \{q_n(x)\}_{n \in \Gamma}$ is an SP chain under $E = \{e_n(x)\}_{n \in \Gamma}$, then P and Q are *equivalent* if there exists $\phi(x)$ as in (23) such that

$$\phi^{-1}[p_n(\phi(x))d_n(\phi(x))] = q_n(x)e_n(x) \text{ for all } n \text{ in } \Gamma.$$

Corollary to Lemma 6a: Suppose $\{u_n(x)\}$ is an SP chain under $\{d_n(x)\}$ and $\phi(x) = rx + s$, where $r \neq 0$. Write

$$\phi^{-1}[p_n(\phi(x))d_n(\phi(x))] \text{ as } q_n(x)e_n(x).$$

[This is always possible, since we may choose $e_n(x) \equiv 1$ for all n in Γ.] Then $\{q_n(x)\}$ is an SP chain under $\{e_n(x)\}$.

If Γ is the sequence of odd positive integers, and $p_n(x)$ is a polynomial of degree $n - 1$ for each n in Γ, and P is an SP chain under D, then P is an *even SP chain under* D. Similarly, if Γ is the sequence of even positive integers, and $p_n(x)$ is a polynomial of degree $n - 1$ for each n in Γ, and P is an SP chain under D, then P is an *odd SP chain under* D. In particular, we define a *Chebyshev even chain* by

$$|p_n(x)| = U_n(x) \quad \text{and} \quad d_n(x) = \sqrt{4 - x^2} \text{ for } n = 1, 3, 5, \ldots;$$

and a *Chebyshev odd chain* by the same symbols, for $n = 2, 4, 6, \ldots$.

Finally, if Γ is the sequence of all the positive integers, and $p_n(x)$ is a polynomial of degree $n - 1$ for each n in Γ, and P is an SP chain under D, then P is a *complete SP chain under D*.

Lemma 6b: Suppose α, a, and e are nonzero, $\beta^2 \neq 4\alpha\gamma$, $F(x) = e\sqrt{\alpha x^2 + \beta x + \gamma}$, and $G(x) = \sqrt{\alpha x^2 + \beta x + \gamma}(ax^2 + bx + c)$. If $F(x)$ and $G(x)$ are semipermutable, then $F(x)$ and $G(x)$ are equivalent [with the same ϕ in (23)], respectively, to the functions

$$\overline{U}_1(x) = \sqrt{4 - x^2} \quad \text{and} \quad a_3\overline{U}_3(x) = a_3(x^2 - 1)\sqrt{4 - x^2}, \text{ where } a_3^2 = 1.$$

Proof:

$$(24)\qquad [F(G(x))]^2 = e^2[\alpha^2a^2x^6 + (2\alpha^2ab + \alpha\beta a^2)x^5 + (\alpha^2b^2 + 2\alpha^2ac + 2\alpha\beta ab + \alpha\gamma a^2)x^4 + (2\alpha^2bc + \alpha\beta b^2 + 2\alpha\beta ac + 2\alpha\gamma ab)x^3 + (\alpha^2c^2 + 2\alpha\beta bc + \alpha\gamma b^2 + 2\alpha\gamma ac)x^2 + (\alpha\beta c^2 + 2\alpha\beta bc)x + (\alpha\gamma c^2 + \gamma) + \beta(ax^2 + bx + c)\sqrt{\alpha x^2 + \beta x + \gamma}],$$

and

$$(25)\qquad [KG(F(x)) + L]^2 = K^2[\alpha^4a^2e^6x^6 + 3\alpha^3\beta a^2e^6x^5 + \alpha^2ae^5(\beta a + 2\alpha b)x^4\sqrt{\alpha x^2 + \beta x + \gamma} + \cdots + 2KL(\quad) + L^2,$$

where the expression indicated parenthetically after $2KL$ contains no nonzero constant multiple of $x^4\sqrt{\alpha x^2 + \beta x + \gamma}$.

In (22), suppose $M \neq 0$. Then, squaring both sides of (22) and writing

$$[MG(F(x)) + N]^2[F(G(x))]^2 = [KG(F(x)) + L]^2,$$

the left side contains for its highest degree term a multiple of x^{12}, whereas the highest degree term on the right side is $K^2\alpha^4a^2e^6x^6$. Therefore, $M = 0$, and there is no loss in assuming that $\Phi(x)$ is simply $Kx + L$.

Equating coefficients of x^6 and x^5 in (24) and (25) gives $K^2\alpha^2e^4 = 1$ and $\alpha b = \beta a$. The assumption $\beta^2 \neq 4\alpha\gamma$ keeps $\sqrt{\alpha x^2 + \beta x + \gamma}$ from being a polynomial, and this implies that the coefficient $(\alpha^2\beta a^2 + 2\alpha^3ab)e^5$ in (25) equals 0; together with $\alpha b = \beta a$ and $\alpha \neq 0$, this means $\beta = b = 0$. Thus,

$$(26)\qquad [F(G(x))]^2 = e^2[\alpha^2a^2x^6 + (2\alpha^2ac + \alpha\gamma a^2)x^4 + (\alpha^2c^2 + 2\alpha\gamma ac)x^2 + \alpha\gamma c^2 + \gamma]$$

and

$$(27)\qquad [KG(F(x))+L]^2 = K^2[\alpha^2e^2x^2 + \gamma(\alpha e^2+1)][\alpha^2a^2e^4x^4 + 2\alpha ae^2(\gamma ae^2+c)x^2 + (\gamma ae^2+c)^2] + 2KL\sqrt{\alpha^2e^2x^2 + \gamma(\alpha e^2+1)}(\alpha ae^2x^2 + \gamma ae^2 + c) + L^2.$$

Again comparing coefficients, we see that either $L = 0$ or $\sqrt{\alpha^2e^2x^2+\gamma(\alpha e^2+1)}$ is a polynomial. The latter implies $\alpha e^2 = -1$, which, by comparison of odd powers of x, leads to $L = 0$.

Multiplying out the right side of (27) and again comparing coefficients with (26), we find

$$(28)\qquad \gamma a(2\alpha e^2 + 1) + 2\alpha c(1 - \alpha e^2) = 0,$$

$$(29)\qquad \alpha^2ce^4(\alpha c + 2\gamma a) - (\gamma ae^2 + c)(3\alpha\gamma ae^2 + \alpha c + 2\gamma a) = 0,$$

$$(30)\qquad c^2(1 - \alpha e^2 + \alpha^3e^6) - \alpha^2e^6 + \gamma ae^2(\alpha e^2 + 1)(\gamma ae^2 + 2c) = 0.$$

Evaluating (26) and (27) at $x^2 = -\gamma/\alpha$ and equating them gives $e^2 = K^2c^2$, so that $c^2 = \alpha^2e^2$. We now rewrite (28), (29), and (30) with $q = \gamma a$ and $\alpha e = \delta c$, where $|\delta| = 1$:

(31) $2c^3e - 2\delta c^2 - 2\delta ce^2q - qe = 0,$

(32) $\delta c^5e^2 + (2qe^3 - \delta)c^3 - 4q\delta c^2e^2 - (3\delta qe^3 + 2)qce - 2q^2e^3 = 0,$

(33) $c^3e(c^2 - 1) + \delta c^2(1 - e^4) + qe^2(ce + \delta)(qe^2 + 2c) = 0.$

If $2\delta ce + 1 = 0$, no q satisfies both (32) and (33). Therefore, $2\delta ce + 1 \neq 0$, and in this case we find

$$q = \frac{2c^2(ce - \delta)}{e(2\delta ce + 1)}$$

from (31) and substitute into (32) to obtain $c^2e^2 = 1$. For $\delta = 1$, we find from $c^2e^2 = 1$ that $ce = -1$, since if $ce = 1$ then $q = 0$, contrary to $\gamma \neq 0 \neq a$. Simplifying the expression for q gives $\gamma ae = 4c^2$. Also, from $\alpha e = \delta c$ comes $\alpha e^2 = -1$. Similarly for $\delta = -1$, we determine $ce = 1$, $\gamma ae = -4c^2$, and $\alpha e^2 = -1$.

Now for $\phi(x) = e\sqrt{\gamma}x/2$, it is easy to verify that

$$\phi^{-1}[F(\phi(x))] = \sqrt{4 - x^2}$$

and, using the fact $\gamma ae^3 = 4\delta$, that

$$\phi^{-1}[G(\phi(x))] = e^{-2}(x^2 - 1)\sqrt{4 - x^2}.$$

Finally, it is easy to check directly that these two functions are semipermutable if and only if $e^2 = \pm 1$, and this completes the proof.

Theorem 6: Every even SP chain under a constant sequence of the form

$$d_n(x) = \sqrt{\alpha x^2 + \beta x + \gamma}$$

is equivalent to a Chebyshev even chain $\{a_nU_n(x)\}$, $a_n^2 = 1$, $n = 1, 3, 5, \ldots$.

Proof: Suppose $\{y_1, y_3, y_5, \ldots\}$ is an even SP chain under $d(x) = d_n(x)$ as above. Let $\overline{y}_n(x) = y_n(x)d(x)$. By Lemma 6b, we may assume that $d(x) = \sqrt{4 - x^2}$. Since every even polynomial $y_n(x)$ of degree $n - 1$ is a linear combination of even $U_i(x)$'s up to degree $n - 1$, we write

$$\overline{y}_n(x) = a_n\overline{U}_n(x) + \sum_{i=1}^{m} b_i\overline{U}_i(x), \quad n > m \geq 1,$$

where $b_i = 0$ for even i. Suppose $b_m \neq 0$. Then

(34) $$[\overline{y}_1(\overline{y}_n(x))]^2 = (4 - \overline{y}_n^2(x)) = \left\{-a_n^2\overline{U}_n^2(x) - 2a_n\overline{U}_n(x)\sum_{i=1}^{m} b_i\overline{U}_i(x) - \left[\sum_{i=1}^{m} b_i\overline{U}_i(x)\right]^2 + 4\right\}$$

and

(35) $$[K\overline{y}_n(\overline{y}_1(x)) + L]^2 = K^2\left[a_n\overline{U}_n(\overline{U}_1(x)) + \sum_{i=1}^{m} b_i\overline{U}_i(\overline{U}_1(x))\right]^2 + 2KL\left[a_n\overline{U}_n(\overline{U}_1(x)) + \sum_{i=1}^{m} b_i\overline{U}_i(\overline{U}_1(x))\right] + L^2.$$

The highest degree term on the right side of (34) is $a_n^2x^{2n}$, while that on the right side of (35) is $(-1)^{n-1}K^2a_n^2x^{2n}$. Thus, $K^2 = 1$, so subtracting (35) from (34) and using Lemma 2a [rewritten as $T_m(x)T_n(x) + \overline{U}_m(x)\overline{U}_n(x) = 2T_{n-m}(x)$ for $0 \leq m \leq n$],

$$0 = [\overline{y}_1(\overline{y}_n(x))]^2 - [K\overline{y}_n(\overline{y}_1(x)) + L]^2 = -a_n^2\overline{U}_n^2(x) - 2a_n\sum_{i=1}^{m} b_i\overline{U}_n(x)\overline{U}_i(x)$$
$$-\left[\sum_{i=1}^{m} b_i\overline{U}_i(x)\right]^2 + 4 - \left\{a_n^2T_n^2(x) + 2a_n\sum_{i=1}^{m} b_iT_n(x)T_i(x) + \left[\sum_{i=1}^{m} b_iT_i(x)\right]^2\right\}$$
$$- 2KL\left[a_nT_n(x) + \sum_{i=1}^{m} b_iT_i(x)\right] - L^2$$
$$= -a_n^2[\overline{U}_n^2(x) + T_n^2(x)] - 2a_n\sum_{i=1}^{m} b_i[\overline{U}_n(x)\overline{U}_i(x) + T_n(x)T_i(x)]$$
$$-\left\{\left[\sum_{i=1}^{m} b_i\overline{U}_i(x)\right]^2 + \left[\sum_{i=1}^{m} b_iT_i(x)\right]^2\right\} - 2KL\left[a_nT_n(x) + \sum_{i=1}^{m} b_iT_i(x)\right]$$
$$- L^2 + 4.$$

Thus,

$$0 = -4a_n^2 - 4a_n\sum_{i=1}^{m} b_iT_{n-i}(x) - \left[2\sum_{i=1}^{m} b_i^2 + 4\sum_{1\le i<j\le m} b_ib_jT_{j-i}(x)\right] \tag{36}$$
$$- 2KL\left[a_nT_n(x) + \sum_{i=1}^{m} b_iT_i(x)\right] - L^2 + 4.$$

If $L \neq 0$, the right side of (36) is a polynomial of degree n. Therefore, $L = 0$. If $b_m \neq 0$, the right side of (36) is a polynomial of degree $n - 1$, again a contradiction. Therefore, $m = 0$, so that

$$\overline{y}_n(x) = a_n\overline{U}_n(x) \text{ for } n > 1,$$

and (36) shows that $a_n^2 = 1$ for $n > 1$.

Lemma 7a: Suppose α, a, and e are nonzero, $\beta^2 \neq 4\alpha\gamma$, $F(x) = (ex + f)\sqrt{\alpha x^2 + \beta x + \gamma}$ and $G(x) = (ax^3 + bx^2 + cx + d)\sqrt{\alpha x^2 + \beta x + \gamma}$.

If $F(x)$ and $G(x)$ are semipermutable, then $F(x)$ and $G(x)$ are equivalent [with the same ϕ in (23)], respectively, to the functions

$$\overline{U}_2(x) = x\sqrt{4 - x^2} \text{ and } a_4\overline{U}_4(x) = a_4(x^3 - 2x)\sqrt{4 - x^2}, \text{ where } a_4^2 = 1.$$

Proof: Write $A = \sqrt{\alpha x^2 + \beta x + \gamma}$ and $B = ax^3 + bx^2 + cx + d$, so that

$$F(x) = (ex + f)A \quad\text{and}\quad G(x) = BA.$$

Direct computations show

$$[F(G(x))]^2 = \alpha e^2G^4(x) + (\alpha f^2 + 2\beta ef + \gamma e^2)G^2(x) + [e(2\alpha f + \beta e)G^2(x) + f(\beta f + 2\gamma e)]BA \tag{37}$$

and

$$[KG(F(x)) + L]^2 = K^2[Q_8F^8(x) + Q_7F^7(x) + \cdots + Q_1F(x) + Q_0] + 2KLG(F(x)) + L^2, \tag{38}$$

where

$Q_8 = \alpha a^2$, $Q_7 = a(2\alpha b + \beta a)$,
$Q_6 = 2\alpha ac + \alpha b^2 + 2\beta ab + \gamma a^2$, $Q_5 = 2\alpha ad + 2\alpha bc + 2\beta ac + \beta b^2 + 2\gamma ab$,
$Q_4 = 2\alpha bd + \alpha c^2 + 2\beta ad + 2\beta bc + 2\gamma ac + \gamma b^2$,
$Q_3 = 2\alpha cd + 2\beta bd + \beta c^2 + 2\gamma ad + 2\gamma bc$, $Q_2 = \alpha d^2 + 2\beta cd + 2\gamma bd + \gamma c^2$,
$Q_1 = d(\beta d + 2\gamma c)$, $Q_0 = \gamma d^2$.

Comparing coefficients of x^{16} in (37) and (38) gives $a^2 = K^2\alpha^2 e^6$. In (38) only the expression $K^2 a(2\alpha b + \beta a)F^7(x)$ comtains a nonzero multiple of $x^{13}A$, and (37) contains no such term. Specifically, (38) contains the term

$$K^2\alpha^3 ae^7(2\alpha b + \beta a)x^{13}A.$$

The condition $\beta^2 \neq 4\alpha\gamma$ keeps A from being a polynomial, and since $K^2\alpha^3 ae^7 \neq 0$, comparison with terms in (37) gives

$$\beta a = -2\alpha b. \tag{39}$$

In (37) only the expression $e(2\alpha f + \beta e)G^2(x)BA$ contains a nonzero multiple of $x^{11}A$, and (38) contains no such term. Writing this expression as

$$e(2\alpha f + \beta e)(\alpha a^2 x^8 + \cdots)(ax^3 + \cdots)A,$$

we find by comparison with (37) that

$$\beta e = -2\alpha f. \tag{40}$$

Since A is not a polynomial, the expression

$$\sqrt{\alpha A^2(ex + f)^2 + \gamma + \beta A}\,[bF^2(x) + d + (aF^2(x) + c)(ex + f)A] \tag{41}$$

for $G(F(x))$ in (38) cannot be of the form $R(x) + Q(x)A$ for any polynomials $R(x)$ and $Q(x)$ unless perhaps $\beta = 0$. Thus, for $\beta \neq 0$, the expression (41) is linearly independent of the other terms in (38) and all those in (37), so that $L = 0$. On the other hand, if $\beta = 0$, then $b = f = 0$ by (39) and (40). Then (37) shows $[F(G(x))]$ to be a polynomial, and (41) reduces to

$$\sqrt{\alpha A^2 e^2 x^2 + \gamma}\,[d + (ae^2 A^2 x^2 + c)ex\sqrt{\alpha x^2 + \gamma}].$$

For this to be a polynomial requires $\gamma = 0$, contrary to $\beta^2 \neq 4\alpha\gamma$. Consequently, for $\beta = 0$, we still have $L = 0$.

Equation (40) shows that no multiple of $x^p A$ occurs in $[F(G)]^2$ for any $p > 3$. Since only $Q_5 F^5(x)$ in (38) contains such a multiple for $p = 9$, we have $Q_5 = 0$. Because of this and the fact that $Q_3 F^3(x)$ alone in (38) contains a multiple of $x^5 A$, we have $Q_3 = 0$. This leaves (38) with no multiple of $x^3 A$, so that the coefficient of $x^3 A$ in (37), namely $f(\beta f + 2\gamma e)$, must equal 0. If $f \neq 0$, then eliminating e from $\beta f + 2\gamma e = 0$ and $\beta e + 2\alpha f = 0$ gives $\beta^2 = 4\alpha\gamma$, which is forbidden. Therefore, $f = 0$. By (40) and (39), $\beta = b = 0$ also.

For x_0 a root of $\alpha x^2 + \beta x + \gamma$,

$$F[G(x_0)] = F(0) = \sqrt{\gamma}f = 0 \quad \text{and} \quad G[F(x_0)] = G(0) = \sqrt{\gamma}d;$$

since $L = 0$, we have $\sqrt{\gamma}d = 0$. The condition $\beta^2 \neq 4\alpha\gamma$ implies $\gamma \neq 0$. We summarize our findings:

$$\beta = 0,\ b = 0,\ f = 0,\ d = 0,\ L = 0,\ Q_5 = 0,\ Q_3 = 0,\ Q_1 = 0. \tag{42}$$

These enable us to simplify (37) and (38) as follows:

$$\begin{aligned}[F(G(x))]^2 = {} & \alpha^3 a^4 e^2 x^{16} + 2\alpha^2 a^3 e^2(2\alpha c + \gamma a)x^{14} \\ & + \alpha a^2 e^2(6\alpha^2 c^2 + 8\alpha\gamma ac + \gamma^2 a^2)x^{12} \\ & + 2\alpha ace^2(5\alpha\gamma ac + 2\gamma^2 a^2 + 2\alpha^2 c^2 + \alpha\gamma ac)x^{10} \\ & + \alpha e^2(6\gamma^2 a^2 c^2 + 8\alpha\gamma ac^3 + \alpha^2 c^4 + \gamma a^2)x^8 \\ & + \gamma e^2(4\alpha\gamma ac^3 + 2\alpha^2 c^2 + 2\alpha ac + \gamma a^2)x^6 \\ & + \gamma ce^2(\alpha\gamma c^3 + 2\gamma a + \alpha c)x^4 + \gamma^2 c^2 e^2 x^2;\end{aligned} \tag{43}$$

$$\begin{aligned}K^2[G(F(x))]^2 = K^2\{ & \alpha^5 a^2 e^8 x^{16} + 4\alpha^4 a^2\gamma e^8 x^{14} + \alpha^3 e^6(6\gamma^2 a^2 e^2 \\ & + 2\alpha ac + \gamma a^2)x^{12} + \alpha^2\gamma e^6(4\gamma^2 a^2 e^2 + 6\alpha ac + 3\gamma a^2)x^{10} \\ & + \alpha e^4(\gamma^4 a^2 e^4 + 6\alpha\gamma^2 ace^2 + 3\gamma^3 a^2 e^2 + \alpha^2 c^2 + 2\alpha\gamma ac)x^8 \\ & + \gamma e^4(2\alpha\gamma^2 ace^2 + \gamma^3 a^2 e^2 + 2\alpha^2 c^2 + 4\alpha\gamma ac)x^6 \\ & + \gamma ce^2(\alpha\gamma ce^2 + 2\gamma^2 ae^2 + \alpha c)x^4 + \gamma^2 c^2 e^2 x^2\}.\end{aligned} \tag{44}$$

Comparing coefficients of x^{16}, x^{14}, ..., x^2, in order, gives

(45) $$a^2 = \alpha^2 e^6 \text{ [because of (52) below]}$$

(46) $$2\alpha c = \gamma a$$

(47) $$13a^2c^2 = e^4(3\gamma^2 a^2 e^2 + 2\alpha ac)$$

(48) $$11\alpha ac = e^4(\gamma^3 a^2 e^2 + 6\alpha^2 c^2)$$

(49) $$41\alpha^2 c^4 + 2\alpha ac = \gamma^4 a^2 e^6 + 24\alpha^2 \gamma c^2 e^4 + 5\alpha^2 c^2 e^2$$

(50) $$5\alpha c^2 + 2a = 4\alpha\gamma c e^4 + 5\alpha c e^2$$

(51) $$\alpha c^3 + 2a = 5\alpha c e^2$$

(52) $$K^2 = 1.$$

Subtracting (51) from (50) gives

(53) $$c^2 = \gamma e^4.$$

Eliminating α from (46) and (47) gives

(54) $$13c^2 = \gamma e^4(3\gamma e^2 + 1).$$

Eliminating c^2 from (53) and (54) gives

(55) $$\gamma e^2 = 4.$$

With (45), (53), and (55) in mind, we now discern four possibilities for given a and e:

(56) $$\alpha = -ae^3 \quad \text{and} \quad c = -2e$$

(57) $$\alpha = ae^3 \quad \text{and} \quad c = -2e$$

(58) $$\alpha = -ae^3 \quad \text{and} \quad c = 2e$$

(59) $$\alpha = ae^3 \quad \text{and} \quad c = 2e.$$

For (56), we have

$$F(x) = x\sqrt{4 - ae^5x^2} \quad \text{and} \quad G(x) = e^{-1}(ax^3 + cx)\sqrt{4 - ae^5x^2}.$$

For $\phi(x) = x/\sqrt{ae^5}$ we find that $\phi^{-1}[F(\phi(x))] = x\sqrt{4 - x^2}$ and, using the assumption $c = -2e$, that

$$\phi^{-1}[G(\phi(x))] = (e^{-6}x^3 - 2x)\sqrt{4 - x^2}.$$

It is easily checked directly that these two functions are semipermutable iff $e^6 = 1$.

Direct checking for semipermutability further shows that (57) gives F and G respectively equivalent to $\overline{U}_2$ and $\overline{U}_4$, while (58) and (59) give functions respectively equivalent to $\overline{U}_2$ and $-\overline{U}_4$ as desired.

Theorem 7: Every odd SP chain under a constant sequence of the form

$$d_n(x) = \sqrt{\alpha x^2 + \beta x + \gamma}$$

is equivalent to a Chebyshev odd chain $\{a_n U_n(x)\}$, $a^2 = 1$, $n = 2, 4, 6, \ldots$.

Proof: Suppose $\{y_2, y_4, \ldots\}$ is an odd SP chain under $d(x) = d_n(x)$ as above. Let $\overline{y}_n(x) = y_n(x)d(x)$. By Lemma 7a, we may assume that $d(x) = \sqrt{4 - x^2}$. Since every odd polynomial $y_n(x)$ of degree $n - 1$ is a linear combination of odd $U_i(x)$'s up to degree $n - 1$, we write

$$\overline{y}_n(x) = a_n\overline{U}_n(x) + \sum_{i=1}^{m} b_i\overline{U}_i(x), \; n > m \geq 1,$$

where $b_i = 0$ for odd i. The rest of the proof follows that of Theorem 6 exactly.

Theorem 8: Suppose $d(x) = \sqrt{\alpha x^2 + \beta x + \gamma}$ where $\alpha \neq 0$ and $\beta^2 \neq 4\alpha\gamma$. There exists no complete SP chain under D.

Proof: Referring to the definitions given just before Lemma 6b, if such a chain $\{p_1(x), p_2(x), \ldots\}$ exists, then the chain $\{p_1(x), p_3(x), \ldots\}$ is an even SP chain. The proof of Lemma 6b shows that we may assume $\Phi(x) = Kx + L$ in (22) and $\alpha = -1$ and $\beta = 0$. Thus, we write

$$\overline{p}_1(x) = a\sqrt{-x^2 + \gamma} \quad \text{and} \quad \overline{p}_2(x) = (bx + c)\sqrt{-x^2 + \gamma}$$

where a, b, and γ are nonzero. Writing out the assumption

$$[\overline{p}_1(\overline{p}_2(x))]^2 = [K\overline{p}_2(\overline{p}_1(x)) + L]^2,$$

we find the term $2K^2a^3bcx^2\sqrt{-x^2 + \gamma}$ on the right side and all other terms in this equation linearly independent of this term. Thus $c = 0$, so that

$$[\overline{p}_1(\overline{p}_2(x))]^2 = a^2b^2x^4 - \gamma a^2b^2x^2 + \gamma a^2.$$

It is easily checked that $L = 0$, so that

$$[K\overline{p}_2(\overline{p}_1(x)) + L]^2 = -a^4b^2K^2x^4 + \gamma a^2b^2K^2(2a^2 - 1)x^2 + K^2\gamma^2a^2b^2(1 - a^2).$$

Comparison of coefficients of x^4 gives $a^2K^2 = -1$, which along with comparison of coefficients of x^2 implies $K^2 = -1$. But this leads to a contradiction, since comparison of constant terms gives $1 = \gamma b^2(K^2 + 1)$.

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