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Thus, by our above argument, if $\alpha(b - 1, b, p) \equiv 0 \pmod 2$, then

$$\alpha(b - 1, b, p) = \mu(b - 1, b, p), \text{ and } \beta(b - 1, b, p) = 1.$$

If $\alpha(b - 1, b, p) \equiv 1 \pmod 2$, then

$$\mu(b - 1, b, p) = 2\alpha(b - 1, b, p), \text{ and } \beta(b - 1, b, p) = 2.$$

The results of parts (i)-(iii) now follows.

(iv)-(vii) These follow from Theorems 9 and 10.

(viii) This follows from Theorems 11 and 12.

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MIXING PROPERTIES OF MIXED CHEBYSHEV POLYNOMIALS

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The *Chebyshev polynomials of the first kind*, defined recursively by

$$t_0(x) = 1,\ t_1(x) = x,\ t_n(x) = 2xt_{n-1}(x) - t_{n-2}(x) \text{ for } n = 2, 3, \ldots,$$

or equivalently, by

$$t_n(x) = \cos(n \cos^{-1} x) \text{ for } n = 0, 1, \ldots,$$

commute with one another under composition; that is

$$t_m(t_n(x)) = t_n(t_m(x)).$$

In [1], Adler and Rivlin use this well-known fact to prove that in an appropriate measure-theoretic setting the mappings $t_1, t_2, \ldots$ are measure-preserving and the sequence $\{t_1, t_2, \ldots\}$ is strongly mixing. In another setting, Johnson and Sklar [2] obtain related results. The purpose of the present note is to establish results analogous to those in [1] for sequences involving not only t_n's but also the *Chebyshev polynomials of the second kind*; these are defined recursively by

$$u_0(x) = 1,\ u_1(x) = 2x,\ u_n(x) = 2xu_{n-1}(x) = u_{n-2}(x) \text{ for } n = 2, 3, \ldots,$$

or equivalently, by

$$u_n(x) = \frac{\sin[(n+1)\cos^{-1} x]}{\sqrt{1-x^2}} \text{ for } n = 0, 1, \ldots .$$

Concerning compositions of Chebyshev polynomials of both kinds, we have the following lemma from [3], where a trigonometric proof may be found.

Lemma 1: Let $\{t_0, t_1, \ldots\}$ and $\{u_0, u_1, \ldots\}$ be the sequences of Chebyshev polynomials of the first and second kinds, respectively. Put $\overline{u}_{-1}(x) \equiv 0$ and define

$$\overline{u}_n(x) = u_n(x)\sqrt{1-x^2} \text{ for } n = 0, 1, \ldots .$$

Then for nonnegative m and n,

(1) $$t_m(t_n) = t_{mn},$$

(2) $$\overline{u}_m(t_n) = \overline{u}_{mn+n-1},$$

(3) $$t_m(\overline{u}_n) = \begin{cases} (-1)^{\frac{m}{2}} t_{mn+n} & \text{for even } m \\ (-1)^{\frac{m-1}{2}} \overline{u}_{mn+m-1} & \text{for odd } m, \end{cases}$$

(4) $$\overline{u}_m(\overline{u}_n) = \begin{cases} (-1)^{\frac{m}{2}} t_{(m+1)(n+1)} & \text{for even } m \\ (-1)^{\frac{m-1}{2}} \overline{u}_{mn+m+n} & \text{for odd } m. \end{cases}$$

We introduce some notation:

I = the closed interval $[-1, 1]$
I' = the closed interval $[0, \pi]$
$\mathcal{B}$ = the family of Borel subsets of I
$\mathcal{B}'$ = the family of Borel subsets of I'
λ = Legesgue measure on $\mathcal{B}$
λ' = Lebesgue measure on $\mathcal{B}'$

Let μ be the measure defined on $\mathcal{B}$ by the Lebesgue integral

$$\mu(B) = \frac{2}{\pi}\int_B \frac{dx}{\sqrt{1-x^2}},\ B \in \mathcal{B}.$$

Rivlin [4] proves that each t_n for $n \geq 1$ preserves the measure μ; that is, the inverse mapping t_n^{-1}, which is an n-valued mapping (except at ±1) from I' onto I, satisfies

$$\mu(t_n^{-1}(B)) = \mu(B),\ B \in \mathcal{B}.$$

Using the same method of proof, we establish the following lemma.

Lemma 2a: Let $\overline{u}_n = u_n(x)\sqrt{1-x^2}$ for $n = 0, 1, \ldots$. For odd n, the mapping $\overline{u}_n$ preserves the measure μ on $\mathcal{B}$.

Proof: Let ϕ be the one-to-one measurable mapping of I onto I' defined by

$$\phi(x) = \theta = \cos^{-1} x,$$

and put $v_n = \phi(\overline{u}_n(\phi^{-1}))$. Then, for odd n and

$$\frac{(2k+1)\pi}{2(n+1)} \le \theta \le \frac{(2k+3)\pi}{2(n+1)}, \quad k = 0, 1, \ldots, n-1,$$

we find

$$v_n(\theta) = \begin{cases} -(n+1)\theta + \frac{\pi}{2}, & 0 \le \theta \le \frac{\pi}{2(n+1)} \\ (n+1)\theta - \frac{2k+1}{2}\pi, & \text{even } k \\ -(n+1)\theta + \frac{2k+3}{2}\pi, & \text{odd } k \\ -(n+1)\theta + \frac{2n+3}{2}\pi, & \frac{(2n+1)\pi}{2(n+1)} \le \theta \le \pi. \end{cases}$$

An open subinterval of $[0, \pi/2]$ or $[\pi/2, \pi]$ having length ℓ is the image under v_n of $n + 1$ subintervals of I' (on the horizontal axis in Figure 1) in case n is odd, where each of these subintervals has length $\ell/(n+1)$. It follows that the mapping v_n preserves the measure λ'. Now, if $-1 \le a < b < 1$, then

$$\int_a^b \frac{dx}{\sqrt{1-x^2}} = \int_{\phi(b)}^{\phi(a)} d\theta,$$

so that $\mu(B) = \frac{2}{\pi}\lambda'(\phi(B))$ for $B \in \mathcal{B}$. Consequently (omitting parentheses),

$$\mu(\overline{u}_n^{-1}(B)) = \frac{2}{\pi}\lambda'(\phi\overline{u}_n^{-1}B) = \frac{2}{\pi}\lambda'(\phi\overline{u}_n^{-1}\phi^{-1}\phi B) = \frac{2}{\pi}\lambda'(v_n^{-1}\phi B) = \frac{2}{\pi}\lambda'(\phi B) = \mu(B).$$

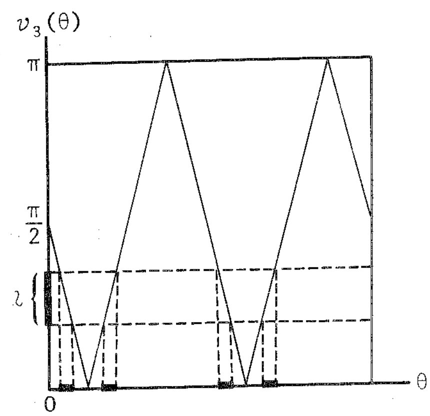

Fig. 1. v_3 preserves λ' on $[0, \pi]$.

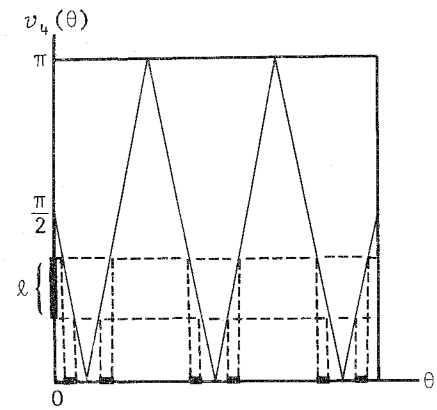

Fig. 2. v_4 preserves λ' on $\left[0, \frac{4\pi}{5}\right]$.

For even n, the result is not so simple, since in this case v_n fails to preserve λ' on all of I'. However, one may prove the following lemma with an argument similar to that just given.

Lemma 2b: Let $\overline{u}_n(x) = u_n(x)\sqrt{1-x^2}$ for $n = 0, 1, \ldots$. For even n, the mapping $\overline{u}_n$ preserves the restriction of the measure μ to the family of Borel sets of the closed interval $\left[\cos^{-1}\frac{n\pi}{n+1}, 1\right]$. (See Figure 2.)

Turning now to orthogonality of Chebyshev polynomials of both kinds, let $L^2(I, \mathcal{B}, \mu)$ denote the set of square μ-integrable functions f which are μ-measurable on $\mathcal{B}$:

$$\int_{-1}^{1} f^2(x)\,d\mu(x) < \infty.$$

For f and g in $L^2(I, \mathcal{B}, \mu)$, let $\langle f, g\rangle$ denote the inner product

$$\frac{2}{\pi}\int_{-1}^{1} f(x)g(x)\,d\mu(x),$$

and let $\|f\|$ denote the norm $\langle f, f\rangle^{1/2}$.

Lemma 3: Let $\{t_0, t_1, \ldots\}$ and $\{u_0, u_1, \ldots\}$ be the sequences of Chebyshev polynomials of the first and second kinds, respectively. Put

$$\overline{u}_n(x) = u_n(x)\sqrt{1 - x^2} \text{ for } n = 0, 1, \ldots .$$

Then for nonnegative m and n,

$$\langle t_m, t_n\rangle = \begin{cases} 0 & m \neq n \\ 1 & m = n \neq 0 \\ 2 & m = n = 0 \end{cases} \tag{5}$$

$$\langle \overline{u}_m, \overline{u}_n\rangle = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases} \tag{6}$$

$$\langle \overline{u}_m, t_n\rangle = \begin{cases} 0 & m + n \text{ odd} \\ \dfrac{4(m + 1)}{\pi[(m + 1)^2 - n^2]} & m + n \text{ even} \end{cases} \tag{7}$$

Proof: Equations (5) and (6) are well known. Proof of (7) follows from

$$\int_0^{\pi} \sin(m + 1)\theta \cos n\theta \, d\theta = \frac{1}{2}\int_0^{\pi} [\sin(m + 1 - n)\theta + \sin(m + 1 + n)\theta]\,d\theta,$$

where $\cos\theta = x$.

Lemma 3 shows that the sequences

$$\left\{\frac{1}{\sqrt{2}}t_0, t_1, t_2, \ldots\right\} \text{ and } \{\overline{u}_0, \overline{u}_1, \overline{u}_2, \ldots\}$$

are orthonormal over I, a well-known fact. It is well known, a fortiori, that these are complete orthonormal sets in the space $L^2(I, \mathcal{B}, \mu)$; i.e., for each f in $L^2(I, \mathcal{B}, \mu)$ and $\varepsilon > 0$, there exists a finite linear combination

$$s_n(x) = \sum_{k=0}^{n} a_k t_k(x)$$

such that $\| f - s_n \| < \varepsilon$ [and similarly for the $\overline{u}_k(x)$'s].

Now let $\{F_n\} = \{F_0, F_1, F_2, \ldots\}$ denote the sequence

$$\frac{1}{\sqrt{2}}t_0, \overline{u}_1, t_2, \overline{u}_3, \ldots$$

and let $\{G_n\} = \{G_0, G_1, G_2, \ldots\}$ denote the sequence

$$\{\overline{u}_0, t_1, \overline{u}_2, t_3, \ldots\}.$$

These are orthonormal sequences by Lemma 3. For f in $L^2(I, \mathcal{B}, \mu)$, we define the *F-Chebyshev series* for f to be the series

$$\sum_{k=0}^{\infty} f_k F_k(x),$$

where the coefficients $f_0, f_1, \ldots$ are given by $f_k = \langle f, F_k \rangle$. Similarly, the *G-Chebyshev series* for given g in $L^2(I, \mathcal{B}, \mu)$ is defined by

$$\sum_{k=0}^{\infty} g_k G_k(x),$$

where $g_k = \langle g, G_k \rangle$ for $k = 0, 1, \ldots$.

Lemma 4: If n is an odd positive integer and $\varepsilon > 0$, then there exists a sum of the form

$$s_m(x) = \sum_{k=0}^{m} a_{2k+1} \overline{u}_{2k+1}(x)$$

such that $\| t_n - s_m \| < \varepsilon$. If n is an even nonnegative integer and $\varepsilon > 0$, then there exists a sum of the form

$$s_m(x) = \sum_{k=0}^{m} a_{2k} t_{2k}$$

such that $\| \overline{u}_n - s_m \| < \varepsilon$.

Proof: Suppose that n is an odd positive integer. It suffices, by the Riesz-Fischer Theorem (see [5], p. 127) to show that the sequence $\tau_{2k+1} = \langle t_n, \overline{u}_{2k+1} \rangle$ satisfies

$$\sum_{k=0}^{\infty} \tau_{2k+1}^2 < \infty.$$

This is clearly the case, since, by (7),

$$\tau_{2k+1} = \frac{8}{\pi} \frac{k+1}{[(2k+2)^2 - n^2]}.$$

Similarly, for even nonnegative n and $\tau_{2k} = \langle \overline{u}_n, t_{2k} \rangle$, we have

$$\tau_{2k} = \frac{4}{\pi} \frac{n+1}{(n+1)^2 - 4k^2}.$$

Theorem 1: The orthonormal sequences $\{F_n\}$ and $\{G_n\}$ for $n = 0, 1, \ldots$ are complete in $L^2(I, \mathcal{B}, \mu)$.

Proof: We deal first with $\{F_n\}$. Suppose $f \in L^2(I, \mathcal{B}, \mu)$ and $\varepsilon > 0$. Since

$$\left\{ \frac{1}{\sqrt{2}} t_0, t_1, t_2, \ldots \right\}$$

is a complete orthonormal sequence in $L^2(I, \mathcal{B}, \mu)$, we choose odd m and numbers $a_0, a_1, \ldots, a_m$ satisfying

$$\left\| f - \sum_{k=0}^{m} a_k t_k \right\| < \varepsilon/2.$$

By Lemma 4, there exist sums $s_k = c_{k1}\overline{u}_1 + c_{k3}\overline{u}_3 + \cdots + c_{kq_k}\overline{u}_{q_k}$ such that

$$\| a_k t_k - a_k s_k \| < \varepsilon/m \text{ for } k = 1, 3, 5, \ldots, m.$$

Let $Q = \max\{q_k : k = 1, 3, 5, \ldots, m\}$ and put

$$q = \begin{cases} Q & \text{if } Q \text{ is odd} \\ Q+1 & \text{if } Q \text{ is even.} \end{cases}$$

Put $c_{kp} = 0$ for $q_k < p \le q$, $k = 1, 3, 5, \ldots, m$. Next, let

$$b_j = \begin{cases} a_1c_{1j} + a_3c_{3j} + \cdots + a_mc_{mj} & \text{for } j = 1, 3, 5, \ldots, q \\ a_j & \text{for even } j < m \\ 0 & \text{for even } j > m. \end{cases}$$

Then,

$$\begin{aligned} \| f - (b_0t_0 + b_1\overline{u}_1 + \cdots + b_q\overline{u}_q) \| \le & \| f - b_0t_0 - a_1t_1 - b_2t_2 - a_3t_3 - \cdots - a_mt_m \| \\ & + \| a_1t_1 - a_1(c_{11}\overline{u}_1 + \cdots + c_{1q}\overline{u}_q) \| \\ & + \| a_3t_3 - a_3(c_{31}\overline{u}_1 + \cdots + c_{3q}\overline{u}_q) \| + \cdots \\ & + \| a_mt_m - a_m(c_{m1}\overline{u}_1 + \cdots + c_{mq}\overline{u}_q) \| < \varepsilon. \end{aligned}$$

This proves completeness of the sequence $\{F_n\}$. The proof for $\{G_n\}$ is quite similar.

We wish to use all the foregoing results to prove that the sequences of mappings $\{F_n^{-1}\}$, $\{G_n^{-1}\}$, and $\{\overline{u}_n^{-1}\}$, when applied to any B in $\mathcal{B}$, *increasingly homogenize* or *mix* B throughout I. This vague description is made precise for a μ-preserving sequence of mappings $\{\tau_n\}$ by the notion that $\{\tau_n\}$ is a *strongly mixing sequence with respect to* μ if

(8) $$\lim_{n\to\infty} \mu[(\tau_n^{-1}A) \cap B] = \frac{\mu(A)\mu(B)}{\mu(I)}$$

for all A and B in $\mathcal{B}$.

Theorem 2: The sequence of mappings $\{F_1, F_2, \ldots\}$ is strongly mixing in $L^2(I, \mathcal{B}, \mu)$ with respect to the measure μ.

Proof: To establish (8), it suffices to prove

(9) $$\lim_{n\to\infty} \langle f(F_n), g\rangle = \frac{1}{2}\langle f, 1\rangle\langle g, 1\rangle$$

for all f and g in $L^2(I, \mathcal{B}, \mu)$, since (9) is merely a restatement of (8) in case f is the characteristic function of A and g is the characteristic function of B. [That is, $f(x) = 1$ for $x \in A$ and $f(x) = 0$ for $x \notin A$; similarly for g and B.] First, assume f and g are terms of the sequence $\{F_0, F_1, \ldots\}$. Then for some $j \ge 0$ and $k \ge 0$, with $n \ge 1$, Lemmas 1 and 3 show that

$$\langle f(F_n), g\rangle = \langle F_j(F_n), F_k\rangle$$

$$= \begin{cases} \langle t_{jn}, F_k\rangle & j \text{ even, } n \text{ even, } j \ne 0 \\ \langle t_0/\sqrt{2}, F_k\rangle & j = 0 \\ (-1)^{j/2}\langle t_{jn+j}, F_k\rangle & j \text{ even, } n \text{ odd, } j \ne 0 \\ \langle \overline{u}_{jn+n-1}, F_k\rangle & j \text{ odd, } n \text{ even} \\ (-1)^{\frac{j-1}{2}}\langle \overline{u}_{jn+j+n}, F_k\rangle & j \text{ odd, } n \text{ odd} \end{cases}$$

$$= \begin{cases} 1 & 0 \ne k = jn, \quad j \text{ even, } n \text{ even} \\ \sqrt{2} & 0 = j = k \\ (-1)^{j/2} & k = (j+1)n, \quad j \text{ even, } n \text{ odd} \\ (-1)^{\frac{j-1}{2}} & k = (j+1)n + j, \; j \text{ odd, } n \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

Thus,

$$\lim_{n\to\infty}\langle f(F_n),\ g\rangle = 0 \text{ for } j > 0,$$

and in this case (9) clearly holds. If $j = 0$, then (9) is satisfied by

$$\langle f(F_n),\ g\rangle = 1 \text{ for all } n \geq 1.$$

We have shown so far that (9) holds if f and g are both terms of the sequence $\{F_0, F_1, \ldots\}$. We continue now as in Rivlin [4, p. 171]: Suppose f and g are *any* functions in $L^2(I, \mathcal{B}, \mu)$ and let $\varepsilon > 0$. By Theorem 1, there exist finite linear combinations u and v of the mappings F_n such that

$$(10) \qquad \| f - u\| < \varepsilon^2 \quad \text{and} \quad \| g - v\| < \varepsilon^2 .$$

We write

$$\begin{aligned} C &= \langle f(F_n),\ g\rangle - \tfrac{1}{2}\langle f,\ 1\rangle\langle g,\ 1\rangle \\ &= [\langle f(F_n) - u(F_n),\ g - v\rangle + \langle v,\ f(F_n) - u(F_n)\rangle + \langle u(F_n),\ g - v\rangle] + \\ &\quad \left[\langle u(F_n),\ v\rangle - \tfrac{1}{2}\langle u,\ 1\rangle\langle v,\ 1\rangle\right] + \left[\tfrac{1}{2}\langle u,\ 1\rangle\langle v,\ 1\rangle - \tfrac{1}{2}\langle f,\ 1\rangle\langle g,\ 1\rangle\right] \\ &= [J] + [K] + [L]. \end{aligned}$$

Since F_n is measure perserving,

$$\| f(F_n) - u(F_n)\| = \| f - u\| \text{ and } \| u(F_n)\| = \| u\| .$$

(See, for example, [4, p. 169].) Thus, the Schwarz inequality with (10) shows that $|J| < j\varepsilon$ for some constant $j > 0$. For large enough n, $|K| < \varepsilon$ since the theorem is already proved for u and v. Now

$$L = \tfrac{1}{2}[\langle f - u,\ 1\rangle\langle g - v,\ 1\rangle - \langle g,\ 1\rangle\langle f - u,\ 1\rangle - \langle f,\ 1\rangle\langle g - v,\ 1\rangle],$$

so that $|L| < \ell\varepsilon$ for some constant $\ell > 0$, again by the Schwarz inequality and (10). Thus $|C| < (1 + j + \ell)\varepsilon$ for large enough n, and this proves the theorem.

Is the sequence $\{G_1, G_2, \ldots\}$ strongly mixing, too? This question is presumptuous, since "strongly mixing" has been defined only for measure-preserving (on I) mappings. However, while no single G_n is measure-preserving on all of I, Lemma 2b shows G_n to be measure-preserving on

$$\left[\cos^{-1}\frac{n\pi}{n+1},\ 1\right],$$

and since "strongly mixing" involves $\lim_{n\to\infty}$, we are led to the following definition:

> A sequence of mappings $\{\tau_n\}$, not necessarily measure-preserving on I, is *limit-strongly mixing* if (8) holds for all f and g in $L^2(I, \mathcal{B}, \mu)$.

One may now prove the following two theorems, using Lemma 2b and a modification of the proof of Theorem 2.

Theorem 3: The sequence $\{G_1, G_2, \ldots\}$ is limit-strongly mixing in $L^2(I, \mathcal{B}, \mu)$ with respect to the measure μ.

Theroem 4: The sequence $\{\overline{u}_1, \overline{u}_2, \ldots\}$ is limit-strongly mixing in $L^2(I, \mathcal{B}, \mu)$ with respect to the measure μ.

Finally, we note that the *mapping* F_n, for $n \geq 1$, is *strongly mixing* and, therefore, *ergodic* in the sense given in [4, p. 169]. In the limiting sense of Theorems 3 and 4 above, the same properties hold for the mappings G_n and $\overline{u}_n$ for $n \geq 1$.

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ON THE CONVERGENCE OF ITERATED EXPONENTIATION—I

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We have investigated the properties of the function $f(x) = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$ with an infinite number of x's in the region $0 < x < e^{1/e}$. We have also defined a class of functions $F_n(x)$ which are a generalization of $f(x)$, and which exhibit the property of "dual convergence," i.e., convergence to different values of $F_n(x)$ as $n \to \infty$, depending upon whether n is even or odd.

An elementary exercise is to find a positive x satisfying

$$x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = 2 \tag{1}$$

when an infinite number of exponentiations is understood [1], [2]. The standard solution is to note that the exponent of the first x must be 2, and thus $x = \sqrt{2}$. Indeed, the sequence f_n defined by

$$f_0 = 1$$
$$f_{n+1} = 2^{f_n/2} \tag{2}$$

does converge to 2 as n goes to infinity. Now consider the problem

$$x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = \frac{1}{3}. \tag{3}$$

By analogy, one might assume that

$$x = \left(\frac{1}{3}\right)^3 = \frac{1}{27}$$

is the solution; however, this is too naive because the sequence f_n defined by

$$f_0 = 1$$
$$f_{n+1} = \left(\frac{1}{27}\right)^{f_n} \tag{4}$$

does not converge.

The purpose of this article is to discuss some criteria for convergence of sequences of the form

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