Graphs and Combinatorics 2012 (vol.28,5), p.p. 597-614

EMBEDDING DISTRIBUTIONS AND CHEBYSHEV POLYNOMIALS

Yichao Chen†[1], Toufik Mansour‡, Qian Zou†

†College of mathematics and econometrics, Hunan University, 410082 Changsha, China

‡Department of Mathematics, University of Haifa, 31905 Haifa, Israel

ycchen@hnu.edu.cn, toufik@math.haifa.ac.il, Joe_king520@qq.com

ABSTRACT

The history of genus distributions began with J. Gross et al. in 1980s. Since then, a lot of study has given to this parameter, and the explicit formulas are obtained for various kinds of graphs. In this paper, we find a new usage of Chebyshev polynomials in the study of genus distribution, using the overlap matrix, we obtain homogeneous recurrence relation for rank distribution polynomial, which can be solved in terms of Chebyshev polynomials of the second kind. The method here can find explicit formula for embedding distribution of some other graphs. As an application, the well known genus distributions of closed-end ladders and cobblestone paths [J. Combin. Ser. B 46 (1989) 22] are derived. The explicit formula for non-orientable embedding distributions of closed-end ladders and cobblestone paths are also obtained.

Key words: Overlap matrix; embedding distribution; Chebyshev polynomials; closed-end ladders; cobblestone path.
2000 Mathematics Subject Classification: 05C10, 30B70, 42C05

1. Introduction

1.1. **Background.** One enumerative branch of topological graph theory is to count genus distributions of a graph. The history of genus distribution began with J. Gross in 1980s. Since then, it has been attracted a lot of attentions. Calculating genus distributions is NP-hard. Up to now, there are three principal approaches which characterize this variable:

(i). The topological approach was developed by Gross which consists in applying to genus distribution a topological operation interpreted in terms of Ringle-White adding edge lemma and face-trace algorithm [6, 10]. The articles based on this kind of approach can be found in the literature [8, 7, 21, 9, 15, 16, 33] etc.

(ii). The combinatorial approach is more recent and developed by Liu [35]. Since a genus embedding corresponding to a rotation system, the main idea of Liu is transfer the rotation system to a linear order of letters. The articles based on this kind of approach can be found in the literature [11, 39, 36, 37] etc.

(iii). The algebraic approach was developed by Jackson [13], Stahl [30] and Mohar [23]. The articles based on this approach can be found in the literature [2, 3, 9, 12, 29, 30, 31, 32]. Permutation-group algebra is a key to calculating the distribution of graphs with high symmetry. Gross, Robbins and Tucker established the equation for the bouquets B_n, $g_h(B_n) = (n-1)!2^{n-1}e_{n-(2h+1)}(n)$, where the quantity $e_k(n)$ is the cardinality of the set of permutations $\pi \in \Sigma_{2l}$, corresponding to an arbitrary fixed cycle ζ of length $2l$ for which there is a full involution β such that $\pi = \zeta \circ \beta$ and such that π has k cycles. The value of $e_k(n)$ is given by

[1]The work was partially supported by NNSFC under Grant No. 10901048

a formula of Jackson [12]. Similarly, Rieper [27] (J. H. Kwak and J. Lee [19] independently) did it for genus distribution of dipoles. Stahl [30, 32] introduced the concept of permutation-partition pair (P, Π) and proved the walkup reduction process that expresses the genera of the embeddings of a pair (P, Π) in terms of those of smaller pairs. He used it to find the genus distributions of H-linear family graphs. Given a general rotation system for a graph and M be its overlap matrix. Mohar [23] shown that the rank of M equals twice the genus, if the corresponding embedding surface is orientable, and it equals the crosscap number otherwise. In [2], Chen, Gross and Rieper firstly used the overlap matrix to calculate the total embedding distribution of necklaces of type $(r, 0)$, closed-end ladders and cobblestone paths. In [3], Chen, Liu and Wang did it for graphs of maximum genus 1. Furthermore, in [4], Chen, Ou and Zou obtained explicit formula for total embedding distributions of Ringel ladders.

1.2. **Total embedding-distribution polynomial.** It is assumed that the reader is somewhat familiar with the basics of topological graph theory as found in Gross and Tucker [10]. A graph $G = (V(G), E(G))$ is permitted to have both loops and multiple edges. A *surface* is a compact closed 2-dimensional manifold without boundary. In topology, surfaces are classified into O_m, the *orientable surface* with $m(m \geq 0)$ handles and N_n, the *nonorientable surface* with $n(n > 0)$ crosscaps. A graph embedding into a surface means a *cellular embedding*.
A *spanning tree* of a graph G is a tree on its edges has the same order as G. The number cotree edges of a spanning tree of G is called the *Betti number*, $\beta(G)$, of G. A *rotation* at a vertex v of a graph G is a cyclic order of all edges incident with v. A *pure rotation system* P of a graph G is the collection of rotations at all vertices of G. A *general rotation system* is a pair (P, λ), where P is a pure rotation system and λ is a mapping $E(G) \to \{0, 1\}$. The edge e is said to be *twisted* (respectively, *untwisted*) if $\lambda(e) = 1$ (respectively, $\lambda(e) = 0$). It is well known that every orientable embedding of a graph G can be described by a general rotation system (P, λ) with $\lambda(e) = 0$ for all $e \in E(G)$. By allowing λ to take the non-zero value, we can describe nonorientable embeddings of G, see [2, 28] for more details. A *T-rotation system* (P, λ) of G is a general rotation system (P, λ) such that $\lambda(e) = 0$, for all $e \in E(T)$.

Theorem 1.1. *(see [2, 28]) Let T be a spanning tree of G and (P, λ) be a general rotation system. Then there exists a general rotation system (P', λ') such that*

(1) *(P', λ') yields the same embedding of G as (P, λ), and*
(2) *$\lambda'(e) = 0$, for all $e \in E(T)$.*

Two embeddings are considered to be the *same* if their T-rotation systems are combinatorially equivalent. Fix a spanning tree T of a graph G. Let Φ_G^T be the set of all T-rotation systems of G. It is known that

$$|\Phi_G^T| = 2^{\beta(G)} \prod_{v \in V(G)} (d_v - 1)!.$$

Suppose that in these $|\Phi_G^T|$ embeddings of G, there are a_i, $i = 0, 1, \ldots$, embeddings into orientable surface O_i and b_j, $j = 1, 2, \ldots$, embeddings into nonorientable surface N_j. We call the polynomial

$$I_G^T(x, y) = \sum_{i=0}^{\infty} a_i x^i + \sum_{j=1}^{\infty} b_j y^j$$

the *T-distribution polynomial* of G. By the *total embedding-distribution polynomial* of G, we shall mean the polynomial

$$I_G(x, y) = I_G^T(x, y).$$

We call the first (respectively, second) part of $I_G(x, y)$ the *genus polynomial* (respectively, *crosscap number polynomial*) of G and denoted by $g_G(x) = \sum_{i=0}^{\infty} a_i x^i$ (respectively, $f_G(y) = \sum_{i=1}^{\infty} b_i y^i$).

Clearly, $I_G(x,y) = g_G(x)+f_G(y)$. This means the number of orientable embeddings of G is $\prod_{v\in G}(d_v-1)!$, while the number of non-orientable embeddings of G is $(2^{\beta(G)}-1)\prod_{v\in G}(d_v-1)!$.

1.3. **Mohar's theorem.** Let T be a spanning tree of G and $(P^{'},\lambda^{'})$ be a T-rotation system. Let $e_1, e_2, \ldots, e_{\beta(G)}$ be the cotree edges of T. The *overlap matrix* of $(P^{'},\lambda^{'})$ is the $\beta\times\beta$ matrix $M=[m_{ij}]$ over $GF(2)$ such that $m_{ij}=1$ if and only if either $i\neq j$ and the restriction of the underlying pure rotation system to $T+e_i+e_j$ is nonplanar, or $i=j$ and e_i is twisted.

Theorem 1.2. *(see* [23]*) Let (P,λ) be a general rotation system for a graph, and let M be the overlap matrix. Then the rank of M equals twice the genus, if the corresponding embedding surface is orientable, and it equals the crosscap number otherwise. It is independent of the choice of a spanning tree.*

1.4. **Homogeneous recurrence relation and Chebyshev polynomials.** To begin with the discussion, we give some concepts of the n-th Chebyshev polynomials of the second kind which is related to the solution of the recurrence relation. Let the recurrence function $U_n(x)$ be

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x)$$

with the initial conditions $U_0(x)=1$, $U_1(x)=2x$, then we derived the *n-th Chebyshev polynomials of the second kind* $U_n(x)$ (see [26]). For instance, $U_2(x)=4x^2-1$, $U_3(x)=8x^3-4x$, $U_4(x)=16x^4-12x^2+1$. Moreover, we have the identity that

$$U_n(x)=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n-k}{k}(-1)^k(2x)^{n-2k}. \tag{1}$$

Now, we will build the relation between the recurrence relation and the Chebyshev polynomials of the second kind. Let $P_n(z)=\sum_{m=0}^{n}C_n(m)z^m$, satisfies

$$P_n(z)=a_1(z)P_{n-1}(z)+a_2(z)P_{n-2}(z),$$

where $a_i(z)=\sum_{k=0}^{q}a_{i,k}z^k$ for $i=1,2$ and the initial condition $P_0(z)=c_0$. Note that $P_1(z)$ and $P_2(z)$ can be derived by the initial values of $C_n(m)$.

Let $Q_n(z)=\frac{P_n(z)}{(\sqrt{a_2(z)}i)^n}$, then it is easy to verify that

$$Q_n(z)=\frac{a_1(z)}{\sqrt{a_2(z)}i}Q_{n-1}(z)-Q_{n-2}(z)$$

with the initial conditions $Q_0(z)=P_0(z)$, $Q_1(z)=\frac{P_1(z)}{\sqrt{a_2(z)}i}$ and $Q_2(z)=\frac{P_2(z)}{-a_2(z)}$. Thus by induction on n, we obtain that

$$Q_n(z)=AU_n\left(\frac{a_1(z)}{2\sqrt{a_2(z)}i}\right)+BU_{n-1}\left(\frac{a_1(z)}{2\sqrt{a_2(z)}i}\right)+CU_{n-2}\left(\frac{a_1(z)}{2\sqrt{a_2(z)}i}\right), \tag{2}$$

where A, B and C are determined by the initial conditions.

Thus we have

$$P_n(z)=(\sqrt{a_2(z)}i)^n\left[AU_n\left(\frac{a_1(z)}{2\sqrt{a_2(z)}i}\right)+BU_{n-1}\left(\frac{a_1(z)}{2\sqrt{a_2(z)}i}\right)+CU_{n-2}\left(\frac{a_1(z)}{2\sqrt{a_2(z)}i}\right)\right]. \tag{3}$$

Using the fact that

$$U_n(x)=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n-k}{k}(-1)^k(2x)^{n-2k},$$

we can derive that

$$(i\sqrt{a_2(z)})^n U_n\left(\frac{a_1(z)}{2\sqrt{a_2(z)}i}\right) = \sum_{j\geq 0}\binom{n-j}{j}(a_1(z))^{n-2j}.$$

Since $a_1(z)$ is a polynomial of degree less than q then $(a_1(z))^{n-2j}$ can be expressed as the type of power series. Plugging the above formula into (3) and comparing the coefficient z^m in both sides, we obtain an explicit formula for $C_n(m)$ for $0 \leq m \leq n$.

2. Total embedding distributions of closed-end ladders

An *n-rung closed-end ladder* L_n can be obtained by taking the graphical cartesian product of an *n-vertex* path with the complete graph K_2, and then doubling both its end edges. Figure 1 presents a 4-rung closed-end ladder.

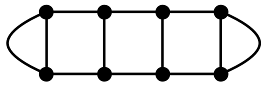

FIGURE 1. The 4-rung closed-end ladder L_4

2.1. **The overlap matrix of L_n.** We adopt the same notation used by Ringel [27, p.17]. A cubic graph at each vertex has two cyclic orderings of its neighbors. One of these two cyclic orderings is denoted as clockwise and the other *counterclockwise*. We color the vertex *black*, if that vertex has the *clockwise* ordering of its neighbors, otherwise, we will color the counterclockwise vertices *white*. This will bring convenient to embed a cubic graph into surfaces, as we can draw an imbedding on the plane and only need to color the vertices black and white.

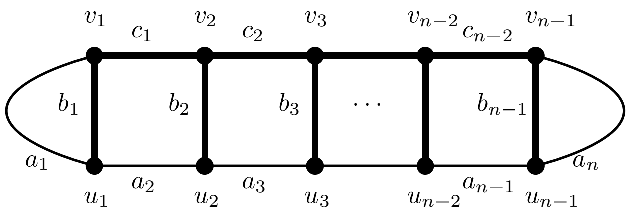

FIGURE 2. The graph L_n

Definition 2.1. *An edge is called matched if it has the same color at both ends, otherwise it is called unmatched.*

We fix a spanning tree T of L_{n-1} shown as the thicker lines in Figure 2, that is, the cotree edges are $a_1, a_2, \ldots, a_n$. Then the overlap matrix M_n of L_{n-1} can be written as

$$M_n = M_n^{X,Y} = \begin{pmatrix} x_1 & y_1 & & & & & \\ y_1 & x_2 & y_2 & & & \mathbf{0} & \\ & y_2 & x_3 & y_3 & & & \\ & & & & & & \\ & \mathbf{0} & & & y_{n-2} & x_{n-1} & y_{n-1} \\ & & & & & y_{n-1} & x_n \end{pmatrix},$$

where $X = (x_1, x_2, \ldots, x_n) \in (GF(2))^n$ and $Y = (y_1, y_2, \ldots, y_{n-1}) \in (GF(2))^{n-1}$.

Each variable y_j corresponds to a unique vertex of the L_{n-1} and has value 1 if and only if the two corresponding cotree edges overlap. Each variable x_j corresponds to one unique cotree edge of L_{n-1} and has value 1 if and only if the edge is twisted.

Property 2.2. *Two cotree edges a_i and a_{i+1}, $i = 1, 2, \cdots, n-1$, overlap if and only if the edge b_i is unmatched.*

Property 2.3. *For a fixed matrix of the form $M_n^{X,Y}$, there are exactly 2^{n-1} different T-rotation systems corresponding to that matrix.*

Proof. Note that there are four different assignments of colors to the edge b_i, two of them are matched and two of them are unmatched. Since the edges $b_1, b_2, \cdots, b_{n-1}$ are independent, the property follows. □

2.2. **The rank-distribution polynomial.** We define

$$\mathscr{A}_n = \{M_n^{X,Y} \mid X \in (GF(2))^n \text{ and } Y \in (GF(2))^{n-1}\},$$

which is the set of all matrices over $GF(2)$ that are of the type $M_n^{X,Y}$. We define the *rank-distribution polynomial* to be the polynomial $P_n(z) = \sum_{j=0}^{n} C_n(j)z^j$, where $C_n(j)$, $j = 0, 1, \ldots, n$, is the number of different assignment of the variables x_j, y_k, where $j \in [n]$ and $k \in [n-1]$, for which the matrix $M_n^{X,Y}$ in $\mathscr{A}_n$ has rank j. Similarly, let

$$\mathscr{O}_n = \{M_n^{0,Y} \mid Y \in (GF(2))^{n-1}\},$$

and $O_n(z) = \sum_{j=0}^{n} O_n(j)z^j$ be the *rank-distribution polynomial* of $\mathscr{O}_n$,where $O_n(j)$, $j = 0, 1, \ldots, n$, is the number of different assignment of the variables y_k, where $k \in \{1, 2, \ldots, n-1\}$, for which the matrix M_n^Y in $\mathscr{A}_n$ has rank j.

Proposition 2.4. *The polynomial $O_n(z)$ satisfies the recurrence relation*

$$O_n(z) = O_{n-1}(z) + 2z^2 O_{n-2}(z)$$

with the initial conditions $O_0(z) = O_1(z) = 1$ and $O_2(z) = z^2 + 1$.

Proof. We consider the matrix $M_n^{0,Y}$ and let us give a recurrence relation for $O_n(z)$.
Case 1. For $y_1 = 0$. It is obvious to obtain $rank(M_n^{0,Y}) = rank(M_{n-1}^{0,Y})$, so it contributes a term $O_{n-1}(z)$.
Case 2. For $y_1 = 1$. If $y_2 = 0$, then $rank(M_n^{0,Y}) = 2 + rank(M_{n-2}^{0,Y})$. Otherwise $y_2 = 1$, we add the first row and first column to the third row and third column respectively, and the rank of $M_n^{0,Y}$ is equal to 2 plus the rank of the lower-right matrix with order $n-2$, which has the form of $M_{n-2}^{0,Y}$, that is, $rank(M_n^{0,Y}) = 2 + rank(M_{n-2}^{0,Y})$. In total, it contributes $2z^2 O_{n-2}(z)$.
Hence, the polynomials $O_n(z)$ satisfy the recurrence relation $O_n(z) = O_{n-1}(z) + 2z^2 O_{n-2}(z)$ for all $n \geq 3$. By studying the rank distributions of $M_j^{0,Y}$ for $j = 0, 1, 2$, we derive the initial conditions of the above recurrence, namely, $O_0(z) = 1$, $O_1(z) = 1$ and $O_2(z) = z^2 + 1$, as claimed. □

Theorem 2.5. *For all $n \geq 1$,*

$$O_n(z) = (\sqrt{2}iz)^n Q_n(z) = \sum_{j\geq 0} \binom{n-j}{j} 2^j\ z^{2j} - \sum_{j\geq 0} \binom{n-2-j}{j} 2^j\ z^{2j+2}.$$

Proof. Proposition 2.4 gives that the polynomial $Q_n(z) = \frac{O_n(z)}{(\sqrt{2}iz)^n}$ satisfies the recurrence relation

$$Q_n(z) = U_n\left(\frac{1}{2\sqrt{2}iz}\right) + \frac{1}{2} U_{n-2}\left(\frac{1}{2\sqrt{2}iz}\right).$$

By (1), we have

$$O_n(z) = (\sqrt{2}iz)^n Q_n(z) = \sum_{j\geq 0}\binom{n-j}{j}2^j\, z^{2j} - \sum_{j\geq 0}\binom{n-2-j}{j}2^j\, z^{2j+2},$$

which completes the proof. □

Recall that $O_n(z) = \sum_{m=0}^{n} O_n(m)z^m$, the next corollary gives an explicit formula for $O_n(m)$. Comparing the coefficient of z^m in the statement of Theorem 2.5, we obtain the following result.

Corollary 2.6. *For all $m \leq [\frac{n}{2}]$,*

$$O_n(2m+1) = 0,$$
$$O_n(2m) = 2^m\binom{n-m}{m} - 2^{m-1}\binom{n-m-1}{m-1}.$$

Proposition 2.7. *The polynomial $P_n(z)$ satisfies the recurrence relation*

$$P_n(z) = (1+2z)P_{n-1}(z) + 4z^2P_{n-2}(z)$$

with the initial conditions $P_0(z) = 1$, $P_1(z) = 1+z$ and $P_2(z) = 4z^2+3z+1$.

Proof. We consider the matrix $M_n^{X,Y}$ and let us write a recurrence relation for $P_n(z)$.
Case 1. For $x_1 = 0, y_1 = 0$. It is obvious to obtain $rank(M_n^{X,Y}) = rank(M_{n-1}^{X,Y})$, so it contributes $P_{n-1}(z)$.
Case 2. For $x_1 = 0, y_1 = 1$. We interchange the first and the second row, then the rank of $M_n^{X,Y}$ is equal to 2 plus the rank of the lower-right matrix with order $n-2$, which has the form of $M_{n-2}^{X,Y}$, that is, $rank(M_n^{X,Y}) = 2 + rank(M_{n-2}^{X,Y})$. Since the rank distribution is independent of the choice of x_2, y_2 and it has four different choices, so it contributes $4z^2P_{n-2}(z)$.
Case 3. For $x_1 = 1, y_1 = 0$. It is easy to check that $rank(M_n^{X,Y}) = rank(M_{n-1}^{X,Y})+1$, so it contributes $zP_{n-1}(z)$.
Case 4. For $x_1 = 1, y_1 = 1$. We firstly add the first row to the second row, with the same discussion, we have that $rank(M_n^{X,Y}) = rank(M_{n-1}^{X,Y}) + 1$, so it contributes $zP_{n-1}(z)$.
Hence, the polynomials $P_n(z)$ satisfy the recurrence relation $P_n(z) = (1+2z)P_{n-1}(z)+4z^2P_{n-2}(z)$ for all $n \geq 3$. By studying the rank distributions of $M_j^{X,Y}$ for $j = 0, 1, 2$, we derive the initial conditions of the above recurrence, namely, $P_0(z) = 1$, $P_1(z) = 1+z$ and $P_2(z) = 4z^2+3z+1$, as claimed. □

Theorem 2.8. *For all $n \geq 1$,*

$$P_n(z) = (2iz)^n\left[U_n\left(\frac{1+2z}{4iz}\right) + \frac{i}{2}U_{n-1}\left(\frac{1+2z}{4iz}\right) - \frac{1}{2}U_{n-2}\left(\frac{1+2z}{4iz}\right)\right],$$

where $U_s(t)$ is the s-th Chebyshev poynomial of the second kind and $i^2 = -1$.

Proof. Proposition 2.7 gives that the polynomial $Q_n(z) = \frac{P_n(z)}{(2iz)^n}$ satisfies the recurrence relation

$$Q_n(z) = \frac{1+2z}{2iz}Q_{n-1}(z) - Q_{n-2}(z)$$

with the initial conditions $Q_1(z) = \frac{1+z}{2iz}$ and $Q_2(z) = -\frac{1+3z+4z^2}{4z^2}$. By Section 1.4 and induction on n, we obtain that

$$Q_n(z) = U_n\left(\frac{1+2z}{4iz}\right) + \frac{i}{2}U_{n-1}\left(\frac{1+2z}{4iz}\right) - \frac{1}{2}U_{n-2}\left(\frac{1+2z}{4iz}\right),$$

which completes the proof. □

Recall that $P_n(z) = \sum_{m=0}^{n} C_n(m)z^m$, the next corollary gives an explicit formula for $C_n(m)$.

Corollary 2.9. *For all $n \geq 1$ and $0 \leq m \leq n$,*

$$C_n(m) = 2^m \sum_{j=0}^{[m/2]} \binom{n-j}{j}\binom{n-2j}{n-m} - 2^{m-1} \sum_{j=0}^{[(m-1)/2]} \binom{n-1-j}{j}\binom{n-1-2j}{n-m} + 2^{m-1} \sum_{j=0}^{[(m-2)/2]} \binom{n-2-j}{j}\binom{n-2-2j}{n-m}.$$

Proof. By (1) we obtain that

$$(2iz)^n U_n\left(\frac{1+2z}{4iz}\right) = \sum_{j\geq 0} \binom{n-j}{j} (2z)^{2j}(1+2z)^{n-2j}.$$

Thus, by Theorem 2.8 we have that

$$P_n(z) = \sum_{j\geq 0} \binom{n-j}{j}(2z)^{2j}(1+2z)^{n-2j} - z\sum_{j\geq 0}\binom{n-1-j}{j}(2z)^{2j}(1+2z)^{n-1-2j} + 2z^2\sum_{j\geq 0}\binom{n-2-j}{j}(2z)^{2j}(1+2z)^{n-2-2j}.$$

Comparing the coefficient of z^m in both sides of the above equation, we obtain the desired result. □

For instance, $C_n(n) = 2^{n-1}F_{n+1}$, where F_s is the s-th Fibonacci number (F_s is defined by the recurrence relation $F_s = F_{s-1} + F_{s-2}$ with the initial conditions $F_0 = 0$ and $F_1 = 1$).

2.3. The total embedding-distribution polynomial of L_n.

Theorem 2.10. *(see [15]) The number of embeddings of closed-end ladders L_{n-1} into orientable surface S of genus i is*

$$g_i(L_{n-1}) = \begin{cases} 2^{n-2+i}\binom{n-i}{i}\frac{2n-3i}{n-i}, & \text{when } i \leq [\frac{n}{2}] \\ 0 & \text{otherwise} \end{cases}$$

Proof. Let the genus polynomial of L_{n-1} be $g_{L_{n-1}}(z) = \sum_{i\geq 0} g_i(L_{n-1})z^i$, By Theorem 2.8 and Corollary 2.6, we have

$$\begin{aligned} G_{L_{n-1}}(z) &= 2^{n-1}T_n(z) \\ &= 2^{n-1}\left\{\sum_{j\geq 0}\binom{n-j}{j}2^j\, z^{2j} - \sum_{j\geq 0}\binom{n-2-j}{j}2^j\, z^{2j+2}\right\} \end{aligned} \tag{4}$$

Note that $g_j(L_{n-1})$ is equal to the coefficients of z^{2j}. By (4), we have

$$g_j(L_{n-1}) = 2^{n-1}\cdot\left\{\binom{n-j}{j}2^j - \binom{n-j-1}{j-1}2^{j-1}\right\}$$

By the Newton's identity $\binom{n-m}{m} = \frac{n-m}{m}\binom{n-m-1}{m-1}$, the formula of Theorem 2.10 is the same as that of [15]. □

By the above discussion, the following theorem follows.

Theorem 2.11. *The total embedding-distribution polynomial of an* $(n-1)$*-rung closed-end ladder* L_{n-1} *is given by*

$$\begin{aligned}\mathbb{I}_{L_{n-1}}(x,y) &= 2^{n-1}\ P_n(y) - \mathbb{I}_0(L_{n-1},\ y^2) + \mathbb{I}_0(L_{n-1},\ x)\\ &= 2^{n-1}\sum_{j\geq 0}\binom{n-j}{j}(2z)^{2j}(1+2z)^{n-2j} - z\sum_{j\geq 0}\binom{n-1-j}{j}(2z)^{2j}(1+2z)^{n-1-2j}\\ &+ 2z^2\sum_{j\geq 0}\binom{n-2-j}{j}(2z)^{2j}(1+2z)^{n-2-2j} - \mathbb{I}_0(L_{n-1},\ y^2) + \mathbb{I}_0(L_{n-1},\ x).\end{aligned}$$

where $\mathbb{I}_0(L_{n-1}, x)$ *is the genus-distribution polynomial of the closed-end ladder* L_{n-1}*, which has been derived by Furst et al* [15]*, that is,*

$$\mathbb{I}_0(L_{n-1}, x) = \sum_{j=0}^{[\frac{n}{2}]} 2^{n-2+j}\binom{n-j}{j}\frac{2n-3j}{n-j}x^j,$$

and

$$\mathbb{I}_0(L_{n-1}, y^2) = \sum_{j=0}^{[\frac{n}{2}]} 2^{n-2+j}\binom{n-j}{j}\frac{2n-3j}{n-j}y^{2j}.$$

Furthermore, by Corollary 2.9, we can obtain the following result.

Corollary 2.12. *Let* $N_{L_{n-1}}(k)$ *be the number of non-orientable embeddings on the surface with genus* j *for the* $(n-1)$*-rung closed-end ladder* L_{n-1}*, then we have*

$$\begin{aligned}N_{L_{n-1}}(1) &= (2n-1)2^{n-1};\\ N_{L_{n-1}}(2) &= 2^{n-1}(2n^2-2n-1);\\ N_{L_{n-1}}(3) &= 2^{n-1}\left(\frac{2n(n+1)(2n+1)}{3} - 24n + 28\right);\\ N_{L_{n-1}}(2p+1) &= 2^{n-1}C_n(2p+1),\quad \textit{for } 0\leq p\leq [\frac{n-1}{2}];\\ N_{L_{n-1}}(2p) &= 2^{n-1}C_n(2p) - 2^{n-2+p}\binom{n-p}{p}\frac{2n-3p}{n-p},\quad \textit{for } 0\leq p\leq [\frac{n}{2}],\end{aligned}$$

where $N_{L_{n-1}}(1)$ *is the number of embeddings on projective plane,* $N_{L_{n-1}}(2)$ *is the number of embeddings on klein bottle, etc.*

The following shows the non-orientable embeddings distribution for small values of n and j.

$$\begin{aligned}&P_{L_1}(y) = 6y^2 + 6y,\\ &P_{L_2}(y) = 48y^3 + 44y^2 + 20y,\\ &P_{L_3}(y) = 304y^4 + 416y^3 + 184y^2 + 56y,\\ &P_{L_4}(y) = 2048y^5 + 3072y^4 + 2048y^3 + 624y^2 + 144y,\\ &P_{L_5}(y) = 13184y^6 + 23552y^5 + 17600y^4 + 13760y^3 + 1888y^2 + 352y.\end{aligned}$$

3. Total embedding distributions of cobblestone paths

A complete similar analysis to what have given for closed-end ladders, we fix a spanning tree T of J_{n-1} shown as the thicker lines in Figure 3, and the total imbedding distribution of J_{n-1} equals the imbedding distribution of T-rotation systems of J_{n-1}.

Note that J_{n-1} has the same overlap matrix M_n as closed-end ladders, and each vertex of J_{n-1} has degree four, thus, there exists six possible rotation at each vertex. Of these six rotations, exactly two lead to the incident cotree edges to overlap, which implies that there are two ways to set each y_j to 1 and four ways to set each y_j to 0. Assume that there are exactly p, $0 \leq p \leq n-1$, elements in Y equal to 1 then $n-1-p$ elements equal to 0 for some fixed matrix of the form M_n, then there are $2^{n-1-p}4^p = 2^{n+p-1}$ different T-rotation systems corresponding to the matrix. It is hard task to derive the total imbedding-distribution of J_n by the same consideration as we deal with the total imbedding distribution of L_n in the former section, since different overlap matrix corresponds to different T-rotation systems.

Suppose that every edge of the n-vertex path P_n is doubled, and that a self-adjacency is then added at each end. Such resulting graph is called a *cobblestone path of order* n, denoted by J_n. Figure 3 presents a cobblestone path of order 5.

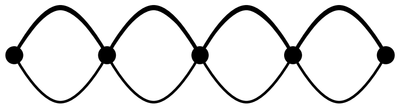

FIGURE 3. The cobblestone path J_5

A complete similar analysis to what have given for closed-end ladders, we fix a spanning tree T of J_{n-1} shown as the thicker lines in Figure 3, and the genus distribution of J_{n-1} equals the genus distribution of rotation systems of J_{n-1}.

Note that J_{n-1} has the same overlap matrix M_n as closed-end ladders, and each vertex of J_{n-1} has degree four, thus, there exists six possible rotations at each vertex. Of these six rotations, exactly two lead to the incident cotree edges to overlap, which implies that there are two ways to set each y_j to 1 and four ways to set each y_j to 0. See Figure 4.

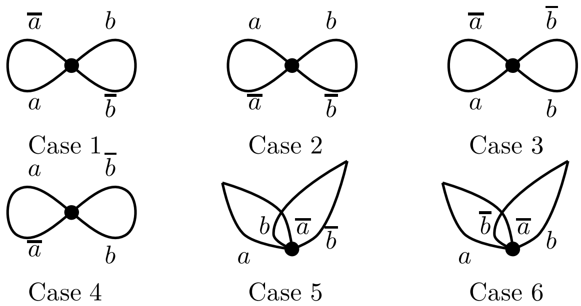

FIGURE 4. The six rotations.

Assume that there are exactly p, $0 \leq p \leq n-1$, elements in Y equal to 1 then $n-1-p$ elements equal to 0 for some fixed matrix of the form M_n, then there are $2^{n-1-p}4^p = 2^{n+p-1}$ different T-rotation systems corresponding to the matrix.

Theorem 3.1. *(see [2]) The genus polynomial J_{n-1} equals*

$$G_{J_{n-1}}(y) = \sum_{\substack{i_1+i_2+\cdots+i_r=n \\ i_1,i_2,\cdots,i_r>0}} 2^{n+r-2} \cdot y^{\sum_{h=1}^{r} \lfloor \frac{i_h}{2} \rfloor}.$$

Let $G_{J_{n-1}}(y) = \sum_{i=0}^{n} g_{J_{n-1}}(i) y^i$, where $g_{J_{n-1}}(i)$ is the number of embeddings of J_{n-1} into orientable surface of genus i. We can get the explicit formula of $g_{J_{n-1}}(i)$ as follows.

Theorem 3.2. *(see [15]) The number of embeddings of J_{n-1} into orientable surface of genus i equals*

$$g_i(J_{n-1}) = 3^i \cdot 4^{n-1-i} \cdot \binom{n-1-i}{i} + 2 \cdot 3^{i-1} \cdot 4^{n-1-i} \cdot \binom{n-1-i}{i-1}.$$

where $i \geq 0, n \geq 2$.

Proof. In Theorem 3.1, we let $R_n(y) = \sum_{i_1,i_2,\cdots,i_r>0}^{i_1+i_2+\cdots+i_r=n} 2^r y^{\sum_{h=1}^{r} \lfloor \frac{i_h}{2} \rfloor}$, it is obvious that $G_{J_{n-1}}(y) = 2^{n-2}R_n(y)$. When $i_r = 1$, we have

$$R_n(y) = \sum_{i_1,i_2,\cdots,i_{r-1}>0}^{i_1+i_2+\cdots+i_{r-1}=n-1} 2^r \cdot y^{\sum_{h=1}^{r-1} \lfloor \frac{i_h}{2} \rfloor} = 2R_{n-1}(y).$$

More generally, if $i_r = p$, $1 \leq p \leq n$, we have

$$R_n(y) = \sum_{i_1,i_2,\cdots,i_p>0}^{i_1+i_2+\cdots+i_p=n-p} 2^r \cdot y^{\sum_{h=1}^{r-1} \lfloor \frac{i_h}{2} \rfloor} = 2y^{\lfloor \frac{p}{2} \rfloor} \cdot R_{n-p}(y).$$

Thus, we have

$$\begin{aligned} R_n(y) &= 2\sum_{j=1}^{n} y^{\lfloor \frac{j}{2} \rfloor} \cdot R_{n-j}(y) \\ &= 2R_{n-1}(y) + 2\sum_{j=1}^{n-1} y^{\lfloor \frac{j+1}{2} \rfloor} \cdot R_{n-j-1}(y) \\ &= 2R_{n-1}(y) + 2yR_{n-2}(y) + 2\sum_{j=1}^{n-2} y^{\lfloor \frac{j+2}{2} \rfloor} \cdot R_{n-j-2}(y) \\ &= 2R_{n-1}(y) + 3yR_{n-2}(y). \end{aligned}$$

By Theorem 3.1, we have $R_1(y) = 2,\ R_2(y) = 2y + 4,\ R_3(y) = 10y + 8$ etc.
Let $H(y) = \frac{R_n(y)}{(\sqrt{3y}\ i)^n}$, by a simple calculation, see Section 1.4, we have

$$H_n(y) = U_n\left(\frac{1}{\sqrt{3y}i}\right) + \frac{1}{3}U_{n-2}\left(\frac{1}{\sqrt{3y}i}\right).$$

Thus, by (1), we have that

$$\begin{aligned} R_n(y) &= ((\sqrt{3y}\ i)^n)^n \cdot H_n(y) \\ &= \sum_{j\geq 0} \binom{n-j}{j} 2^{n-2j} \cdot 3^j \cdot y^j - \sum_{j\geq 0} \binom{n-2-j}{j} \cdot 2^{n-2-2j} \cdot 3^j \cdot y^{j+1}. \end{aligned}$$

Comparing the coefficient of y^m in both sides of the above equation, we have

$$g_m(J_{n-1}) = \binom{n-m}{m} \cdot 2^{n-2m} \cdot 3^m - \binom{n-m-1}{m-1} \cdot 2^{n-2m} \cdot 3^{m-1}$$

By the Pascal's Identity $\binom{n-m}{m} = \binom{n-m-1}{m} + \binom{n-m-1}{m-1}$, the result here is equal to that of [15]. □

Theorem 3.3. *The total imbedding-distribution polynomial of the cobblestone path J_{n-1} is given by*

$$\mathbb{I}_{J_{n-1}}(x, y) = 2^{n-2}S_n(y) - \mathbb{I}_0(J_{n-1}, y^2) + \mathbb{I}_0(J_{n-1}, x),$$

where $\mathbb{I}_0(J_{n-1}, x)$ *is the genus-distribution polynomial of the cobblestone path* J_{n-1}*, which is already known by Furst et al* [15]*, that is, for* $i \geq 0, n \geq 2$*,*

$$\mathbb{I}_0(J_{n-1}, x) = \sum_{j=0}^{[\frac{n-1}{2}]} \left\{ 3^j 4^{n-1-j} \binom{n-1-j}{j} + 2 \cdot 3^{j-1} 4^{n-1-j} \binom{n-1-j}{j-1} \right\} x^j,$$

$$\mathbb{I}_0(J_{n-1}, y^2) = \sum_{j=0}^{[\frac{n-1}{2}]} \left\{ 3^j 4^{n-1-j} \binom{n-1-j}{j} + 2 \cdot 3^{j-1} 4^{n-1-j} \binom{n-1-j}{j-1} \right\} y^{2j},$$

and

$$S_n(y) = (i\sqrt{6}y)^{n-1} \left[2(1+y) U_{n-1} \left(\frac{2+3y}{2i\sqrt{6}y} \right) + \frac{4y}{i\sqrt{6}} U_{n-2} \left(\frac{2+3y}{2i\sqrt{6}y} \right) \right],$$

where U_s *is the* s*-th Chebyshev polynomial of the second kind, and* $i^2 = -1$*.*

Proof. In [2], obtained the rank distribution of the overlap matrix M_n as

$$P_n(y) = \sum_{j_1, j_2, \ldots, j_r > 0}^{j_1+j_2+\ldots+j_r=n} y^{n-r} \prod_{h=1}^{r} R_{j_h}(y).$$

Furthermore, the total imbedding-distribution polynomial of J_{n-1} is also obtained as follows:

$$\begin{aligned} \mathbb{I}_{J_{n-1}}(x,y) &= 2^{n-2} \sum_{j_1, j_2, \ldots, j_r > 0}^{j_1+j_2+\ldots+j_r=n} 2^r y^{n-r} \prod_{h=1}^{r} R_{j_h}(y) - \mathbb{I}_0(J_{n-1},\ y^2) + \mathbb{I}_0(J_{n-1}, x) \\ &\triangleq 2^{n-2} S_n(y) - \mathbb{I}_0(J_{n-1},\ y^2) + \mathbb{I}_0(J_{n-1}, x). \end{aligned}$$

For the function $S_n(y)$, we consider the value j_r for fixed r. Note that if $j_r = 1$, then

$$\begin{aligned} S_n(y) &= \sum_{j_1, j_2, \ldots, j_{r-1} > 0}^{j_1+j_2+\ldots+j_{r-1}+1=n} 2^r y^{n-r} \prod_{h=1}^{r-1} R_{j_h}(y) R_1(y) \\ &= 2R_1(y) \sum_{j_1, j_2, \ldots, j_{r-1} > 0}^{j_1+j_2+\ldots+j_{r-1}=n-1} 2^{r-1} y^{n-1-(r-1)} \prod_{h=1}^{r-1} R_{j_h}(y) \\ &= 2(1+y) S_{n-1}(y). \end{aligned}$$

Generally, if $j_r = p$, where $1 \leq p \leq n$, we have

$$\begin{aligned} S_n(y) &= \sum_{j_1, j_2, \ldots, j_{r-1} > 0}^{j_1+j_2+\ldots+j_{r-1}+p=n} 2^r y^{n-r} \prod_{h=1}^{r-1} R_{j_h}(y) R_p(y) \\ &= 2y^{p-1} R_p(y) \sum_{j_1, j_2, \ldots, j_{r-1} > 0}^{j_1+j_2+\ldots+j_{r-1}=n-p} 2^{r-1} y^{n-p-(r-1)} \prod_{h=1}^{r-1} R_{j_h}(y) \\ &= 2y^{p-1} R_p(y) S_{n-p}(y). \end{aligned}$$

Synthesize the above discussions, we obtain

$$S_n(y) = 2 \sum_{j=1}^{n} y^{j-1} R_j(y) S_{n-j}(y) \tag{5}$$

with the initial conditions $S_0(y) = 1$, $S_1(y) = 2 + 2y$ and $S_2(y) = 10y^2 + 10y + 4$, where

$$R_p(y) = round(2^p/3) + round(2^{p+1}/3)y$$

with

$$round(2^p/3) = \begin{cases} \frac{1}{3}(2^p+1), & if\ p\ is\ odd, \\ \frac{1}{3}(2^p-1), & if\ p\ is\ even. \end{cases}$$

$$round(2^{p+1}/3) = \begin{cases} \frac{2}{3}(2^p+1)-1, & if\ p\ is\ odd, \\ \frac{2}{3}(2^p-1)+1, & if\ p\ is\ even. \end{cases}$$

Now let us find explicit formula for $S_n(y)$. By induction on p it is not hard to see that the polynomials $R_p(y)$ satisfy the recurrence relation $R_p(y) = 2R_{p-1}(y) + (1-y)(-1)^{p-1}$ with the initial condition $R_1(y) = 1 + y$. Hence, from (5) we obtain that

$$\begin{aligned} S_n(y) &= 2\sum_{j=1}^{n} y^{j-1}R_j(y)S_{n-j}(y) \\ &= 2(1+y)S_{n-1}(y) + 2\sum_{j=1}^{n-1} y^j R_{j+1}(y)S_{n-1-j}(y) \\ &= 2(1+y)S_{n-1}(y) + 2\sum_{j=1}^{n-1} y^j(2R_j(y) + (1-y)(-1)^j)S_{n-1-j}(y) \\ &= 2(1+y)S_{n-1}(y) + 2(1-y)\sum_{j=1}^{n-1}(-y)^j S_{n-1-j}(y) + 4y\sum_{j=1}^{n-1} y^{j-1}R_j(y)S_{n-1-j}(y) \\ &= (2+4y)S_{n-1}(y) + 2(1-y)\sum_{j=0}^{n-2}(-y)^{n-1-j}S_j(y), \end{aligned}$$

which implies that

$$S_n(y) + yS_{n-1}(y) = (2+4y)S_{n-1}(y) + y(2+4y)S_{n-2}(y) - 2y(1-y)S_{n-2}(y).$$

Hence, we have that $S_0(y) = 1$, $S_1(y) = 2 + 2y$, $S_2(y) = 10y^2 + 10y + 4$, and for all $n \geq 3$,

$$S_n(y) = (2+3y)S_{n-1}(y) + 6y^2 S_{n-2}(y).$$

Define $T_n(y) = \frac{S_n(y)}{(i\sqrt{6}y)^n}$ with $i^2 = -1$, so $T_n(y)$ satisfies the recurrence relation

$$T_n(y) = \frac{2+3y}{i\sqrt{6}z}T_{n-1}(y) - T_{n-2}(y)$$

with the initial conditions $T_0(y) = 1$, $T_1(y) = \frac{2+3y}{i\sqrt{6}z}$ and $T_2(y) = -\frac{2+5y+5y^2}{3y^2}$. By Section 1.4 and induction on n, we obtain that

$$T_n(y) = \frac{1}{i\sqrt{6}y}\left[2(1+y)U_{n-1}\left(\frac{2+3y}{2i\sqrt{6}y}\right) + \frac{4y}{i\sqrt{6}}U_{n-2}\left(\frac{2+3y}{2i\sqrt{6}y}\right)\right],$$

which gives that

$$S_n(y) = (i\sqrt{6}y)^{n-1}\left[2(1+y)U_{n-1}\left(\frac{2+3y}{2i\sqrt{6}y}\right) + \frac{4y}{i\sqrt{6}}U_{n-2}\left(\frac{2+3y}{2i\sqrt{6}y}\right)\right]. \tag{6}$$

Plug the relation into $\mathbb{I}_{J_{n-1}}(x, y)$, it completes the proof. □

Let $S_n(y) = \sum_{m=0}^{n} D_n(m)y^m$, and the following corollary derives an explicit formula for $D_n(m)$.

Corollary 3.4. *For all $n \geq 1$ and $0 \leq m \leq n$,*

$$\begin{aligned} D_n(m) = & \sum_{j=0}^{\lfloor m/2 \rfloor} \left\{ \binom{n-1-j}{j} \binom{n-1-2j}{n-1-m} \cdot 2^{n-m} \cdot 3^{m-2j} \cdot 6^j \right\} \\ & + \sum_{j=0}^{\lfloor (m-1)/2 \rfloor} \left\{ \binom{n-1-j}{j} \binom{n-1-2j}{n-m} \cdot 2^{n-m+1} \cdot 3^{m-2j-1} \cdot 6^j \right\} \\ & + \sum_{j=0}^{\lfloor (m-2)/2 \rfloor} \left\{ \binom{n-2-j}{j} \binom{n-2-2j}{n-m} \cdot 2^{n-m+2} \cdot 3^{m-2j-2} \cdot 6^j \right\}. \end{aligned}$$

Proof. By (1) we obtain that

$$\begin{aligned} & (i\sqrt{6}y)^{n-1} U_{n-1}\left(\frac{2+3y}{2i\sqrt{6}y}\right) \\ & = \sum_{j \geq 0} \left\{ (-1)^j \binom{n-1-j}{j} \cdot (2+3y)^{n-1-2j} \cdot (i\sqrt{6}y)^{2j} \right\} \\ & = \sum_{j \geq 0} \left\{ \binom{n-1-j}{j} \sum_{s=0}^{n-1-2j} \binom{n-1-2j}{s} \cdot 2^{n-1-2j-s} \cdot 3^s \cdot 6^j \cdot y^{s+2j} \right\}. \end{aligned}$$

Plug the formulae into (6), and comparing the coefficient of y^m in both sides, then it leads to the result. □

For instance, the crosscap number polynomial $P_{J_n}(y)$ for the cobblestone J_n with $n = 1, 2, 3, 4, 5$ is given by

$$\begin{aligned} P_{J_1}(y) &= 8y^2 + 10y, \\ P_{J_2}(y) &= 84y^3 + 104y^2 + 64y, \\ P_{J_3}(y) &= 720y^4 + 1320y^3 + 848y^2 + 352y, \\ P_{J_4}(y) &= 6480y^5 + 13536y^4 + 12672y^3 + 5696y^2 + 1792y, \\ P_{J_5}(y) &= 56448y^6 + 140832y^5 + 152064y^4 + 97536y^3 + 34304y^2 + 8704y. \end{aligned}$$

4. CONCLUDED REMARKS

In this paper, we find a new usage of Chebyshev polynomials in the study of genus distribution, using the overlap matrix, we obtain homogeneous recurrence relation for rank distribution polynomial, which can be solved in terms of Chebyshev polynomials of the second kind. We think that the method here can also be used to find explicit formula of embedding distributions for some other graphs in [32, 37].

References

[1] D. Archdeacon, Calculations on the average genus and genus distribution of graphs, *Congr. Numer.* **67** (1988) 114–124.

[2] J. Chen, J. Gross and R. G. Rieper, Overlap matrices and total embeddings, *Discrete Math.* **128** (1994) 73–94.

[3] Y. Chen, Y. Liu and T. Wang, The total embedding distributions of cacti and necklaces, *Acta Mathematica Sinica* **22**(5) (2006) 1583–1590.

[4] Y. Chen, L. Ou and Q. Zou, Embedding distributions of Ringel ladders, *Discrete Math.*, Accepted.

[5] Y. Chen T. Mansour and Q. Zou, Embedding distributions of generalized fan graphs, *Canad. Math. Bull.*, Accepted.

[6] J. Edmonds, A combinatorial representation for polyhedral surfaces, *Notices Amer. Math. Soc.* **7** (1960) 646.

[7] J. L. Gross and M. L. Furst, Hierarchy for imbedding-distribution invariants of a graph, *J. Graph Theory* **11** (1987) 205–220.
[8] J. L. Gross, I. F. Khan, and M. I. Poshni, Genus distribution of graph amalgamations: Pasting at root-vertices, *Ars Combin.* **94** (2010) 33–53.
[9] J. L. Gross, D. P. Robbins and T. W. Tucker, Genus distributions for bouquets of circles, *J. Combin. Theory* (B) **47** (1989) 292–306.
[10] J. L. Gross and T. W. Tucker, Topological Graph Theory, Dover, 2001 (original edn. Wiley, 1987).
[11] Rongxia Hao and Yanpei Liu, The genus distributions of directed antiladders in orientable surfaces, *Appl. Math. Lett.* bf 21(2) (2008) 161–164.
[12] D. M. Jackson, Counting cycles in permutations by group characters with an application to a topological problem, *Trans. Amer. Math. Soc.* **299** (1987) 785–801.
[13] D. M. Jackson and T. I. Visentin, A character-theoretic approach to embeddings of rooted maps in an orientable surface of given genus, *Trans. Amer. Math. Soc.* **322** (1990) 343–363.
[14] D. M. Jackson and T. I. Visentin, An Atlas of the Smaller Maps in Orientable and Nonorientable Surfaces, Chapman and Hall/CRC Press, 2001.
[15] M. Furst, J. Gross and R. Statman, Genus distributions for two classes of graphs, *J. Combin. Ser. B* **46** (1989) 22–36.
[16] I. F. Khan, M. I. Poshni, and J. L. Gross, Genus distribution of graph amalgamations: pasting when one root has higher degree, *Ars Math. Contemporanea* (2010), to appear.
[17] V. P. Korzhik and H-J Voss, Exponential families of non-isomorphic non-triangular orientable genus embeddings of complete graphs, *J. Combin. Theory* (B) **86** (2002) 86–211.
[18] J. H. Kwak and J. Lee, Genus polynomials of dipoles, *Kyungpook Math. J.* **33** (1993) 115–125.
[19] J. H. Kwak and J. Lee, Enumeration of graph embeddings, *Discrete Math.* **135** (1994) 129–151.
[20] J. H. Kwak and S. H. Shim, Total embedding distributions for bouquets of circles, *Discrete Math.* **248** (2002) 93–108.
[21] L. A. McGeoch, Algorithms for two graph problems: computing maximum-genus imbedding and the two-server problem, PhD thesis, Carnegie-Mellon University, 1987.
[22] B. Mohar and C. Thomassen, Graphs on Surfaces, Johns Hopkins Press, 2001.
[23] B. Mohar, An obstruction to embedding graphs in surface, *Discrete Math.* **78** (1989) 135–142.
[24] B. P. Mull, Enumerating the orientable 2-cell imbeddings of complete bipartite graphs, *J. Graph Theory* **30** (1999) 77–90.
[25] M. I. Poshni, I. F. Khan, and J. L. Gross, Genus distribution of edge amalgamations, *Ars Math. Contemporanea* **3** (2010) 69–86.
[26] Th. Rivlin, Chebyshev polynomials. From approximation theory to algebra and number theory, John Wiley, New York, 1990.
[27] R. G. Rieper, The enumeration of graph imbeddings, PhD thesis, Western Michigan University, 1990.
[28] S. Stahl, Generalized embedding schemes, *J. Graph Theory* **2** (1978) 41–52.
[29] S. Stahl, Region distributions of graph embeddings and Stirling numbers, *Discrete Math.* **82** (1990) 57–78.
[30] S. Stahl, Permutation-partition pairs III: Embedding distributions of linear families of graphs, *J. Combin. Theory* (B) **52** (1991) 191–218.
[31] S. Stahl, Region distributions of some small diameter graphs, *Discrete Math.* **89** (1991) 281–299.
[32] S. Stahl, On the zeros of some polynomial, *Canad. J. Math.* **49** (1996) 617–640.
[33] E. H. Tesar, Genus distribution of Ringel ladders, *Discrete Math.* **216** (2000) 235–252.
[34] T. I. Visentin and S. W. Wieler, On the genus distribution of (p, q, n)-dipoles, *Electronic J. of Combin.* **14** (2007) #R12.
[35] Y. P. Liu, Advances in Combinatorial Maps (in Chinese), Northern Jiaotong University Press, Beijing, 2003.
[36] L. X. Wan and Y. P. Liu, Orientable embedding distributions by genus for certain types of graphs, *Ars Combin.* **79** (2006) 97–105.
[37] L. X. Wan and Y. P. Liu, Orientable embedding genus distribution for certain types of graphs, *J. Combin. Theory* (B) **47** (2008) 19–32.
[38] A. T. White, Graphs of Groups on Surfaces, North-Holland, 2001.
[39] Y. Yang and Y. Liu, Classification of (p,q,n)-dipoles on nonorientable surfaces, *Electron. J. Combin.* **17** (2010) #N12.