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TSCHEBYSCHEFF AND OTHER FUNCTIONS ASSOCIATED WITH THE SEQUENCE $\{w_n(a,b;\ p,q)\}$

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1. INTRODUCTION

Previously in this journal [5] and [6], I have defined a generalized sequence $\{w_n(a,b;\ p,q)\}$ and established its fundamental general arithmetical properties, as well as certain special properties of it. In this article, the sequence is related to Tschebyscheff functions and to some combinatorial functions used by Riordan [8]. This is the third of a series of articles developing the theory of $\{w_n(a,b;\ p,q)\}$, as envisaged in [5]. Notation and content of [5] and [6] are assumed when the occasion warrants.

For subsequent reference, we reproduce the Lucas results [7]

$$(1.1)\qquad u_n(p,q) = \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{p} p^{n-2k} q^k$$

and

$$(1.2)\qquad v_n(p,q) = \sum_{k=0}^{[n/2]} (-1)^k \frac{n}{n-k} \binom{n-k}{k} p^{n-2k} q^k$$

with reciprocals [3]

$$(1.3)\qquad p_n = \sum_{k=0}^{[n/2]} \left[\binom{n}{k} - \binom{n}{k-1}\right] u_{n-2k}(p,q) q^k$$

and

$$(1.4)\qquad p^n = \sum_{k=0}^{[n/2]} \binom{n}{k} v_{n-2k}(p,q)q^k \quad (v_0(p,q) = 1) ,$$

respectively. Consequently, it follows that $(p = -q = 1)$.

$$(1.5)\qquad f_n = \sum_{k=0}^{[n/2]} \binom{n-k}{k}$$

from (1.1), and

$$(1.6)\qquad l_n = \sum_{k=0}^{[n/2]} \frac{n}{n-k}\binom{n-k}{k}$$

from (1.2), with appropriate reciprocals from (1.3) and (1.4).

Making use of (1.1) above together with the first of the forms given in (2.14) [5], we may express w_n as

$$(1.7)\qquad w_n(a,b,p,q) = a\sum_{k=0}^{\left[\frac{n}{2}\right]}(-1)^k\binom{n-k}{k}p^{n-2k}q^k + (b-pa)\sum_{k=0}^{\left[\frac{n-1}{2}\right]}(-1)^k\binom{n-1-k}{k}p^{n-1-2k}q^k$$

2. TSCHEBYSCHEFF FUNCTIONS

Write

$$(2.1)\qquad x = \cos\theta$$

$$(2.2)\qquad p = 2x ,\quad q = 1$$

so that

$$(2.3)\qquad d = 2i\sin\theta \quad (i = \sqrt{-1}) .$$

Define

$$(2.4)\qquad w_n = w_n(a,\ 2x;\ 2x,\ 1) = a\cos n\theta + (2 - a)\sin n\theta\cot\theta\ .$$

Using Simpson's formulae (reference Lucas [7]),

$$(2.5)\qquad \begin{cases} \sin(n+2)\theta = 2\cos\theta\ \sin(n+1)\theta - \sin n\theta \\ \cos(n+2)\theta = 2\cos\theta\ \cos(n+1)\theta - \cos n\theta \end{cases},$$

we deduce that

$$(2.6)\qquad w_{n+2} = pw_{n+1} - w_n\ ,$$

as required by the definition of $w_n(a,b;\ p,q)$ given in [5], in conjunction with (2.1) and (2.2). Notice that (2.1) and (2.2) ensure [5] that

$$(2.7)\qquad e = 4(a-1)\cos^2\theta - a^2\ ,$$

whence, for $\{u_n\}$, for which $a = 1$,

$$(2.8)\qquad e = -1\ ,$$

while for $\{v_n\}$, for which $a = 2$,

$$(2.9)\qquad e = -4\sin^2\theta\ .$$

Immediately from (2.4) we have the Lucas substitutions [7]

$$(2.10)\qquad u_n(2x,1) = \frac{\sin(n+1)\theta}{\sin\theta} = \sum_{k=0}^{[n/2]}(-1)^k\binom{n-k}{k}(2x)^{n-2k}$$

and

$$v_n(2x,1) = 2\cos n\theta = \sum_{k=0}^{[n/2]} (-1)^k \frac{n}{n-k}\binom{n-k}{k}(2x)^{n-2k} \tag{2.11}$$

with reciprocals

$$(2x)^n = \sum_{k=0}^{[n/2]} \left[\binom{n}{k} - \binom{n}{k-1}\right] u_{n-2k}(2x,1) \tag{2.12}$$

and

$$(2x)^n = \sum_{k=0}^{[n/2]} \binom{n}{k} v_{n-2k}(2x,1)\ , \tag{2.13}$$

where we have used (1.1)-(1.4).

But, the expressions in (2.10) and (2.11) exactly describe the Tscheby-scheff functions $U_n(x)$ and $2T_n(x) = t_n(x)$ respectively ($T_0 = \frac{1}{2}t_0 = 1$). That is,

$$w_n(1,2x;\,2x,1) = u_n(2x,1) = U_n(x) = 2xU_{n-1}(x) - U_{n-1}(x) \tag{2.14}$$

and

$$w_n(2,2x;\,2x,1) = v_n(2x,1) = 2T_n(x) = 2(xU_{n-1}(x) - U_{n-2}(x))\ . \tag{2.15}$$

Special cases are

$$w_n(1,1;\,1,1) = u_n(1,1) = U_n(\tfrac{1}{2}) = U_{n-1}(\tfrac{1}{2}) - U_{n-2}(\tfrac{1}{2}) \tag{2.16}$$

and

$$w_n(2,1;\,1,1) = v_n(1,1) = 2T_n(\tfrac{1}{2}) = U_{n-1}(\tfrac{1}{2}) - 2U_{n-2}(\tfrac{1}{2})\ . \tag{2.17}$$

Generally,

$$w_n(a,b;\ 2x,1) = bU_{n-1}(x) - aU_{n-2}(x) \quad . \tag{2.18}$$

By means of the w_n-notation, relationships among Tschebyscheff polynomials may be conveniently expressed. Recalling the known result [8], for instance, that

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x) \tag{2.19}$$

we may, writing for brevity,

$$w_n = w_n(2, 2x;\ 2x,\ 1) \quad , \tag{2.20}$$

express it in the form

$$\omega_n - 2x\omega_{n-1} - \omega_{n-2} \cdot \tag{2.21}$$

Equations (2.4), (2.10) and (2.11) enable us to express every formula in the theory of our second-order recurrences as a corresponding formula involving trigonometrical functions. [Observe that $q = 1$ invalidates any specialized application to the sequences $\{h_n\}$, $\{f_n\}$ and $\{l_n\}$, for all of which $q = -1$.]

Corresponding to the fundamental formula $w_{n+r}w_{n-r} - w_n^2 = eq^{n-r}u_{r-1}^2$ ((4.5) in [5]), for instance, we have

$$\begin{aligned} &a^2\{\cos(n+r)\theta\ \cos(n-r)\theta - \cos^2 n\theta\} \\ &+ (2-a)^2\cot^2\theta\ \{\sin(n+r)\theta\ \sin(n-r)\theta - \sin^2 n\theta\} = e\,\frac{\sin^2 r\theta}{\sin^2\theta} \end{aligned} \tag{2.22}$$

where e is given by (2.7). For $\{u_n\}$ and $\{v_n\}$, we obtain

$$\sin(n+r+1)\theta\ \sin(n-r+1)\theta - \sin^2(n+1)\theta = -\sin^2 r\theta \tag{2.23}$$

and

$$\cos(n+r)\theta\ \cos(n-r)\theta - \cos^2 n\theta = -\sin^2 r\theta \quad , \tag{2.24}$$

in which e is given by (2.8) and (2.9), respectively. Both results (2.23) and (2.24) are easy to verify. The particular result $w_n^2 + eu_{n-1}^2 = aw_{2n}$ ((4.6) [5]) derived by setting $r = m$ implies the identity

$$\cos 2n\theta - \cos^2 n\theta = -\sin^2 n\theta$$

in (2.24).

Other trigonometrical identities are not hard to detect, but it is interesting to discover just how they are disguised. As further examples, we note that

$$pw_{n+2} - (p^2 - q)w_{n+1} + q^2 w_{n-1} = 0$$

((3.3) [5]), and

$$\frac{w_{n+r} + q^r w_{n-r}}{w_n} = v_r$$

((3.16) [5]) lead to, respectively,

$$\left\{\begin{array}{ll} 2\cos\theta\ \sin(n+3)\theta - (4\cos^2\theta - 1)\sin(n+2)\theta + \sin n\theta = 0 & (2.25) \\ 2\cos\theta\ \cos(n+2)\theta - (4\cos^2\theta - 1)\cos(n+1)\theta + \cos(n-1)\theta = 0 & (2.26) \end{array}\right.$$

and

$$\left\{\begin{array}{ll} \dfrac{\sin(n+r+1)\theta + \sin(n-r+1)\theta}{\sin(n+1)\theta} = 2\cos r\theta & (2.27) \\[2ex] \dfrac{\cos(n+r)\theta + \cos(n-r)\theta}{\cos n\theta} = 2\cos r\theta & (2.28) \end{array}\right.$$

where, in each pair of identities, the first refers to $\{u_n\}$ and the second to $\{v_n\}$. A formula also worth investigation is

$$aw_{m+n} + (b - pa)w_{m+n-1} = w_m w_n - qw_{m-1} w_{n-1}$$

((4.1) [5]). Furthermore, the summation formula (3.4) [5] indicates expressions for

$$\sum_{k=0}^{n-1} \cos k\theta$$

and

$$\sum_{k=0}^{n-1} \sin (k+1)\theta \quad .$$

Similar remarks apply to the formulae for sums of squares and cubes.

Instead of (2.1)-(2.3), we may put

$$y = \cosh\phi = \cos i\phi \tag{2.29}$$

$$p = 2y, \quad q = 1 \tag{2.30}$$

so that

$$d = 2\sinh\phi = -2i \sin i\phi \tag{2.31}$$

and hence derive a set of parallel results for hyperbolic functions.

Apart from the Carlitz [3] reference quoted earlier, other sources of information regarding the relationships among Tschebyscheff polynomials and Fibonacci-type sequences are, say, Buschman [1] and Gould [4].

3. COMBINATORIAL FUNCTIONS

From (1.1), we have, using the combinatorial function $L_n(x)$ used in Riordan [8], (3.1) $$u_n(1,-x) = \sum_{k=0}^{[n/2]} \binom{n-k}{k} x^k = L_{n-1}(x) .$$

Then, by the second half of the expression (2.14) [5],

(3.2) $$w_n(a,b;1,-x) = b\sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n-1-k}{k} x^k + ax\sum_{k=0}^{\left[\frac{n-2}{2}\right]} \binom{n-2-k}{k} x^k$$
$$= bL_{n-2}(x) + axL_{n-3}(x)$$
$$= b\left\{\frac{(1+g)^n - (1-g)^n}{2^n g^2}\right\} + ax\left\{\frac{(1+g)^{n-1} - (1-g)^{n-1}}{2^{n-1} g^2}\right\}$$

where, for brevity, $g = (1+4x)^{\frac{1}{2}}$.

More particularly, notice that

(3.3) $$w_n(1,1;1,-x) = u_n(1,-x)$$

affords an alternative expression for the known recurrence relation [8].

(3.4) $$L_{n-1}(x) = L_{n-2}(x) + xL_{n-3}(x) \qquad [L_0 = 1, L_1 = 1+x]$$

while

(3.5) $$w_{2n}(2,1;1,-x) = v_{2n}(1,-x)$$

is an alternative expression for the combinatorial function [8]

(3.6) $$M_n(x) = L_{2n-1}(x) + xL_{2n-3}(x) \qquad (n > 1) .$$

Of course,

(3.7) $$L_{n-1}(1) = f_n$$

(3.8) $$M_n(1) = l_{2n} .$$

4. OTHER FUNCTIONS

Besides these combinatorial functions and Tschebyscheff functions (themselves involving trigonometrical and hyperbolic functions), other functions are related to the Fibonacci-type recurrences. In this respect, a recent article by Byrd [2] is worth emphasizing, particularly as, it seems, his work offers possibilities for generalization. In this article, Byrd considers the expansion of analytical functions in a certain set of polynomials which can be associated with Fibonacci numbers. Bessel functions and modified Bessel functions are involved in the process.

Throughout, we have assumed that $p^2 \neq 4q$. The degenerate case $p^2 = 4q$ has been discussed by Carlitz [3], who relates it to the Eulerian polynomial, and, briefly, by the author [5].

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