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ON DISCRETE q-EXTENSIONS OF CHEBYSHEV POLYNOMIALS

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Abstract

We study in detail main properties of two families of the basic hypergeometric ${}_2\phi_1$-polynomials, which are natural q-extensions of the classical Chebyshev polynomials $T_n(x)$ and $U_n(x)$. In particular, we show that they are expressible as special cases of the big q-Jacobi polynomials $P_n(x;a,b,c;q)$ with some chosen parameters a, b and c. We derive quadratic transformations that relate these polynomials to the little q-Jacobi polynomials $p_n(x;a,b\,|q)$. Explicit forms of discrete orthogonality relations on a finite interval, q-difference equations and Rodrigues-type difference formulas for these q-Chebyshev polynomials are also given.

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1 Introduction

The Chebyshev polynomials find frequent and profound applications in many areas of mathematical analysis such as approximation, series expansions, interpolation, quadrature and integral equations [1, 2]. Hence it is of considerable interest to inquire into the defining of explicit q-extensions of the Chebyshev polynomials, which may be similarly useful in analysis of q-special functions. The interest in this study is motivated by the following circumstance. It is well known that the Chebyshev polynomials $T_n(x)$ and $U_n(x)$ may be regarded as special cases of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ with parameters $\alpha=\beta=-1/2$ and $\alpha=\beta=1/2$, respectively. Therefore it appears at first that the continuous q-Jacobi polynomials $P_n^{(\alpha,\beta)}(x|\,q)$ (which evidently represent q-extensions of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$) with the particular values of the parameters $\alpha=\beta=-1/2$ and $\alpha=\beta=1/2$ would be natural q-extensions of the Chebyshev polynomials $T_n(x)$ and $U_n(x)$. Under closer examination however, it turns out that the continuous q-Jacobi polynomials $P_n^{(-1/2,-1/2)}(x|\,q)$ and $P_n^{(1/2,1/2)}(x|\,q)$ *are only constant (but q-dependent) multiples* of the Chebyshev polynomials $T_n(x)$ and $U_n(x)$. In other words, the continuous q-Jacobi polynomials $P_n^{(-1/2,-1/2)}(x|\,q)$ and $P_n^{(1/2,1/2)}(x|\,q)$ are, in fact, rescalings of the Chebyshev polynomials $T_n(x)$ and $U_n(x)$; therefore the former two polynomial families are just trivial

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q-extensions of the latter ones. This curious "q-degeneracy" of the continuous q-Jacobi polynomials $P_n^{(\alpha,\beta)}(x|q)$ for the values of the parameters $\alpha=\beta=-1/2$ and $\alpha=\beta=1/2$ had been already noticed by R.Askey and J.A.Wilson in their seminal work [3]. Observe also that nothing essentially changes when one tries to use the connection with the *monic form*[1] of the continuous Rogers q-ultraspherical polynomials $C_n^{(M)}(x;q^\lambda|q)$, rather than with the continuous q-Jacobi polynomials $P_n^{(\alpha,\beta)}(x|q)$. The q-polynomials $C_n^{(M)}(x;1|q)$ are known to provide a q-extension of the Chebyshev polynomials $T_n(x)$, whereas the $C_n^{(M)}(x;q|q)$ represent a q-extension of the Chebyshev polynomials $U_n(x)$. But both of these q-extensions are trivial in the above-mentioned sense.

This work is an attempt to explore properties of q-extensions of the Chebyshev polynomials $T_n(x)$ and $U_n(x)$ in terms of the basic hypergeometric ${}_2\phi_1$-polynomials, which were introduced in a recent paper [4] devoted to the study of Fourier integral transforms for the q-Fibonacci and q-Lucas polynomials. We prove that these two q-Chebyshev families are expressible as special cases of the big q-Jacobi polynomials $P_n(x;a,b,c;q)$ with particularly chosen parameters a, b and c. Thus it becomes apparent that the required q-Chebyshev polynomials have been "in hiding" within the Askey q-scheme at one level higher than the continuous q-Jacobi polynomials $P_n^{(\alpha,\beta)}(x|q)$. We use this connection with the big q-Jacobi polynomials $P_n(x;a,b,c;q)$ in order to establish an explicit form of the discrete orthogonality relation for these q-Chebyshev polynomials.

The paper is organized as follows. In section 2 we determine three-term recurrence relations for the q-Chebyshev polynomials under study in order to clarify their connections with the big q-Jacobi polynomials. Quadratic transformations, relating them with the little q-Jacobi polynomials are derived in section 3. In section 4 we present explicit forms of discrete orthogonality relations on a finite interval, q-difference equations and Rodrigues-type difference formulas for these q-Chebyshev polynomials. Some conclusions are offered in section 5. The Appendix contains the derivation of two transformation formulas between basic hypergeometric ${}_2\phi_1$ and ${}_3\phi_2$ polynomials, associated with q-extensions of the Chebyshev polynomials $T_n(x)$ and $U_n(x)$.

Throughout this exposition we employ standard notation of the theory of special functions (see, for example, [5]–[7]).

2 Connections with Big q-Jacobi Polynomials

Recall that the Chebyshev polynomials of the first kind $T_n(x)$ and of the second kind $U_n(x)$ are explicitly given in terms of the hypergeometric ${}_2F_1$-polynomials as

$$T_0(z)=1\,, \qquad T_n(z)={}_2F_1\Big(-n,n;\,1/2\,\Big|\,\frac{1-z}{2}\Big)=2^{n-1}z^n\,{}_2F_1\left(-\frac{n}{2},\frac{1-n}{2};1-n\,\Big|\,1/z^2\right), \qquad n\ge 1, \tag{2.1}$$

and

$$U_n(z)=(n+1)\,{}_2F_1\Big(-n,n+2;\,3/2\,\Big|\,\frac{1-z}{2}\Big)=(2z)^n\,{}_2F_1\left(-\frac{n}{2},\frac{1-n}{2};-n\,\Big|\,1/z^2\right), \qquad n\ge 0\,, \tag{2.2}$$

respectively. The Chebyshev polynomials $T_n(x)$ are generated by the three-term recurrence relation

$$2z\,T_n(z)\,=\,T_{n+1}(z)\,+\,T_{n-1}(z)\,, \qquad n\ge 1\,, \tag{2.3}$$

with the initial conditions $T_0(z)=1$ and $T_1(z)=z$; whereas the Chebyshev polynomials $U_n(x)$ are governed by the same recurrence (2.3) but for $n\ge 0$ and initial assignment $U_{-1}(z)=0$ and $U_0(z)=1$.

As was noticed in [4], two q-polynomial families of degree n in the variable x, defined by

$$p_n^{(T)}(x\,|\,q)=2^{n-1}x^n\,{}_2\phi_1\Big(q^{-n},q^{1-n};\,q^{2(1-n)}\,\Big|\,q^2;\,q^2x^{-2}\Big), \qquad n\ge 1, \qquad p_0^{(T)}(x\,|\,q)=1\,, \tag{2.4}$$

[1]We recall that an arbitrary polynomial $p_n(x)=\sum_{k=0}^n c_{n,k}\,x^k$ of degree n in the variable x can be written in the *monic form* $p_n^{(M)}(x)=c_{n,n}^{-1}\,p_n(x)=x^n+c_{n,n}^{-1}\sum_{k=0}^{n-1}c_{n,k}\,x^k$ just by changing its normalization.

$$p_n^{(U)}(x\,|\,q) = (2x)^n\,{}_2\phi_1\Big(q^{-n}, q^{1-n}; q^{-2n}\,\Big|\,q^2\,; q^2x^{-2}\Big), \qquad n\ge 0\,, \qquad 0<q<1\,, \tag{2.5}$$

represent very natural q-extensions of the Chebyshev polynomials of the first kind $T_n(x)$ and of the second kind $U_n(x)$, respectively. For checking this statement one just has to bear in mind the well-known limit property

$$\lim_{q\to 1}\,{}_2\phi_1\Big(q^{-n}, q^a\,; q^b\,\Big|q;z\Big) = {}_2F_1(-n,a\,;b\,|z) \tag{2.6}$$

of the q-hypergeometric ${}_2\phi_1$-polynomials (see, for example, section 1.10, p. 15 in [7]). Then from (2.6) it follows at once that the polynomials $p_n^{(T)}(x|\,q)$ and $p_n^{(U)}(x|\,q)$ coincide in the limit as $q\to 1$ with the $T_n(x)$ and $U_n(x)$, given by the second lines in (2.1) and (2.2), respectively.

Note that from (2.4) and (2.5) it is evident that both of these q-polynomials are either reflection symmetric (when degree n is even) or antisymmetric (when degree n is odd), that is,

$$p_n^{(T)}(-x\,|\,q) = (-1)^n\,p_n^{(T)}(x\,|\,q)\,, \qquad p_n^{(U)}(-x\,|\,q) = (-1)^n\,p_n^{(U)}(x\,|\,q)\,. \tag{2.7}$$

The best route to determine whether these q-polynomials (2.4) and (2.5) are related to some “named” families of basic hypergeometric orthogonal polynomials from the Askey q-scheme [7], is first to find three-term recurrence relations, associated with them.

Let us start with (2.4) and slightly simplify its explicit form,

$$\begin{aligned} p_n^{(T)}(x\,|\,q) &= 2^{(n-1)}x^n\sum_{k=0}^{\lfloor n/2\rfloor}\frac{(q^{-n},q^{1-n};q^2)_k}{(q^{2(1-n)},q^2;q^2)_k}\,q^{2k}x^{-2k} = 2^{(n-1)}x^n\sum_{k=0}^{\lfloor n/2\rfloor}\frac{(q^{-n};q)_{2k}\,q^{2k}}{(q^{2(1-n)},q^2;q^2)_k}\,x^{-2k} \\ &= (q;q)_n\,2^{(n-1)}x^n\sum_{k=0}^{\lfloor n/2\rfloor}\frac{q^{k(2k-2n+1)}\,x^{-2k}}{(q;q)_{n-2k}\,(q^{2(1-n)},q^2;q^2)_k}\,, \end{aligned} \tag{2.8}$$

by using the relation $(z,qz;q^2) = (z;q)_{2n}$ at the first step and the identity

$$(q^{-n};q)_{2k} = \frac{(q;q)_n}{(q;q)_{n-2k}}\,q^{k(2k-2n-1)}\,, \qquad 0\le k\le \lfloor n/2\rfloor\,,$$

at the second one. Observe that the symbol $\lfloor x\rfloor$ in (2.8) denotes the greatest integer in x and we have employed the conventional notation $(z_1,z_2,...,z_k\,;q)_n := \prod_{j=1}^k(z_j\,;q)_n$ for products of q-shifted factorials $(z_j\,;q)_n$, $j=1,2,...,k$.

Let us assume now that n is odd, $n=2m+1$. Then from (2.8) one obtains that

$$\begin{aligned} p_{2m+1}^{(T)}(x\,|\,q) &= (q;q)_{2m+1}\,x(2x)^{2m}\sum_{k=0}^{m}\frac{q^{k(2k-4m-1)}\,x^{-2k}}{(q;q)_{2m+1-2k}\,(q^{-4m},q^2;q^2)_k} \\ &= (q;q)_{2m}\,x(2x)^{2m}\sum_{k=0}^{m}\frac{(1-q^{2m+1})(1-q^{2k-4m})}{(1-q^{-4m})(1-q^{2m-2k+1})}\,\frac{q^{k(2k-4m-1)}\,x^{-2k}}{(q;q)_{2(m-k)}\,(q^{2(1-2m)},q^2;q^2)_k}\,, \end{aligned} \tag{2.9}$$

upon employing the relations

$$(1-z)(z\,q;q)_k = (z;q)_{k+1} = (1-z\,q^k)(z;q)_k\,. \tag{2.10}$$

Finally, use a readily verified identity

$$\frac{(1-q^{2m+1})(1-q^{2k-4m})}{(1-q^{-4m})(1-q^{2m-2k+1})} = q^{2k} + \frac{(1-q^{1-2m})(1-q^{2k})}{(1-q^{-4m})(1-q^{2m-2k+1})}\,, \qquad 0\le k\le m\,,$$

to represent (2.9) as

$$p^{(T)}_{2m+1}(x\,|\,q) \,=\, (q;q)_{2m}\,x(2x)^{2m}\sum_{k=0}^{m}\frac{q^{k(2k-4m+1)}\,x^{-2k}}{(q;q)_{2(m-k)}\,(q^{2(1-2m)},q^2;q^2)_k}$$

$$-\frac{q^{6m-1}\,(q;q)_{2m-1}}{(1+q^{2m})(1+q^{2m-1})}\,x(2x)^{2m}\sum_{k=1}^{m}\frac{q^{k(2k-4m-1)}\,x^{-2k}}{(q;q)_{2(m-k)+1}\,(q^{4(1-m)},q^2;q^2)_{k-1}}$$

$$=2\,x\,p^{(T)}_{2m}(x\,|\,q)-\frac{2\,q^{2m}\,(q;q)_{2m-1}\,(2x)^{2m-1}}{(1+q^{2m})(1+q^{2m-1})}\sum_{l=0}^{m-1}\frac{q^{l[2l-2(2m-1)+1]}\,x^{-2l}}{(q;q)_{2m-1-2l}\,(q^{2[1-(2m-1)]},q^2;q^2)_l}$$

$$=2\,x\,p^{(T)}_{2m}(x\,|\,q)\,-\,\frac{4\,q^{2m}}{(1+q^{2m})(1+q^{2m-1})}\,p^{(T)}_{2m-1}(x\,|\,q)\,. \tag{2.11}$$

Similarly, if one assumes that the degree n in (2.8) is even, $n = 2m$, then by the same reasoning one arrives at the three-term recurrence relation between the polynomials $p^{(T)}_{2m}(x|\,q)$, $p^{(T)}_{2m-1}(x|\,q)$ and $p^{(T)}_{2m-2}(x|\,q)$. Thus we conclude that the general (*i.e.*, valid for both even and odd degrees n) recurrence formula for the q-polynomials (2.4) is

$$p^{(T)}_{n+1}(x\,|\,q) \,=\, 2x\,p^{(T)}_{n}(x\,|\,q)\,-\,\frac{4\,q^{n}}{(1+q^{n})(1+q^{n-1})}\,p^{(T)}_{n-1}(x\,|\,q), \qquad n\geq 1\,. \tag{2.12}$$

Using the same considerations *mutatis mutandis*, one derives the three-term recurrence relation for the second family of q-polynomials (2.5):

$$p^{(U)}_{n+1}(x\,|\,q) \,=\, 2x\,p^{(U)}_{n}(x\,|\,q)\,-\,\frac{4\,q^{n-1}}{(1+q^{n})(1+q^{n+1})}\,p^{(U)}_{n-1}(x\,|\,q)\,, \qquad n\geq 0\,, \qquad p^{(U)}_{-1}(x\,|\,q)=0\,. \tag{2.13}$$

Now we are in a position to establish that the q-extensions (2.4) and (2.5) of the Chebyshev polynomials $T_n(x)$ and $U_n(x)$ are in fact connected with the big q-Jacobi polynomials

$$P_n(x;a,b,c;q) := {}_3\phi_2\Big(q^{-n},ab\,q^{n+1},x\,;aq,cq\,\Big|\,q;q\Big) \tag{2.14}$$

with some particularly chosen parameters a,b and c. Indeed, recall that the *monic form*

$$P^{(M)}_{n}(x;a,a,-a;q) \,=\, \frac{(a^2q^2;q^2)_n}{(a^2q^{n+1};q)_n}\,P_n(x;a,a,-a;q) \tag{2.15}$$

of the big q-Jacobi polynomials (2.14) with the parameters $a=b=-\,c$ satisfies the three-term recurrence relation

$$P^{(M)}_{n+1}(x;a,a,-a;q) = x\,P^{(M)}_{n}(x;a,a,-a;q) - \gamma_n(a;q)\,P^{(M)}_{n-1}(x;a,a,-a;q) \tag{2.16}$$

with the coefficients (see (14.5.4), p. 439 in [7])

$$\gamma_n(a;q) \,=\, \frac{a^2\,q^{n+1}(1-q^n)(1-a^2q^n)}{(1-a^2q^{2n-1})(1-a^2q^{2n+1})}\,.$$

For $a=q^{-1/2}$ the recurrence (2.16) clearly reduces to

$$P^{(M)}_{n+1}(x;q^{-1/2},q^{-1/2},-\,q^{-1/2};q) = x\,P^{(M)}_{n}(x;q^{-1/2},q^{-1/2},-\,q^{-1/2};q)$$

$$-\frac{q^{n}}{(1+q^{n})(1+q^{n-1})}\,P^{(M)}_{n-1}(x;q^{-1/2},q^{-1/2},-\,q^{-1/2};q)\,, \tag{2.17}$$

whereas the choice of $a = q^{1/2}$ in (2.16) leads to

$$P^{(M)}_{n+1}(x;q^{1/2},q^{1/2},-q^{1/2};q) = x\,P^{(M)}_n(x;q^{1/2},q^{1/2},-q^{1/2};q) - \frac{q^{n-1}}{(1+q^n)(1+q^{n+1})}\,P^{(M)}_{n-1}(x;q^{1/2},q^{1/2},-q^{1/2};q)\,. \tag{2.18}$$

On comparing (2.17) and (2.18) with (2.12) and (2.13), respectively, one thus concludes that

$$p^{(T)}_0(x\,|\,q) = 1, \qquad p^{(T)}_n(x\,|\,q) = 2^{n-1}\,P^{(M)}_n(x;q^{-1/2},q^{-1/2},-q^{-1/2};q) = 2^{n-1}\,\frac{(q;q^2)_n}{(q^n;q)_n}\,{}_3\phi_2\Big(q^{-n},\,q^n,x;q^{1/2},-q^{1/2}\,\Big|\,q;q\Big)\,, \qquad n\ge 1\,, \tag{2.19}$$

and

$$p^{(U)}_n(x\,|\,q) \,=\, 2^n\,P^{(M)}_n(x;q^{1/2},q^{1/2},-q^{1/2};q) = 2^n\,\frac{(q^3;q^2)_n}{(q^{n+2};q)_n}\,{}_3\phi_2\Big(q^{-n},\,q^{n+2},x;q^{3/2},-q^{3/2}\,\Big|\,q;q\Big)\,, \quad n\ge 0\,. \tag{2.20}$$

Evidently, these representations (2.19) and (2.20) in terms of the big q-Jacobi polynomials (2.14) agree with the initial definitions (2.4) and (2.5) of the q-polynomials $p^{(T)}_n(x|\,q)$ and $p^{(U)}_n(x|\,q)$, only if two transformation formulas

$$x^n{}_2\phi_1\Big(q^{-n},q^{1-n};q^{2(1-n)}\,\Big|\,q^2;q^2x^{-2}\Big) = \frac{(q;q^2)_n}{(q^n;q)_n}\,{}_3\phi_2\Big(q^{-n},q^n,x;q^{1/2},-q^{1/2}\,\Big|\,q;q\Big), \tag{2.21}$$

$$x^n{}_2\phi_1\Big(q^{-n},q^{1-n};q^{-2n}\,\Big|\,q^2;q^2x^{-2}\Big) = \frac{(q^3;q^2)_n}{(q^{n+2};q)_n}\,{}_3\phi_2\Big(q^{-n},q^{n+2},x;q^{3/2},-q^{3/2}\,\Big|\,q;q\Big), \tag{2.22}$$

between ${}_2\phi_1$ (with the base q^2) and ${}_3\phi_2$ (with the base q) basic polynomials are valid. Direct proofs of these identities are given in Appendix.

3 Quadratic Transformations

It turns out that, in addition to (2.19) and (2.20), both symmetric or antisymmetric cases of the q-polynomial families (2.4) and (2.5) can be separately expressed in terms of the little q-Jacobi polynomials, defined as (see, for example, (14.12.1), p. 482 in [7])

$$p_n(x;a,b\,|\,q) := {}_2\phi_1(q^{-n},ab\,q^{n+1};aq\,|\,q;qx)\,. \tag{3.1}$$

Indeed, let us apply first the transformation of terminating ${}_2\phi_1$ series (see (1.13.15), p. 20 in [7])

$${}_2\phi_1(q^{-n},a;b|\,q;z) = \frac{(a;q)_n}{(b;q)_n}\,q^{-n(n+1)/2}(-z)^n{}_2\phi_1\left(q^{-n},q^{1-n}/b;q^{1-n}/a\,\middle|\,q;\frac{bq^{n+1}}{az}\right) \tag{3.2}$$

to the q-polynomials of even degree $p^{(T)}_{2m}(x|\,q)$, where m is an arbitrary nonnegative integer. This results in the relation

$$\begin{aligned} p^{(T)}_{2m}(x\,|\,q) &= x(2x)^{2m-1}\,{}_2\phi_1\Big(q^{-2m},q^{1-2m};\,q^{2(1-2m)}\,\Big|\,q^2\,;\,q^2\,x^{-2}\Big) \\ &= (-4)^m\,q^{-m(m-1)}\,\frac{(q^{1-2m};q^2)_m}{2(q^{2(1-2m)};q^2)_m}\,{}_2\phi_1\Big(q^{-2m},q^{2m};\,q\,\Big|\,q^2\,;qx^2\Big) \\ &= (-4\,q^m)^m\,\frac{(q;q^2)_m}{2(q^{2m};q^2)_m}\,p_m\Big(q^{-1}x^2;q^{-1},q^{-1}\Big|\,q^2\Big)\,, \qquad m\ge 1\,. \end{aligned} \tag{3.3}$$

Similarly, in the case of the q-polynomials of odd degree $p^{(T)}_{2m+1}(x|q)$ one obtains, by using (3.2), that

$$\begin{aligned} p^{(T)}_{2m+1}(x\,|\,q) &= x(2x)^{2m}\,{}_2\phi_1\left(q^{-2m-1}, q^{-2m};\, q^{-4m}\,\middle|\, q^2\,;\, q^2\,x^{-2}\right) \\ &= (-4)^m\,q^{-m(m-1)}\,\frac{(q^{-1-2m};q^2)_m}{(q^{-4m};q^2)_m}\,x\,{}_2\phi_1\left(q^{-2m}, q^{2m};\, q\,\middle|\, q^2\,;q^2x^2\right) \\ &= (-4\,q^m)^m\,\frac{(q^3;q^2)_m}{(q^{2(m+1)};q^2)_m}\,x\,p_m\left(q^{-1}x^2;q,q^{-1}\middle|\, q^2\right), \qquad m \ge 0\,. \end{aligned} \tag{3.4}$$

Thus, q-extensions (2.4) of the Chebyshev polynomials $T_n(x)$ can be written it terms of the little q-Jacobi polynomials (3.1) as

$$\begin{aligned} p^{(T)}_{2m}(x\,|\,q) &= (-4\,q^m)^m\,\frac{(q;q^2)_m}{2(q^{2m};q^2)_m}\,p_m\left(q^{-1}x^2;q^{-1},q^{-1}\middle|\, q^2\right), \\ p^{(T)}_{2m+1}(x\,|\,q) &= (-4\,q^m)^m\,\frac{(q^3;q^2)_m}{(q^{2(m+1)};q^2)_m}\,x\,p_m\left(q^{-1}x^2;q,q^{-1}\middle|\, q^2\right). \end{aligned} \tag{3.5}$$

Exactly in the same manner one obtains that q-extensions (2.5) of the Chebyshev polynomials $U_n(x)$ can be represented as

$$\begin{aligned} p^{(U)}_{2n}(x\,|\,q) &= (-4)^n\,q^{n(n+2)}\,\frac{(q;q^2)_n}{(q^{2(n+1)};q^2)_n}\,p_n\left(q^{-3}x^2;q^{-1},q\middle|\, q^2\right), \\ p^{(U)}_{2n+1}(x\,|\,q) &= (-4)^n\,q^{n(n+2)}\,\frac{2(q^3;q^2)_n}{(q^{2(n+2)};q^2)_n}\,x\,p_n\left(q^{-3}x^2;q,q\middle|\, q^2\right). \end{aligned} \tag{3.6}$$

Notice that from the well-known limit property (*cf.* (14.12.15) on p. 485 in [7])

$$\lim_{q\to 1}\,p_n\left(x;\,q^a,q^b\,\middle|\,q\right) \;=\; \frac{n!}{(\alpha+1)_n}\,P^{(\alpha,\beta)}_n(1-2x) \tag{3.7}$$

of the little q-Jacobi polynomials (3.1), it follows that in the limit as $q \to 1$ the quadratic transformations (3.5) and (3.6) reduce to the relations

$$T_{2m}(x) \;=\; \frac{m!}{(1/2)_m}\,P^{(-1/2,-1/2)}_m(2x^2-1)\,, \qquad T_{2m+1}(x) \;=\; \frac{m!}{(1/2)_m}\,x\,P^{(-1/2,1/2)}_m(2x^2-1)\,, \tag{3.8}$$

and

$$U_{2m}(x) \;=\; \frac{m!}{(1/2)_m}\,P^{(1/2,-1/2)}_m(2x^2-1)\,, \qquad U_{2m+1}(x) \;=\; \frac{2(m+1)!}{(3/2)_m}\,x\,P^{(1/2,1/2)}_m(2x^2-1)\,, \tag{3.9}$$

respectively. It should also be observed that the transformations (3.8) and (3.9) for the Chebyshev polynomials $T_n(x)$ and $U_n(x)$ are special cases of the quadratic transformation (*cf.* Remarks on p. 224 in [7])

$$C^{(\lambda;M)}_{2n}(x) \;=\; \frac{n!}{(\lambda+n)_n}\,P^{(\lambda-1/2,-1/2)}_n(2x^2-1)\,, \qquad C^{(\lambda;M)}_{2n+1}(x) \;=\; \frac{n!}{(\lambda+n+1)_n}\,x\,P^{(\lambda-1/2,1/2)}_n(2x^2-1)\,, \tag{3.10}$$

for the *monic* Gegenbauer (or ultraspherical) polynomials $C^{(\lambda;M)}_n(x)$, defined as (see (9.8.19) and (9.8.22) on p. 222 in [7])

$$C^{(\lambda;M)}_n(x) := \frac{n!}{2^n(\lambda)_n}\,C^{(\lambda)}_n(x) = \frac{(\lambda+n)_\lambda}{2^{2\lambda+n-1}(1/2)_\lambda}\,{}_2F_1\left(-n, n+2\lambda; \lambda+1/2\,\middle|\,\frac{1-x}{2}\right). \tag{3.11}$$

Indeed, taking into account that $C^{(0;M)}_n(x) = 2^{1-n}\,T_n(x)$ and $C^{(1;M)}_n(x) = 2^{-n}\,U_n(x)$ by the defintion (3.11), it is readily checked that (3.8) is a special case of (3.10) with $\lambda = 0$ and (3.9) is a special case of (3.10) with $\lambda = 1$.

It should also be noted that the quadratic transformations (3.5) and (3.6) in terms of the little q-Jacobi polynomials were already mentioned in [4], but without proofs and their limits (3.8) and (3.9) as $q \to 1$; a brief proof of (3.5) and (3.6) is given above for the sake of completeness.

4 Main Characteristics of q-Chebyshev Polynomials

A benefit from establishing the representations (2.19) and (2.20) for the q-Chebyshev polynomials $p_n^{(T)}(x|q)$ and $p_n^{(U)}(x|q)$ in terms of the big q-Jacobi polynomials (2.14) is that these connections enable one to deduce their main properties from the well-known properties of the latter ones, $P_n(x;a,b,c;q)$. To illustrate this point, we touch on here only three important characteristics of the q-Chebyshev polynomials $p_n^{(T)}(x|q)$ and $p_n^{(U)}(x|q)$: explicit forms of q-difference equations, discrete orthogonality relations and Rodrigues-type formulas.

It is known that the big q-Jacobi polynomials $P_n(x;a,b,c;q)$ with the parameters $a=b=-c$ are solutions of a q-difference equation:

$$\left[\left(a^2q^{n+1}+q^{-n}\right)x^2-a^2q(1+q)\right]p_n(x)=a^2q(x^2-1)\,p_n(qx)+(x^2-a^2q^2)\,p_n(q^{-1}x)\,, \tag{4.1}$$

where $p_n(x)=P_n(x;a,b,c;q)$ (see (14.5.5) on p. 439 in [7]). Hence q-difference equations for the q-Chebyshev polynomials $p_n^{(T)}(x|q)$ and $p_n^{(U)}(x|q)$ are special cases of (4.1) with the parameter $a=q^{-1/2}$ and $a=q^{1/2}$, respectively; that is,

$$\left[(q^n+q^{-n})\,x^2-(1+q)\right]p_n^{(T)}(x\,|\,q)=(x^2-1)\,p_n^{(T)}(qx\,|\,q)+(x^2-q)\,p_n^{(T)}(q^{-1}x\,|\,q)\,,$$

$$\left[(q^{n+2}+q^{-n})\,x^2-q^2(1+q)\right]p_n^{(U)}(x\,|\,q)=q^2(x^2-1)\,p_n^{(U)}(qx\,|\,q)+(x^2-q^3)\,p_n^{(U)}(q^{-1}x\,|\,q). \tag{4.2}$$

Recall also that the big q-Jacobi polynomials $P_n(x;a,b,c;q)$ with the parameters $a=b=-c$ satisfy the discrete orthogonality relation

$$\int_{-aq}^{aq}\frac{(x^2/a^2;q^2)_\infty}{(x^2;q^2)_\infty}\,P_m(x;a,a,-a;q)\,P_n(x;a,a,-a;q)\,d_qx$$

$$=2(1-q^2)\,q^{(n+1)(n+2)/2}\,\frac{(q^2;q^2)_\infty}{(a^2q^2;q^2)_\infty^2}\,(a^2q^2,-q^2;q)_\infty\,\frac{a^{2n+1}(1-a^2q)(q;q)_n}{(1-a^2q^{2n+1})(a^2q;q)_n}\,\delta_{mn}\,, \tag{4.3}$$

where the q-integral is defined as (see (14.5.2) and (1.15.7) in [7])

$$\int_{-a}^{a}f(x)\,d_qx:=a(1-q)\sum_{n=0}^{\infty}\left[f(aq^n)+f(-aq^n)\right]q^n\,.$$

For $a=q^{-1/2}$ from (4.3) one now gets at once, by employing (2.19) and (2.15), that

$$\int_{-q^{1/2}}^{q^{1/2}}\frac{(qx^2;q^2)_\infty}{(x^2;q^2)_\infty}\,p_m^{(T)}(x\,|\,q)\,p_n^{(T)}(x\,|\,q)\,d_qx=2q^{1/2}\,\frac{(-q;q)_\infty}{(q^3;q^2)_\infty}\,(q^2;q^2)_\infty^2\,c_n\,\delta_{mn}\,, \tag{4.4}$$

where

$$c_0=1\,,\qquad c_n=4^{n-1}q^{n(n+1)/2}\,\frac{(1-q^n)(q;q^2)_n^2}{(1+q^n)(q^n;q)_n^2}\,,\qquad n\ge 1\,.$$

In a like manner, when $a=q^{1/2}$ one finds from (4.3), by employing (2.20) and (2.15), that

$$\int_{-q^{3/2}}^{q^{3/2}}\frac{(q^{-1}x^2;q^2)_\infty}{(x^2;q^2)_\infty}\,p_m^{(U)}(x\,|\,q)\,p_n^{(U)}(x\,|\,q)\,d_qx=2\,q^{3/2}\,\frac{(-q;q)_\infty}{(q^3;q^2)_\infty}\,(q^2;q^2)_\infty^2\,c_n\,\delta_{mn}\,, \tag{4.5}$$

where

$$c_n=4^nq^{n(n+5)/2}\,\frac{(q;q^2)_{n+1}^2}{(1+q^{n+1})(q^{n+1};q)_{n+1}^2}\,,\qquad n\ge 0\,.$$

Another important property of the big q-Jacobi polynomials $P_n(x;a,b,c;q)$ is described by the Rodrigues-type formula

$$P_n(x;a,b,c;q)\,w(x;a,b,c;q) \,=\, \frac{[ac(1-q)]^n}{(aq,cq;q)_n}\,q^{n(n+1)}\left(\mathcal{D}_q\right)^n w(x;aq^n,bq^n,cq^n;q)\,, \tag{4.6}$$

where $\mathcal{D}_q$ is the q-derivative operator (see (1.15.1) on p. 24 in [7]) and the orthogonality weight function $w(x;a,b,c;q)$ is defined as ((14.5.10), p. 440 in [7])

$$w(x;a,b,c;q) \,:=\, \frac{(qx^2;q^2)_\infty}{(x^2;q^2)_\infty}\,. \tag{4.7}$$

Hence, from (4.6) and (4.7) it follows, upon using (2.19) and (2.20), that the Rodrigues-type formulas for the q-Chebyshev polynomials $p_n^{(T)}(x\,|\,q)$ and $p_n^{(U)}(x\,|\,q)$ are

$$p_n^{(T)}(x\,|\,q)\,\frac{(qx^2;q^2)_\infty}{(x^2;q^2)_\infty} = \Big(-2q^n\Big)^n\,\frac{(1-q)^n}{2(q^n;q)_n}\Big(\mathcal{D}_q\Big)^n\,\frac{(q^{1-2n}x^2;q^2)_\infty}{(x^2;q^2)_\infty}\,, \qquad n\geq 1\,,$$

$$p_n^{(U)}(x\,|\,q)\,\frac{(q^{-1}x^2;q^2)_\infty}{(x^2;q^2)_\infty} = \Big(-2q^{n+2}\Big)^n\,\frac{(1-q)^n}{(q^{n+2};q)_n}\Big(\mathcal{D}_q\Big)^n\,\frac{(q^{-1-2n}x^2;q^2)_\infty}{(x^2;q^2)_\infty}, \qquad n\geq 0\,. \tag{4.8}$$

In closing this section, we remark of the following. First, note that it is not difficult to determine also forward and backward shift operators and generating functions for the q-Chebyshev polynomials $p_n^{(T)}(x\,|\,q)$ and $p_n^{(U)}(x\,|\,q)$ in exactly the same way as above, but this task is left to the reader. Second, since the classical Chebyshev polynomials $T_n(x)$ and $U_n(x)$ satisfy the same three-term recurrence relation (2.3) but with different initial assignments, they are known to be interconnected by the relation

$$2\,T_n(x) \,=\, U_n(x) - U_{n-2}(x)\,, \qquad n\geq 1\,, \qquad U_{-1}(x)=0\,. \tag{4.9}$$

Hence one may wonder whether the q-Chebyshev polynomials $p_n^{(T)}(x\,|\,q)$ and $p_n^{(U)}(x\,|\,q)$ also enjoy the similar property of type (4.9), although they are governed by two distinct three-term recurrence relations (2.12) and (2.13), respectively. A link in question between the q-Chebyshev polynomials $p_n^{(T)}(x\,|\,q)$ and $p_n^{(U)}(x\,|\,q)$ turns out to be of the form

$$2\,p_n^{(T)}(x\,|\,q) \,=\, p_n^{(U)}(x\,|\,q) \,-\, \frac{4\,q}{(1+q^n)(1+q^{n-1})}\,p_{n-2}^{(U)}(x\,|\,q)\,, \qquad n\geq 1\,, \qquad p_{-1}^{(U)}(x\,|\,q)=0\,. \tag{4.10}$$

This q-extension of the classical relation (4.9) is not difficult to derive by using the explicit forms (2.4) and (2.5) of the q-polynomials $p_n^{(T)}(x\,|\,q)$ and $p_n^{(U)}(x\,|\,q)$, and the identities

$$\Big(q^{-n};q\Big)_{2l+2} = \Big(1-q^{-n}\Big)\Big(1-q^{1-n}\Big)\Big(q^{2-n};q\Big)_{2l}\,,$$

$$\Big(q^{-2n};q^2\Big)_{l+2} = \Big(1-q^{-2n}\Big)\Big(1-q^{2(1-n)}\Big)\Big(q^{2(2-n)};q^2\Big)_{l}\,,$$

for the q-shifted factorial $(z;q)_n$.

5 Concluding Remarks

We have studied in detail the main properties of two families of the basic hypergeometric ${}_2\phi_1$-polynomials, defined by (2.4) and (2.5), which represent compact forms of q-extensions of the classical Chebyshev polynomials $T_n(x)$ and $U_n(x)$. They are shown to satisfy the discrete orthogonality relations (4.4) and (4.5) on a finite interval. It should be noted that although these discrete q-Chebyshev polynomials $p_n^{(T)}(x\,|\,q)$ and $p_n^{(U)}(x\,|\,q)$ are of clear interest on their own, there is an additional motivation to study them. As we have already remarked, the q-polynomials $p_n^{(T)}(x\,|\,q)$ and $p_n^{(U)}(x\,|\,q)$ were first arisen in a paper [4], devoted mainly to the evaluation of Fourier integral transforms for q-Fibonacci and q-Lucas polynomials. It is worthwhile to emphasize that the q-Chebyshev polynomials $p_n^{(T)}(x\,|\,q)$ and $p_n^{(U)}(x\,|\,q)$ had emerged in [4] only because they are intimately associated with the very natural extensions of the Fibonacci and Lucas polynomials $p_n^{(F)}(x)$ and $p_n^{(L)}(x)$, defined as

$$p_n^{(F)}(x\,|\,q) = \mathrm{i}^{-n}\, p_n^{(U)}(\mathrm{i}x\,|\,q)\,, \qquad p_n^{(L)}(x\,|\,q) = \mathrm{i}^{-n}\, p_n^{(T)}(\mathrm{i}x\,|\,q)\,, \tag{5.1}$$

respectively. These q-extensions of the Fibonacci and Lucas polynomials are different from and simpler than those q-families, introduced and studied recently by Cigler and Zeng in [8]-[10]. Obviously, the present results also provide us with an insight into corresponding properties of the q-Fibonacci and q-Lucas polynomials $p_n^{(F)}(x\,|\,q)$ and $p_n^{(L)}(x\,|\,q)$, which are direct consequences of the links (5.1).

6 Appendix

I. In order to give a direct proof of a transformation formula

$$x^n\,{}_2\phi_1\left(q^{-n}, q^{1-n}; q^{2(1-n)} \,\middle|\, q^2; q^2x^{-2}\right) = \frac{(q;q^2)_n}{(q^n;q)_n}\,{}_3\phi_2\left(q^{-n}, q^n, x; q^{1/2}, -q^{1/2} \,\middle|\, q;q\right) \tag{6.1}$$

between ${}_2\phi_1$ (with the base q^2) and ${}_3\phi_2$ (with the base q) basic polynomials, which was stated in section 2, we start with the defining relation for the hypergeometric ${}_3\phi_2$-polynomial on the right-hand side of (6.1) and represent it first as

$${}_3\phi_2\left(q^{-n}, q^n, x; q^{1/2}, -q^{1/2} \,\middle|\, q;q\right) := \sum_{k=0}^{n} \frac{(q^{-n}, q^n, x;q)_k}{(q^{1/2}, -q^{1/2}, q;q)_k}\, q^k = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(q^n, x;q)_k}{(q;q^2)_k}\, q^{k(k+1-2n)/2}\,, \tag{6.2}$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q$ stands for the q-binomial coefficient,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}\,, \tag{6.3}$$

and we have employed the identities $(z, -z;q)_n = (z^2;q^2)_n$ and

$$\frac{(q^{-n};q)_k}{(q;q)_k} = (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1-2n)/2}\,. \tag{6.4}$$

The next step is to use the expansion

$$(x;q)_k = \sum_{l=0}^{k} q^{l(l-1)/2} \begin{bmatrix} k \\ l \end{bmatrix}_q (-x)^l \tag{6.5}$$

on the right-hand side of (6.2) and then to reverse the order of summation in it with respect to the indices k and l. This results in the relation

$$\begin{aligned}
{}_3\phi_2\left(q^{-n}, q^n, x; q^{1/2}, -q^{1/2} \,\middle|\, q;q\right) &= (q;q)_n \sum_{k=0}^{n} \frac{(-1)^k\,(q^n;q)_k}{(q;q)_{n-k}\,(q;q^2)_k}\, q^{k(k+1-2n)/2} \sum_{l=0}^{k} \frac{(-x)^l\, q^{l(l-1)/2}}{(q;q)_l\,(q;q)_{k-l}} \\
&= (q;q)_n \sum_{l=0}^{n} \frac{(-x)^l}{(q;q)_l}\, q^{l(l-1)/2} \sum_{k=l}^{n} \frac{(-1)^k\,(q^n;q)_k\, q^{k(k+1-2n)/2}}{(q;q)_{n-k}\,(q;q)_{k-l}\,(q;q^2)_k} \\
&= (q;q)_n \sum_{l=0}^{n} \frac{q^{l(l-n)}}{(q;q)_l}\, x^l \sum_{j=0}^{n-l} \frac{(-1)^j\, q^{j[j+1-2(n-l)]/2}\,(q^n;q)_{l+j}}{(q;q)_j\,(q;q)_{n-l-j}\,(q;q^2)_{l+j}}\,.
\end{aligned} \tag{6.6}$$

The last sum over the index j in (6.6) can be simplified by use of the identity (see, for example, (1.8.10) on p. 12 in [7]) $(z;q)_{n+k}=(z;q)_n\,(zq^n;q)_k$ in order to represent factors $(q^n;q)_{l+j}$ and $(q;q^2)_{l+j}$ as

$$(q^n;q)_{l+j}\,=\,(q^n;q)_l\,(q^{n+l};q)_j\,,\qquad (q;q^2)_{l+j}\,=\,(q;q^2)_l\,(q^{2l+1};q^2)_j\,.$$

Consequently,

$$\begin{aligned}
{}_3\phi_2\Big(q^{-n},q^n,x;q^{1/2},-q^{1/2}\Big|q;q\Big) &= \sum_{l=0}^{n}\begin{bmatrix} n\\ l\end{bmatrix}_q \frac{(q^n;q)_l}{(q;q^2)_l}\Big(x\,q^{l-n}\Big)^l\sum_{j=0}^{n-l}\frac{(q^{n+l},q^{l-n};q)_j\,q^j}{(q^{l+1/2},-q^{l+1/2},q;q)_j}\\
&= \sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q \frac{(q^n;q)_{n-k}}{(q;q^2)_{n-k}}\Big(x\,q^{-k}\Big)^{n-k}\sum_{j=0}^{k}\frac{(q^{2n-k},q^{-k};q)_j\,q^j}{(q^{n-k+1/2},-q^{n-k+1/2},q;q)_j}\\
&= \frac{(q^n;q)_n}{(q;q^2)_n}\sum_{k=0}^{n}q^{k(k+1-2n)/2}\begin{bmatrix} n\\ k\end{bmatrix}_q\frac{(q^{1-2n};q^2)_k}{(q^{1-2n};q)_k}\,x^{n-k}\sum_{j=0}^{k}\frac{(q^{2n-k},q^{-k};q)_j\,q^j}{(q^{n-k+1/2},-q^{n-k+1/2},q;q)_j},
\end{aligned}\tag{6.7}$$

where at the last step we have employed the identity

$$(z;q)_{n-k}\,=\,(-1)^k\,q^{k(k+1-2n)/2}\,\frac{(z;q)_n\,z^{-k}}{(q^{1-n}/z;q)_k}\,.$$

The sum over the index j in (6.7) can be now evaluated by an Andrew's terminating q-analogue of ${}_3F_2$ sum (see (II.17), p. 355 in [5])

$${}_3\phi_2\Big(q^{-k},\,a^2q^{k+1},\,0;\,a\,q,-a\,q\,\Big|\,q;q\Big)=\begin{cases}\Big(-a^2q^{m+1}\Big)^m\,\dfrac{(q;q^2)_m}{(a^2q^2;q^2)_m}\,, & k=2m\,,\\[2ex] 0\,, & k=2m+1\,,\end{cases}\tag{6.8}$$

with $a=q^{n-k-1/2}$ in the case of (6.7). Thus in the sum over the index k on the right-hand side of (6.7) only terms with the even $k=2m$, $0\le m\le\lfloor n/2\rfloor$, do give nonzero contributions and therefore

$$\begin{aligned}
{}_3\phi_2\Big(q^{-n},q^n,x;q^{1/2},-q^{1/2}\Big|q;q\Big) &= \frac{(q^n;q)_n}{(q;q^2)_n}\sum_{m=0}^{\lfloor n/2\rfloor}(-1)^m\,q^{m(1-m)}\begin{bmatrix} n\\ 2m\end{bmatrix}_q\frac{(q^{1-2n};q^2)_{2m}}{(q^{1-2n};q)_{2m}}\,\frac{(q;q^2)_m\,x^{n-2m}}{(q^{2n-4m+1};q^2)_m}\\
&= \frac{(q^n;q)_n}{(q;q^2)_n}\,x^n\sum_{m=0}^{\lfloor n/2\rfloor}(-1)^m\,q^{m(2n-3m)}\,\frac{(q^{2(m-n)+1};q^2)_m}{(q^{2n-4m+1};q^2)_m}\,\frac{(q^{-n},q^{1-n};q^2)_m}{(q^{2(1-n)},q^2;q^2)_m}\left(\frac{q^2}{x^2}\right)^m\\
&= \frac{(q^n;q)_n}{(q;q^2)_n}\,x^n\sum_{m=0}^{\lfloor n/2\rfloor}\frac{(q^{-n},q^{1-n};q^2)_m}{(q^{2(1-n)},q^2;q^2)_m}\left(\frac{q^2}{x^2}\right)^m\\
&= \frac{(q^n;q)_n}{(q;q^2)_n}\,x^n\,{}_2\phi_1\Big(q^{-n},q^{1-n};q^{2(1-n)}\Big|q^2;q^2x^{-2}\Big)\,,
\end{aligned}\tag{6.9}$$

where we have repeatedly used the relation $(z;q)_{2m}=(z,qz;q^2)_m$ at the second step and a readily verified identity

$$(-1)^m\,q^{m(2n-3m)}\,(q^{2(m-n)+1};q^2)_m\,=\,(q^{2n-4m+1};q^2)_m\tag{6.10}$$

at the third one. This completes the proof of required transformation formula (6.1).

II. In a similar vein, to prove a second transformation formula

$$x^n {}_2\phi_1\left(q^{-n}, q^{1-n}; q^{-2n} \middle| q^2; q^2 x^{-2}\right) = \frac{(q^3;q^2)_n}{(q^{n+2};q)_n} {}_3\phi_2\left(q^{-n}, q^{n+2}, x; q^{3/2}, -q^{3/2} \middle| q; q\right), \tag{6.11}$$

we start with the defining relation for the basic hypergeometric polynomial ${}_3\phi_2$ on the right-hand side of (6.11) and evaluate first that

$$\begin{aligned}{}_3\phi_2\left(q^{-n}, q^{n+2}, x; q^{3/2}, -q^{3/2} \middle| q; q\right) &:= \sum_{k=0}^{n} \frac{(q^{-n}, q^{n+2}, x; q)_k}{(q^{3/2}, -q^{3/2}, q; q)_k} q^k \\ &= \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(q^{n+2}, x; q)_k}{(q^3;q^2)_k} q^{k(k+1-2n)/2}, \end{aligned} \tag{6.12}$$

by using the relations (6.3) and (6.4). So the next step is to employ the expansion (6.5) on the right-hand side of (6.12) and then to reverse the order of summation in it with respect to the indices k and l. This gives

$$\begin{aligned}{}_3\phi_2\left(q^{-n}, q^{n+2}, x; q^{3/2}, -q^{3/2} \middle| q; q\right) &= (q;q)_n \sum_{k=0}^{n} \frac{(-1)^k (q^{n+2};q)_k}{(q;q)_{n-k} (q^3;q^2)_k} q^{k(k+1-2n)/2} \sum_{l=0}^{k} \frac{(-x)^l q^{l(l-1)/2}}{(q;q)_l (q;q)_{k-l}} \\ &= (q;q)_n \sum_{l=0}^{n} \frac{(-x)^l}{(q;q)_l} q^{l(l-1)/2} \sum_{k=l}^{n} \frac{(-1)^k (q^{n+2};q)_k q^{k(k+1-2n)/2}}{(q;q)_{n-k} (q;q)_{k-l} (q^3;q^2)_k} \\ &= (q;q)_n \sum_{l=0}^{n} \frac{q^{l(l-n)}}{(q;q)_l} x^l \sum_{j=0}^{n-l} \frac{(-1)^j q^{j[j+1-2(n-l)]/2} (q^{n+2};q)_{l+j}}{(q;q)_j (q;q)_{n-l-j} (q^3;q^2)_{l+j}} \\ &= \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \frac{(q^{n+2};q)_l}{(q^3;q^2)_l} \left(x q^{l-n}\right)^l \sum_{j=0}^{n-l} \frac{(q^{n+l+2}, q^{l-n}; q)_j q^j}{(q^{l+3/2}, -q^{l+3/2}, q; q)_j} \\ &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(q^{n+2};q)_{n-k}}{(q^3;q^2)_{n-k}} \left(x q^{-k}\right)^{n-k} \sum_{j=0}^{k} \frac{(q^{2n+2-k}, q^{-k}; q)_j q^j}{(q^{n-k+3/2}, -q^{n-k+3/2}, q; q)_j} \\ &= \frac{(q^{n+2};q)_n}{(q^3;q^2)_n} \sum_{k=0}^{n} q^{k(k+1-2n)/2} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(q^{-2n-1};q^2)_k}{(q^{-2n-1};q)_k} x^{n-k} \sum_{j=0}^{k} \frac{(q^{2n+2-k}, q^{-k}; q)_j q^j}{(q^{n-k+3/2}, -q^{n-k+3/2}, q; q)_j}. \end{aligned} \tag{6.13}$$

The last sum over the index j represents

$${}_3\phi_2\left(q^{-k}, q^{2n+2-k}, 0; q^{n-k+3/2}, -q^{n-k+3/2} \middle| q; q\right)$$

and can be therefore evaluated by (6.8), but with the parameter $a = q^{n-k+1/2}$. Hence only terms with the even $k = 2m$ do contribute into the second sum over the index k in (6.13) and it thus reduces to the expression

$$\begin{aligned}{}_3\phi_2\left(q^{-n}, q^{n+2}, x; q^{3/2}, -q^{3/2} \middle| q; q\right) &= \frac{(q^{n+2};q)_n}{(q^3;q^2)_n} \sum_{m=0}^{\lfloor n/2 \rfloor} (-1)^m q^{m(3-m)} \begin{bmatrix} n \\ 2m \end{bmatrix}_q \frac{(q^{-2n-1};q^2)_{2m}}{(q^{-2n-1};q)_{2m}} \frac{(q;q^2)_m x^{n-2m}}{(q^{2n-4m+3};q^2)_m} \\ &= \frac{(q^{n+2};q)_n}{(q^3;q^2)_n} x^n \sum_{m=0}^{\lfloor n/2 \rfloor} \left(-q^{2n+2-3m}\right)^m \frac{(q^{2m-2n-1};q^2)_m}{(q^{2n-4m+3};q^2)_m} \frac{(q^{-n}, q^{1-n}; q^2)_m}{(q^{-2n}, q^2; q^2)_m} \left(\frac{q^2}{x^2}\right)^m \\ &= \frac{(q^{n+2};q)_n}{(q^3;q^2)_n} x^n \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{(q^{-n}, q^{1-n}; q^2)_m}{(q^{-2n}, q^2; q^2)_m} \left(\frac{q^2}{x^2}\right)^m = \frac{(q^{n+2};q)_n}{(q^3;q^2)_n} x^n {}_2\phi_1\left(q^{-n}, q^{1-n}; q^{-2n} \middle| q^2; q^2 x^{-2}\right), \end{aligned} \tag{6.14}$$

where at the penultimate step we have used the same identity (6.10), but with n replaced by $n+1$. This completes the proof of the transformation formula (6.11).

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