Fibonacci, Chebyshev, and Orthogonal Polynomials

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1. INTRODUCTION. In 1202 Leonardo of Pisa, otherwise known as Fibonacci, published the text *Liber abaci* in which he posed the the following problem: *A man puts one pair of rabbits in a certain place entirely surrounded by a wall. How many pairs of rabbits can be produced from that pair in a year, if the nature of these rabbits is such that every month each pair bears a new pair which from the second month on becomes productive*? Assuming that the initial pair starts breeding only in the second month, the solution of this problem leads to what is now known as the *Fibonacci sequence* $\langle F_n \rangle$ defined by

$$F_{-1} = 1, \quad F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_n + F_{n+1} \quad (n \geq 0).$$

It is well known that the F_n satisfy many remarkable identities; for example,

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n, \tag{1.1}$$

$$F_{m+n+1} = F_{m+1}F_{n+1} + F_m F_n, \tag{1.2}$$

$$F_{n+2} = 1 + F_1 + F_2 + \cdots + F_n. \tag{1.3}$$

The first of these was proved by Cassini in 1680, and the second, which is the basis of many other identities, is sometimes called the *Fibonacci shift formula*. The Fibonacci numbers are also known to have many interesting divisibility properties, the simplest of which is

$$\gcd(F_{n+2}, F_{n+1}) = \gcd(F_{n+1}, F_n) = \cdots = \gcd(F_2, F_1) = 1.$$

Also, when $r \geq 1$ but $r \neq 2$, F_r divides F_s if and only if r divides s.

In this paper we ask to what extent the identities and the divisibility properties enjoyed by the Fibonacci numbers are also shared by solutions of other recurrence relations. Our discussion encompasses recurrence relations whose coefficients depend on n, recurrence relations whose coefficients are independent of n but depend on a parameter x (and so have polynomial solutions), and a combination of both of these. Among the best known examples of polynomial solutions are the *Chebyshev polynomials*, namely, the polynomials $T_n(x)$ and $U_n(x)$ for which

$$T_n(\cos\theta) = \cos(n\theta), \qquad U_n(\cos\theta) = \frac{\sin[(n+1)\theta]}{\sin\theta},$$

and the *Legendre polynomials*, which are given by the recurrence relation

$$(n+1)p_{n+1}(x) = (2n+1)xp_n(x) + np_{n-1}(x). \tag{1.4}$$

More generally, as any sequence of orthogonal polynomials satisfies some second-order recurrence relation, we consider these as well. We shall see that, although the

Fibonacci sequence is simpler than other recurrence relations, it is perhaps not quite so special as is sometimes made to appear. Throughout, we shall pay special attention to the *primary solution* of a recurrence relation: this is the solution x_n ($n \geq 0$) of a recurrence relation with initial values $x_0 = 0$ and $x_1 = 1$. If we consider a second-order recurrence relation with constant coefficients in $\mathbb{C}$ and if the auxiliary equation has roots α and β, then the primary solution is $(\alpha^n - \beta^n)/(\alpha - \beta)$ if $\alpha \neq \beta$ and $n\alpha^{n-1}$ when $\alpha = \beta$.

In section 2 we examine the general solution of the Fibonacci relation $x_{n+2} = x_{n+1} + x_n$ and, while this section could be omitted, we feel that it will help the reader to appreciate some of the ideas in the paper. Section 3 is a short discussion of Chebyshev polynomials; these play a fundamental role in this work. In section 4 we introduce our main result, which gives the shift formula for solutions of recurrence relations whose coefficients depend on n, and we prove this in section 5. Sections 6, 7, and 8 are concerned with divisibility properties of solutions of recurrence relations. Finally, section 9 contains a brief summary of the theory of orthogonal polynomials.

2. THE FIBONACCI RELATION. Instead of restricting ourselves to the Fibonacci sequence, we shall consider all solutions of the *Fibonacci relation*

$$x_{n+2} = x_n + x_{n+1}. \tag{2.1}$$

The Fibonacci sequence F_n is the *primary solution* of this relation; that is (by definition), the solution with initial values $x_0 = 0$ and $x_1 = 1$. It is easy to see that, given any x_0 and x_1, we have

$$x_n = x_0 F_{n-1} + x_1 F_n, \tag{2.2}$$

for the right-hand side satisfies (2.1) and takes the values x_0 and x_1 when n is 0 and 1, respectively. The converse result (namely, the expression that gives F_n in terms of two consecutive x_j from a given solution x_n) is more subtle. If $x_0 x_2 \neq x_1^2$, then a similar argument gives

$$F_n = \left(\frac{x_0}{x_0 x_2 - x_1^2}\right) x_{n+1} - \left(\frac{x_1}{x_0 x_2 - x_1^2}\right) x_n. \tag{2.3}$$

The condition $x_0 x_2 - x_1^2 \neq 0$ is precisely the condition that the two sequences $\langle x_n \rangle$ and $\langle y_n \rangle$, where $y_n = x_{n+1}$, be linearly independent solutions of (2.1). If $x_0 x_2 = x_1^2$ then, since $x_2 = x_0 + x_1$, we see that $x_n = \sigma^n$, where σ is either the golden ratio or its negative reciprocal, in which case F_n cannot be expressed as a linear combination of x_n and x_{n+1}.

In the nineteenth century Édouard Lucas studied the Fibonacci sequence and introduced what are now known as the *Lucas numbers* L_n, which are given by $L_0 = 2$, $L_1 = 1$, and $L_{n+2} = L_n + L_{n+1}$. (On a historical note, Lucas died a somewhat bizarre death as the result of a freak accident at a banquet when a plate was dropped and a piece flew up and cut his cheek. He died a few days later of erysipelas, an acute infection of the skin.) The relationship between the Lucas numbers L_n and the Fibonacci numbers F_n is given by $L_n = F_{n-1} + F_{n+1}$ and $5F_n = L_{n+1} + L_{n-1}$, so it is easy to transfer information between F_n and L_n. More generally, it is obvious (by induction) that the analogue of (1.3) for any solution of (2.1) is

$$x_{n+2} = x_2 + (x_1 + x_2 + \cdots + x_n).$$

In general, however, the identities satisfied by an arbitrary solution x_n of (2.1) tend to be more complicated than the corresponding identity for the Fibonacci numbers, and we shall soon see why this is so. For a discussion of the very many identities satisfied by any solution $\langle x_n \rangle$ of (2.1), see [**10**].

It is instructive to see the identities (1.1), (1.2), and (1.3) proved for general solutions of the general constant coefficient recurrence relation

$$x_{n+2} = a\,x_{n+1} + b\,x_n \quad (n \geq 0) \tag{2.4}$$

using ideas from dynamical systems rather than by the proofs based on induction or combinatorics (see, for example, [**2**]) that are usually used for the Fibonacci numbers. First, any solution $\langle x_n \rangle$ of (2.4) satisfies

$$\begin{pmatrix} x_{n+2} & x_{n+1} \\ x_{n+1} & x_n \end{pmatrix} = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_{n+1} & x_n \\ x_n & x_{n-1} \end{pmatrix},$$

so that

$$\begin{pmatrix} x_{n+1} & x_n \\ x_n & x_{n-1} \end{pmatrix} = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} x_2 & x_1 \\ x_1 & x_0 \end{pmatrix}. \tag{2.5}$$

This provides the generalization of (1.1) to any solution of (2.4), namely,

$$x_{n+1}x_{n-1} - x_n^2 = (-b)^{n-1}(x_0x_2 - x_1^2).$$

Identity (2.5) also shows why the Fibonacci sequence (along with certain other primary solutions) plays a special role here. The primary solution $\langle x_n \rangle$ of (2.4) satisfies

$$\begin{pmatrix} x_2 & x_1 \\ x_1 & x_0 \end{pmatrix} = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}.$$

When $b = 1$, *the primary solution is the only solution for which the matrix of initial values coincides with the iterated matrix*, that is, for which

$$\begin{pmatrix} x_2 & x_1 \\ x_1 & x_0 \end{pmatrix} = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix},$$

and when this happens we have the simpler formula

$$\begin{pmatrix} x_{n+1} & x_n \\ x_n & x_{n-1} \end{pmatrix} = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}^{n}. \tag{2.6}$$

For example, whereas

$$\begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n}, \tag{2.7}$$

we have

$$\begin{pmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}.$$

The identity (2.7) contains the shift formula (1.2) for the Fibonacci sequence, for it implies that

$$\begin{pmatrix} F_{n+q+1} & F_{n+q} \\ F_{n+q} & F_{n+q-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^q = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} \begin{pmatrix} F_{q+1} & F_q \\ F_q & F_{q-1} \end{pmatrix}.$$

By (2.6), the same reasoning holds for the primary solution of (2.4) when $b = 1$.

There is also a shift formula for an arbitrary solution $\langle x_n \rangle$ of (2.4), provided that $x_0 x_2 - x_1^2 \neq 0$. Indeed, from (2.5) we obtain the identity

$$\begin{pmatrix} x_{m+n+2} & x_{m+n+1} \\ x_{m+n+1} & x_{m+n} \end{pmatrix} = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}^{m+n} \begin{pmatrix} x_2 & x_1 \\ x_1 & x_0 \end{pmatrix}$$
$$= \begin{pmatrix} x_{n+2} & x_{n+1} \\ x_{n+1} & x_n \end{pmatrix} \begin{pmatrix} x_2 & x_1 \\ x_1 & x_0 \end{pmatrix}^{-1} \begin{pmatrix} x_{m+2} & x_{m+1} \\ x_{m+1} & x_m \end{pmatrix},$$

which yields

$$(x_0 x_2 - x_1^2) \begin{pmatrix} x_{m+n+2} & * \\ * & * \end{pmatrix} = \begin{pmatrix} x_{n+2} & x_{n+1} \\ * & * \end{pmatrix} \begin{pmatrix} x_0 & -x_1 \\ -x_1 & x_2 \end{pmatrix} \begin{pmatrix} x_{m+2} & * \\ x_{m+1} & * \end{pmatrix}.$$

This gives the shift formula for the general solution $\langle x_n \rangle$ of (2.4), namely (with $q = m + 1$),

$$(x_0 x_2 - x_1^2) x_{n+q+1} = -b\big[x_{-1} x_{n+1} x_{q+1} + x_1 x_n x_q\big] + b x_0 \big[x_n x_{q+1} + x_q x_{n+1}\big].$$

If $x_0 = 0$ and $x_1 = 1$, then $x_0 x_2 - x_1^2 = -1$ and $x_{-1} = 1/b$, which yields

$$x_{q+n+1} = x_{q+1} x_{n+1} + b x_q x_n. \tag{2.8}$$

This is the shift formula for the primary solution of the general constant coefficient recurrence relation (2.4). Obviously, there are even more identities that can be derived in a similar way from the identity $A^{\ell+m+n} = A^\ell A^m A^n$, and so on, for any matrix A.

Finally, we obtain the analogue of (1.3) for the general solution of (2.4). We write

$$P = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}, \qquad Q = a^{-1} P,$$

and use the identity

$$(I + Q + \cdots + Q^m)(I - Q) = I - Q^{m+1}.$$

The Cayley-Hamilton theorem gives $(I - Q)^{-1} = -(a/b)P = -(a^2/b)Q$, so

$$I + Q + \cdots + Q^m = (a^2/b)\big(Q^{m+2} - Q\big). \tag{2.9}$$

We now rewrite (2.5) in the form

$$X_{n+2} = P^n X_2 = a^n Q^n X_2, \qquad X_{n+2} = \begin{pmatrix} x_{n+2} & x_{n+1} \\ x_{n+1} & x_n \end{pmatrix},$$

and infer from (2.9) that

$$\sum_{k=0}^{m} a^{-k} X_{k+2} = (1/a^m b) X_{m+4} - (a/b) X_3.$$

We let $m = n - 2$ and consider only the (1, 1)-entries; this gives

$$x_2 + a^{-1} x_3 + \cdots a^{-(n-2)} x_n = (1/a^{n-2} b) x_{n+2} - (1/ab) x_3.$$

We next divide through by a^2 and use $x_3 = ax_2 + bx_1$ to obtain

$$\frac{ax_2 + bx_1}{ab} + \frac{x_2}{a^2} + \cdots + \frac{x_n}{a^n} = \frac{x_{n+2}}{a^n b},$$

which we express as a generalized version of (1.3), specifically,

$$\frac{x_{n+2}}{a^n b} = \frac{x_2}{b} + \left(\frac{x_1}{a} + \frac{x_2}{a^2} + \cdots + \frac{x_n}{a^n}\right).$$

For completeness, we end this section with a proof that, for positive integers r and s with $r \neq 2$, F_r divides F_s if and only if r divides s. First, we extend the definition of F_n in the obvious way to all integers n. Then any divisor of F_m and F_{m+1} is also a divisor of F_{m-1} and F_{m+2}, hence a divisor of all F_n. As $F_1 = 1$, we see that F_m and F_{m+1} are coprime. Next, choose a positive integer k. We claim that the set $\mathbb{Z}(k)$ of integers n such that F_n is a multiple of k is a subgroup of $\mathbb{Z}$. Because $F_0 = 0$, which is a multiple of k, we see that 0 belongs to $\mathbb{Z}(k)$. Next, we write (1.2) in the form

$$F_{m+n} = F_m F_{n+1} + F_{m-1} F_n,$$

and this shows that $m + n$ lies in $\mathbb{Z}(k)$ whenever m and n do. The same formula demonstrates that, if m and $m + n$ are members of $\mathbb{Z}(k)$, then k divides $F_{m-1}F_n$. However, k divides F_m, and F_m and F_{m+1} are coprime. We conclude that k divides F_n, which proves that $\mathbb{Z}(k)$ is closed under differences. Thus $\mathbb{Z}(k)$ is a subgroup of $\mathbb{Z}$ and so is of the form $d\mathbb{Z}$ for some positive integer d. We have now shown that for each k there exists a d such that k divides F_n if and only if d divides n. Now let $k = F_s$, and let q be the corresponding value of d. Then F_s divides F_n if and only if q divides n. Clearly this implies that q divides s and that F_s divides F_q. These observations imply that $s \geq q$ and that, unless $s = 2$, $q \geq s$. We have now verified that, when $s \neq 2$, F_s divides F_n if and only if s divides n.

3. CHEBYSHEV POLYNOMIALS. There is a good reason why we should expect solutions of different constant coefficient recurrence relations to have similar properties, and to see this we must look at Chebyshev polynomials. Recall from the introduction that the nth Chebyshev polynomials of the first and second kinds are the polynomials $T_n(x)$ and $U_n(x)$, respectively, such that

$$T_n(\cos\theta) = \cos(n\theta), \qquad U_n(\cos\theta) = \frac{\sin[(n+1)\theta]}{\sin\theta}.$$

The formulas for $\cos(n\theta \pm \theta)$ and $\sin(n\theta \pm \theta)$ show that both $\langle T_n \rangle$ and $\langle U_n \rangle$ satisfy the constant coefficient recurrence relation (see section 4 for clarification of our use of "constant coefficient" in this context)

$$y_{n+1}(x) = 2x y_n(x) - y_{n-1}(x), \tag{3.1}$$

with the respective initial conditions

$$T_{-1}(x) = x, \quad T_0(x) = 1, \quad T_1(x) = x; \quad U_{-1}(x) = 0, \quad U_0(x) = 1, \quad U_1(x) = 2x.$$

We now show that *the Chebyshev polynomials of the second kind are the universal primary solution of any constant coefficient second-order recurrence relation in an arbitrary integral domain.* Put more simply, this means that every constant coefficient recurrence relation can be identified with the relation satisfied by the Chebyshev polynomials of the second kind with parameter x *at a particular value of the parameter* x.

It is more convenient to consider the polynomials $\hat{U}_n$, where $\hat{U}_n(x) = U_{n-1}(x)$, for $\hat{U}_n$ is the primary solution of the recurrence relation (3.1). We also put $\hat{T}_n(x) = T_{n-1}(x)$. We then have

$$\hat{T}_{n+q+1}(x) = \hat{U}_{q+1}(x)\hat{T}_{n+1}(x) - \hat{U}_q(x)\hat{T}_n(x)$$

and

$$\hat{U}_{n+q+1}(x) = \hat{U}_{q+1}(x)\hat{U}_{n+1}(x) - \hat{U}_q(x)\hat{U}_n(x).$$

These two equations are just the shift formula (2.8) when we work (as we shall do later) in the ring of complex polynomials rather than in $\mathbb{C}$. In fact, these equations are equivalent to the trigonometric identities

$$\sin\theta\ \cos[(n+q+1)\theta] = \sin[(q+1)\theta]\ \cos[(n+1)\theta] - \sin(q\theta)\ \cos(n\theta),$$
$$\sin\theta\ \sin[(n+q+1)\theta] = \sin[(q+1)\theta]\ \sin[(n+1)\theta] - \sin(q\theta)\ \sin(n\theta).$$

Let us now consider the primary solution $\langle z_n \rangle$ of the constant coefficient relation $x_{n+2} = ax_{n+1} + bx_n$ $(n \geq 0)$. We assume that $b \neq 0$ and also that we are working in an algebraically closed field $\mathbb{F}$. Then there exists a ρ in $\mathbb{F}$ such that $\rho^2 = -b$, and since $\rho \neq 0$, we see that ρ^{-1} exists. We write $y_n = \rho^{1-n} z_n$ and note that

$$y_{n+2} = (a/\rho)y_{n+1} - y_n, \quad y_0 = 0, \quad y_1 = 1.$$

On the basis of (3.1) we conclude that $y_n = \hat{U}_n(a/\rho)$. Thus $z_n = \rho^{n-1}\hat{U}_n(a/2\rho)$, which establishes the next result.

Theorem 3.1. *The primary solution $\langle z_n \rangle$ of the constant coefficient recurrence relation $x_{n+2} = ax_{n+1} + bx_n$ $(n \geq 0)$ in an integral domain D is given by $z_{n+1} = \rho^n U_n(a/2\rho)$, where ρ is the solution of $\rho^2 = -b$ in the algebraic closure of the field of fractions of D.*

In view of Theorem 3.1 it is of interest to note that

$$U_n(x) = \sum_{j=0}^{[n/2]} (-1)^j \binom{n-j}{j} x^{n-2j}.$$

As an example, the Fibonacci sequence is the primary solution when $a = b = 1$ and $\rho = i$, so $F_{n+1} = i^n U_n(-i/2)$, which simplifies to

$$F_{n+1} = \sum_{j=0}^{[n/2]} \binom{n-j}{j}.$$

More generally, we see that the primary solution $\langle z_n \rangle$ of the relation $x_{n+2} = ax_{n+1} + bx_n$ is given by

$$z_{n+1} = \sum_{j=0}^{[n/2]} (-1)^j \binom{n-j}{j} a^{n-2j} \rho^j.$$

4. SOME NOTATION AND TERMINOLOGY. We now consider second-order recurrence relations of the type

$$x_{n+2} = a_n(x)x_{n+1} + b_n(x)x_n, \tag{4.1}$$

where the coefficients a_n and b_n may depend on n but may also be polynomials in the variable x. We say that this relation has *constant coefficients* when a_n and b_n are independent of n, but not necessarily independent of x. For example, the relation

$$x_{n+2}(x) = (x^4 - 3x)\, x_{n+1}(x) + (x^2 + x)\, x_n(x)$$

has constant coefficients. Of course, from an analytic point of view these coefficients are not constant, but from a dynamical point of view they are, for the process by which we construct an x_j from the previous x_i is independent of j. Moreover, from an algebraic point of view, if we work within the ring of polynomials in x and if the coefficients are independent of n, then they are fixed elements in this ring. By contrast, the recurrence relation (1.4) satisfied by the Legendre polynomials does not have constant coefficients. Throughout the discussion that follows, we always extend a solution $\langle x_n \rangle$ of a constant coefficient recurrence relation to all integers n by using the fact that if we have two consecutive values x_n and x_{n+1}, the constant coefficient recurrence relation generates all previous values $x_{n-1}, x_{n-2}, \dots$ and all subsequent values $x_{n+2}, x_{n+3}, \dots$. If a recurrence relation has variable coefficients, then such an extension will depend on how we extend the sequences of coefficients to all values of n.

From now on we shall write the coefficients in (4.1) as a_n and b_n with the implicit understanding that these may be polynomials in some variable x. Each choice of values for x_0 and x_1 (which may also be polynomials in x) provides a solution $x_0, x_1, x_2, \dots$ of (4.1) that is defined inductively by (4.1). The solution with initial values $x_0 = 0$ and $x_1 = 1$ has a special role to play in our discussion; this is the *primary solution* of (4.1). Throughout, we reserve the symbol z_n for the primary solution.

Sometimes it will be helpful to refer explicitly to the sequences $A = (a_0, a_1, \dots)$ and $B = (b_0, b_1, \dots)$ of coefficients in (4.1). In this case, the primary solution, for example, will be denoted by $\langle z_n(x; A, B) \rangle$ and the general solution by $\langle x_n(x; A, B) \rangle$. Later we shall have reason to pass from the given recurrence relation (4.1) to the related recurrence relation

$$x_{n+2} = a_{n+1}(x)x_{n+1} + b_{n+1}(x)x_n, \tag{4.2}$$

a process that is best described in terms of the shift map σ on the space of sequences. The map σ is defined by

$$(u_0, u_1, u_2, \dots) \mapsto (u_1, u_2, u_3, \dots),$$

and if we write $A^r = \sigma^r(A)$, and similarly for B, we find that the primary solution of (4.2) is $\langle z_n(x; A^1, B^1) \rangle$. More generally, we can apply the shift map any number of times and in this way arrive at the primary solution $\langle z_n(x; A^r, B^r)\rangle$ of the recurrence relation

$$x_{n+2} = a_{n+r}(x)x_{n+1} + b_{n+r}(x)x_n.$$

To avoid possible confusion, we note that $A^1 \neq A$ (unless A is a constant sequence); in fact, $A^0 = A$.

Our major objective in this paper is to obtain a far-reaching generalization of the Fibonacci shift formula (1.2) that is applicable to any linear second-order recurrence relation. Our generalization of (1.2) involves terms of the form $\langle z_n(x; A^r, B^r)\rangle$ (which is why we have just introduced them), and the result that we prove is as follows:

Theorem 4.1. *If R is an integral domain, if the coefficients in* (4.1) *belong to the polynomial ring $R[x]$, and if $\langle x_n(x; A, B)\rangle$ is a solution of* (4.1) *in $R[x]$ with initial values x_0 and x_1, then*

$$x_{n+q+1}(A, B) = z_{q+1}(A^n, B^n)\, x_{n+1}(A, B) + b_n\, z_q(A^{n+1}, B^{n+1})\, x_n(A, B),$$

where $\langle z_n(x; A, B)\rangle$ is the primary solution of (4.1).

Of course, in the constant coefficient case, say $a_n = a$ and $b_n = b$ for all n, this shift map has no effect and we obtain a general form of the shift formula that involves two solutions, namely,

$$x_{n+q+1} = z_{q+1}\, x_{n+1} + b\, z_q\, x_n.$$

This shows, for example, that the Lucas numbers L_n satisfy the relation

$$L_{n+q+1} = L_{q+1} F_{n+1} + L_q F_q.$$

For example, $76 = L_9 = L_5F_5 + L_4F_4 = (11 \times 5) + (7 \times 3)$. Moreover, if we take $x_n = z_n$ and $b = 1$ in Theorem 4.1, we obtain

$$z_{n+q+1} = z_{q+1}\, z_{n+1} + z_q\, z_n,$$

and even this is more general than (1.2) because $z_n = F_n$ only if $a = 1$.

Since any orthogonal sequence of polynomials satisfies a linear second-order recurrence relation with polynomial coefficients (see section 9), it follows that Theorem 4.1 is applicable to any sequence of orthogonal polynomials. In particular, Theorem 4.1 provides us with a "shift formula" for sequences of orthogonal polynomials, and these include the Chebyshev and Legendre polynomials. It is an intriguing observation that obtaining a new recurrence relation by translating the sequences of coefficients is known to be a fruitful idea in other contexts within the theory of orthogonal polynomials, where the solutions of the new relations are known as the *associated polynomials*.

We now comment on the algebraic structures that underlie our discussion. If we are studying the Fibonacci sequence we can work entirely within the ring $\mathbb{Z}$ of integers. However, $\mathbb{Z}$ is embedded in the algebraically closed field $\mathbb{C}$ of complex numbers, and if we work in $\mathbb{C}$ we can establish such results as *Binet's formula* (published in 1843):

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

where $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$. This gives an explicit expression for F_n in terms of elements in $\mathbb{C}$ that are not in $\mathbb{Z}$; nevertheless, we know that the F_n are in $\mathbb{Z}$, despite the fact that α and β are not. In the general case we can take a given ring R and work within the ring $R[x]$ of polynomials in the variable x whose coefficients lie in R. Throughout, we assume that "ring" signifies a commutative ring with a multiplicative identity 1. An integral domain D is a ring with no zero divisors, and if D is an integral domain, then so is $D[x]$. Now it is known (i) that every integral domain can be embedded in a field and (ii) that every field can be embedded in an algebraically closed field (see [**5**, pp. 213, 317]). Thus, from this perspective, we may consider an integral domain D to be embedded in an algebraically closed field $\mathbb{F}$, and we may work within $\mathbb{F}$. This means that the standard elementary methods for solving difference equations are valid in these more general circumstances; in particular, they hold when the coefficients of (4.1) are, say, complex polynomials in x.

5. RECURRENCE RELATIONS IN A RING. We now study the second-order recurrence relation

$$x_{n+2} = a_n(x)\,x_{n+1} + b_n(x)\,x_n \quad (n \geq 0), \tag{5.1}$$

(with variable coefficients) in the context of a ring $R[x]$ of polynomials in a variable x in the sense indicated earlier, and prove Theorem 4.1. Although in general we cannot "solve" the recurrence relation (5.1) with variable coefficients, Theorem 4.1 does give some valuable information about its solutions.

Proof of Theorem 4.1. The relation (5.1) is equivalent to

$$\begin{pmatrix} x_{n+2} & x_{n+1} \\ x_{n+1} & x_n \end{pmatrix} = M_n \begin{pmatrix} x_{n+1} & x_n \\ x_n & x_{n-1} \end{pmatrix}, \qquad M_n = \begin{pmatrix} a_n & b_n \\ 1 & 0 \end{pmatrix},$$

so

$$\begin{pmatrix} x_{n+2} & x_{n+1} \\ x_{n+1} & x_n \end{pmatrix} = M_n \cdots M_1 \begin{pmatrix} x_2 & x_1 \\ x_1 & x_0 \end{pmatrix}.$$

For a fixed n and for $q = 1, 2, \ldots$ we write

$$M_{n+q-1} \cdots M_n = \begin{pmatrix} \alpha_q & \beta_q \\ \gamma_q & \delta_q \end{pmatrix}.$$

Then

$$\begin{pmatrix} x_{n+q+1} & x_{n+q} \\ x_{n+q} & x_{n+q-1} \end{pmatrix} = \begin{pmatrix} \alpha_q & \beta_q \\ \gamma_q & \delta_q \end{pmatrix} \begin{pmatrix} x_{n+1} & x_n \\ x_n & x_{n-1} \end{pmatrix},$$

ensuring that any solution $\langle x_n \rangle$ satisfies

$$x_{n+q+1} = \alpha_q\, x_{n+1} + \beta_q\, x_n. \tag{5.2}$$

We now identify α_q and β_q in terms of the primary solution $\langle z_n \rangle$ of (5.1). Recall that the latter satisfies

$$z_0 = 0, \quad z_1 = 1, \quad z_{n+2} = a_n(x)\,z_{n+1} + b_n(x)\,z_n \quad (n = 0, 1, \ldots),$$

and

$$\begin{pmatrix} z_{n+2} & z_{n+1} \\ z_{n+1} & z_n \end{pmatrix} = M_n \cdots M_1 \begin{pmatrix} z_2 & 1 \\ 1 & 0 \end{pmatrix} = M_n \cdots M_0 \begin{pmatrix} 1 & 0 \\ 0 & z_{-1} \end{pmatrix},$$

where we set $z_{-1} = 0$. This shows that

$$\begin{pmatrix} z_{n+2} \\ z_{n+1} \end{pmatrix} = \begin{pmatrix} z_{n+2} & z_{n+1} \\ z_{n+1} & z_n \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = M_n \cdots M_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

We have already introduced the notation $z_n(A^r, B^r)$ in section 4. In a similar way we write $M_j(A, B)$ for M_j, so that $M_j(A^n, B^n) = M_{j+n}(A, B)$. As

$$\begin{pmatrix} \alpha_q & \beta_q \\ \gamma_q & \delta_q \end{pmatrix} = M_{n+q-1}(A, B) \cdots M_n(A, B) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

we see that

$$\begin{aligned} \begin{pmatrix} \alpha_q \\ \gamma_q \end{pmatrix} &= \begin{pmatrix} \alpha_q & \beta_q \\ \gamma_q & \delta_q \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= M_{n+q-1}(A, B) \cdots M_n(A, B) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= M_{q-1}(A^n, B^n) \cdots M_0(A^n, B^n) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} z_{q+1}(A^n, B^n) \\ z_q(A^n, B^n) \end{pmatrix}, \end{aligned}$$

whence

$$\alpha_q = z_{q+1}(A^n, B^n). \tag{5.3}$$

Similarly,

$$\begin{pmatrix} \alpha_q & \beta_q \\ \gamma_q & \delta_q \end{pmatrix} = M_{n+q-1}(A, B) \cdots M_{n+1}(A, B) \begin{pmatrix} a_n & b_n \\ 1 & 0 \end{pmatrix},$$

which leads to

$$\begin{aligned} \begin{pmatrix} \beta_q \\ \delta_q \end{pmatrix} &= \begin{pmatrix} \alpha_q & \beta_q \\ \gamma_q & \delta_q \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= M_{n+q-1}(A, B) \cdots M_{n+1}(A, B) \begin{pmatrix} b_n \\ 0 \end{pmatrix} \\ &= b_n \, M_{n+q-1}(A, B) \cdots M_{n+1}(A, B) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= b_n \, M_{q-2}(A^{n+1}, B^{n+1}) \cdots M_0(A^{n+1}, B^{n+1}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= b_n \begin{pmatrix} z_q(A^{n+1}, B^{n+1}) \\ z_{q-1}(A^{n+1}, B^{n+1}) \end{pmatrix}. \end{aligned}$$

We conclude that $\beta_q = b_n z_q(A^{n+1}, B^{n+1})$. This, along with (5.2) and (5.3), completes the proof of Theorem 4.1. ■

As we have already mentioned in section 4, Theorem 4.1 has the following corollary:

Corollary 5.1. *Let $\langle z_n \rangle$ be the primary solution and $\langle x_n \rangle$ an arbitrary solution of the recurrence relation $y_{n+2} = ay_{n+1} + by_n$ in a ring R. Then $x_{n+q+1} = z_{q+1}\, x_{n+1} + b\, z_q\, x_n$ for $q \geq 1$.*

6. DIVISIBILITY PROPERTIES. The ideas in this second part of the paper originated in a study of the divisibility properties of sequences of orthogonal polynomials, especially the Chebyshev polynomials of the second kind. We know that for the Fibonacci sequence F_r divides F_s if r divides s. Surely it is not an accident that this property is also shared by the modified sequence $\hat{U}_n$ of Chebyshev polynomials of the second kind? We now examine these divisibility properties in detail and, once again, we derive a general result that applies to certain linear second-order recurrence relations with variable coefficients and, in particular, to orthogonal polynomials. We focus on the property given in the following definition:

Definition 6.1. A solution $\langle x_n \rangle$ of (5.1) in a ring R has the *divisibility property* if x_r divides x_s whenever r divides s.

Our objective is to find conditions under which the primary solution of (5.1) has the divisibility property and, to a lesser extent, when the converse property (namely, if z_r divides z_s then r divides s) holds. For the Fibonacci sequence, if F_r divides F_s, then r divides s, except possibly when $F_r = \pm 1$.

As the motivation for this investigation came from known results on the Chebyshev polynomials of the second kind, we consider these first. Suppose that $m+1$ divides $n+1$, say $n+1 = k(m+1)$. It is evident that

$$U_n(\cos\theta) = U_m(\cos\theta)U_{k-1}\big(\cos[(m+1)\theta]\big)$$

and this identity translates to

$$U_n(x) = U_m(x)U_{k-1}\big(T_{m+1}(x)\big).$$

Thus U_m divides U_n. Conversely, suppose that U_m divides U_n. Then

$$\sin[(n+1)\theta] = \sin[(m+1)\theta]Q(\cos\theta)$$

for some polynomial Q. Put $\theta = \pi/(m+1)$; then $(n+1)\theta = k\pi$ for some k, hence $m+1$ divides $n+1$. Thus U_m divides U_n if and only if $m+1$ divides $n+1$. This was the reason for introducing $\hat{U}_n$ in section 3: in parallel with the Fibonacci sequence, $\hat{U}_m$ divides $\hat{U}_n$ if and only if m divides n. Thus we obtain the formula

$$\hat{U}_{mn}(x) = \hat{U}_m(x)\hat{U}_n\big(T_m(x)\big). \tag{6.1}$$

Our first result shows that the primary solution of any recurrence relation with constant coefficients (in a ring R) has the divisibility property.

Theorem 6.2. *Let $\langle z_n \rangle$ be the primary solution of the recurrence relation $x_{n+2} = ax_{n+1} + bx_n$ $(n \geq 0)$ in a ring R. If r divides s, then z_r divides z_s.*

Proof. It is sufficient to show that z_n divides z_{kn} for $k = 1, 2, \ldots$. We prove this by induction, noting that it is trivially true when $k = 1$. Now put $x_j = z_j$ and $q = (m-1)n - 1$ in Theorem 4.1. As we are in the constant coefficient case, this result gives

$$z_{mn} = z_{(m-1)n}\, z_{n+1} + b\, z_{(m-1)n-1} z_n,$$

where all terms involve the constant sequences $A = (a, a, \ldots)$ and $B = (b, b, \ldots)$. It is now clear that, if z_n divides $z_{(m-1)n}$, then it also divides z_{mn}, so the proof is complete. ■

Finally (6.1), together with Theorem 3.1, leads immediately to the next result, which establishes the universality of the Chebyshev polynomials with respect to divisibility for second-order recurrence relations with constant coefficients.

Theorem 6.3. *The primary solution $\langle z_n \rangle$ of the recurrence relation $x_{n+2} = ax_{n+1} + bx_n$ in an integral domain D satisfies*

$$z_{mn} = z_m \rho^{m(n-1)} \hat{U}_n\big(T_m(a/2\rho)\big).$$

7. PERIODIC COEFFICIENTS. We now give an example to show that the conclusion of Theorem 6.2 does not hold for second-order recurrence relations in which both sequences (a_n) and (b_n) are periodic with period two.

Example 7.1. We consider the primary solution $\langle z_n \rangle$ of (5.1) in $\mathbb{Z} + i\mathbb{Z}$, where

$$a_n = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 1+i & \text{if } n \text{ is odd;} \end{cases} \qquad b_n = \begin{cases} 1 & \text{if } n \text{ is even,} \\ i & \text{if } n \text{ is odd.} \end{cases}$$

As $z_0 = 0$ and $z_1 = 1$, we find that $z_3 = 1 + 2i$ and $z_6 = 7i$, so that z_3 does not divide z_6. Thus $\langle z_n \rangle$ does not possess the divisibility property. ■

Theorem 6.2 and Example 7.1 together raise the question of divisibility when one of the sequences is periodic with period two, and the other sequence is constant. The next result deals with one of these two possible cases.

Theorem 7.2. *Consider the relation* (5.1) *in a ring R. If the sequence $\langle a_n \rangle$ is periodic with period two and $b_n = b$, a nonzero constant from R, for all n, then the primary solution of* (5.1) *has the divisibility property.*

Example 7.1 shows that, in some sense, Theorem 7.2 is best possible. As an example of the situation covered by Theorem 7.2, consider the primary solution $\langle z_n \rangle$ of (5.1) in the ring of Gaussian integers, where $b_n = i$ for all n and

$$a_n = \begin{cases} 1 & \text{if } n \text{ is even,} \\ -2i & \text{if } n \text{ is odd.} \end{cases}$$

Then z_0, z_1, z_2, z_3, z_4, and z_5 are 0, 1, 1, $-i$, 0, 1, respectively, so the sequence z_n has period four. It is easy to see that here z_n has the divisibility property. Indeed, it is obvious that z_r divides any z_s whenever r is not a multiple of four. If $r = 4k$ and r divides s, then $z_r = z_s = 0$, so again z_r divides z_s. Finally, we prove the following partial converse to Theorem 7.2:

Theorem 7.3. *If the primary solution $\langle z_n \rangle$ of* (5.1) *in the ring $\mathbb{R}[x]$ of real polynomials has the divisibility property, then $\langle a_n \rangle$ is periodic with period two and $\langle b_n \rangle$ is a constant sequence.*

We take a slightly broader view and consider the recurrence relation (5.1) in the situation where both of the sequences a_n and b_n have period two. First, we give an explicit formula for the primary solution in this case.

Lemma 7.4. *Let $\langle z_n \rangle$ be the primary solution of the recurrence relation* (5.1) *when*

$$\begin{cases} a_n = a, \ b_n = b & \text{if } n \text{ is even;} \\ a_n = a', \ b_n = b' & \text{if } n \text{ is odd.} \end{cases} \tag{7.1}$$

Then $z_{2m} = aw_m$ and $z_{2m+1} = w_{m+1} - bw_m$, where w_n is the primary solution of the constant coefficient relation $y_{n+2} = U\,y_{n+1} + V\,y_n$, with $U = aa' + b + b'$ and $V = -bb'$. In particular,

$$\begin{aligned} \begin{pmatrix} z_{2m+1} \\ z_{2m} \end{pmatrix} &= \begin{pmatrix} 1 & -b \\ 0 & a \end{pmatrix} \begin{pmatrix} w_{m+1} \\ w_m \end{pmatrix} \\ &= \begin{pmatrix} 1 & -b \\ 0 & a \end{pmatrix} \begin{pmatrix} U & V \\ 1 & 0 \end{pmatrix}^m \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{aligned} \tag{7.2}$$

Proof. We define the sequence $\langle \zeta_n \rangle$ by $\zeta_{2m} = aw_m$ and $\zeta_{2m+1} = w_{m+1} - bw_m$, and we show that $z_n = \zeta_n$. As $\zeta_0 = 0 = z_0$ and $\zeta_1 = 1 = z_1$, it is necessary to show only that the ζ_n satisfy the recurrence relation (5.1), and this is easy. First, we have

$$\begin{aligned} \zeta_{2m+2} &= aw_{m+1} \\ &= a(w_{m+1} - bw_m) + abw_m \\ &= a\zeta_{2m+1} + b\zeta_{2m}; \end{aligned}$$

second, we obtain

$$\begin{aligned} \zeta_{2m+1} &= w_{m+1} - bw_m \\ &= Uw_m + Vw_{m-1} - bw_m \\ &= Uw_m - bw_m - bb'w_{m-1} \\ &= a'(aw_m) + b'(w_m - bw_{m-1}) \\ &= a'\zeta_{2m} + b'\zeta_{2m-1}. \end{aligned}$$

Since the last statement in Lemma 7.4 is obvious, the proof is complete. ∎

For instance, in Example 7.1 considered earlier, each of the sequences $\langle a_n \rangle$ and $\langle b_n \rangle$ has period two with, in the notation of Lemma 7.4, $a = b = 1$, $a' = 1 + i$, and $b' = i$. Thus in this example,

$$\begin{pmatrix} z_{2m+1} \\ z_{2m} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2(1+i) & -i \\ 1 & 0 \end{pmatrix}^m \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

We can now give the proof of Theorem 7.2.

Proof of Theorem 7.2. We use the notation $z_n(A, B)$ introduced earlier, and we take $x_j = z_j$, $n = vm$, and $q = m - 1$ in Theorem 4.1. This gives

$$z_{(v+1)m}(A, B) = z_m(A^{vm}, B^{vm})\, z_{vm+1}(A, B) + b_{vm}\, z_{m-1}(A^{vm+1}, B^{vm+1})\, z_{vm}(A, B),$$

from which it becomes clear (by induction) that, if

$$z_m(A, B) = z_m(A^{km}, B^{km}) \tag{7.3}$$

for all k and all m, then $z_m(A, B)$ divides $z_m(A^{km}, B^{km})$ for $k = 1, 2, \ldots$. Therefore the sequence $\langle z_n \rangle$ has the divisibility property.

We now prove that (7.3) holds. Under the assumptions in Theorem 7.2, we have, say, $A = (a, a', a, a', \ldots)$ and $B = (b, b, b, \ldots)$. Obviously, $B^r = B$ and $A^{2r} = A$ for every r, so (7.3) holds when km is even. We may assume, then, that m is odd and write $m = 2k + 1$. Now (7.2) holds with $U = aa' + 2b$ and $V = -b^2$, and from this it is evident that $z_{2m+1}(A, B)$ is a polynomial in the variables a, a', and b that is symmetric in a and a'. Because the change from A^r to A^{r+1} is achieved by interchanging a and a', it is now clear that $z_{2m+1}(A, B) = z_{2m+1}(A^r, B^r)$ for every r. We conclude that (7.3) holds for all k and m, completing the proof of Theorem 7.2. ■

We recall that Example 7.1 shows that, if the sequences A and B are periodic with period two, then the primary solution of (5.1) does not have the divisibility property. However, the next result demonstrates that it does have a partial divisibility property.

Theorem 7.5. *Let $\langle z_n \rangle$ be the primary solution of the relation* (5.1) *in an integral domain D, where the sequences $\langle a_n \rangle$ and $\langle b_n \rangle$ are both periodic with period two and no b_n is zero. Then z_r divides z_{rs} for every even r and every s.*

Proof. The discussion at the start of this section showed that if we take

$$A = (a, a', a, a', \ldots), \qquad B = (b, b', b, b', \ldots),$$

then $z_{2m} = a w_m$, where w_n is the primary solution of the constant coefficient relation $x_{n+2} = U\, x_{n+1} + V\, x_n$, with $U = aa' + b + b'$ and $V = -bb'$. By Theorem 6.2, w_r divides w_{rs} for every s, and as a is not a divisor of zero, this implies that z_{2r} divides z_{2rs} for every s. ■

It is natural to ask whether a result similar to Theorem 7.5 holds for periodic sequences $\langle a_n \rangle$ and $\langle b_n \rangle$ with other periods. The proof of Theorem 7.5 is based on the fact that if the two sequences have period two, then the subsequence $\langle x_{2n} \rangle$ of the solution $\langle x_n \rangle$ satisfies the second-order recurrence relation with constant coefficients. A result analogous to this holds for larger periods but, as the resulting recurrence relation has order greater than two, it is not relevant to this discussion. For a related result, see [**6**].

8. THE PROOF OF THEOREM 7.3. In this section we restrict our attention to recurrence relations in the ring $\mathbb{R}[x]$ of real polynomials in the variable x, and our sole objective is to give a proof of Theorem 7.3. We begin with two lemmas, after which we prove the theorem. The first lemma is (when $a_n(x)$ is a linear polynomial and $b < 0$) a standard result in the theory of orthogonal polynomials, although it has nothing per se to do with orthogonality.

Lemma 8.1. *Let $\langle z_n \rangle$ be the primary solution of the relation $x_{n+2} = a_n(x)x_{n+1} + b\,x_n$ in $\mathbb{R}[x]$, where b is a nonzero real number. Then z_m and z_{m+1} have no common zeros.*

Proof. Each z_m is a polynomial in x. The recurrence relation shows that a common zero of z_{n+2} and z_{n+1} is also a common zero of z_{n+1} and z_n (because b is a nonzero real number), hence a common zero of z_n and z_{n-1}, and so on, until it is seen to be a common zero of z_2 and z_1. This is impossible, for $z_1 = 1$ and has no zeros. ■

Lemma 8.2. *Let $z_n(A, B)$ be the primary solution of the relation*

$$x_{n+2} = a_n(x)x_{n+1} + bx_n,$$

where b is a nonzero real number and the $a_n(x)$ are monic polynomials of degree d with $d \geq 1$. If $z_t(A, B)$ divides each $z_{kt}(A, B)$, then $z_t(A, B) = z_t(A^{kt}, B^{kt})$ for each k.

Proof. Take $n = vt$ and $q = t - 1$ in Theorem 4.1. This yields the relation

$$z_{(v+1)t}(A, B) = z_t^{vt}\, z_{vt+1}(A, B) + b\, z_{t-1}^{vt+1}\, z_{vt}(A, B).$$

Because $z_t(A, B)$ divides each $z_{vt}(A, B)$, we see that $z_t(A, B)$ divides each $z_t^{vt}\, z_{vt+1}(A, B)$. Now $z_t(A, B)$ divides $z_{vt}(A, B)$, while $z_{vt}(A, B)$ and $z_{vt+1}(A, B)$ are coprime (Lemma 8.1). Thus $z_t(A, B)$ divides each $z_t^{vt}(A, B)$. But these polynomials have the same degree and they are both monic. Accordingly, they are equal. ■

Proof of Theorem 7.3. We must show that, if z_n has Chebyshev divisibility, then the sequence $\langle a_n \rangle$ is periodic with period two and $\langle b_n \rangle$ is a nonzero constant sequence. Our main tool is Lemma 8.2, which shows that under this divisibility assumption

$$z_n = z_n^{mn}. \tag{8.1}$$

for all m and n. First, $z_2(x) = x + a_0$. Using this fact and (8.1) with $n = 2$, we see that $x + a_0 = x + a_{2m}$ for every m. Thus

$$a_0 = a_2 = a_4 = a_6 = \cdots. \tag{8.2}$$

We proceed to show that

$$a_1 = a_3 = a_5 = a_7 = \cdots. \tag{8.3}$$

It is easy to prove (by induction) that

$$z_n(x) = x^{n-1} + (a_0 + \cdots + a_{n-2})x^{n-2} + O(x^{n-3}). \tag{8.4}$$

Appealing to (8.4) in tandem with (8.1), we find that, for every m and every n,

$$a_0 + \cdots + a_{n-2} = a_{mn} + \cdots + a_{mn+n-2}. \tag{8.5}$$

We now put $n = 4$ in (8.5), invoke (8.2), and conclude that

$$a_1 = a_5 = a_9 = a_{13} = \cdots. \tag{8.6}$$

Next, we consider (8.5), first with $m = 1, n = 2t + 2$ and then with $m = 1, n = 2t + 1$ to arrive at

$$a_0 + \cdots + a_{2t} = a_{2t+2} + \cdots + a_{4t+2}$$

and

$$a_0 + \cdots + a_{2t-1} = a_{2t+1} + \cdots + a_{4t},$$

respectively. Subtracting, and recalling (8.2) and (8.6), we find that (8.3) holds. Thus the sequence $\langle a_n \rangle$ is periodic with period two.

We now use a similar, but longer, proof to show that $\langle b_n \rangle$ is a constant sequence. Our proof is based on a stronger version of (8.4), namely, the following:

$$\begin{aligned} x_{n+2}(x) = x^{n+1} &+ (a_0 + \cdots + a_n)x^n \\ &+ \left[\sum_{0 \le i < j \le n} a_i a_j + (b_1 + \cdots + b_n) \right] x^{n-1} + O(x^{n-2}). \end{aligned}$$

This can also be proved by induction (we omit the details). Because $x_{n+2} = x_{n+2}^{m(n+2)}$, this yields

$$\begin{aligned} &\sum_{0 \le i < j \le n} a_i a_j + (b_1 + \cdots + b_n) \\ &\quad = \sum_{0 \le i < j \le n} a_{i+m(n+2)} a_{j+m(n+2)} + (b_{1+m(n+2)} + \cdots + b_{n+m(n+2)}). \end{aligned} \tag{8.7}$$

We verify that

$$\sum_{0 \le i < j \le n} a_i a_j = \sum_{0 \le i < j \le n} a_{i+m(n+2)} a_{j+m(n+2)}, \tag{8.8}$$

from which it follows that

$$b_1 + \cdots + b_n = b_{1+m(n+2)} + \cdots + b_{n+m(n+2)}. \tag{8.9}$$

We exploit (8.9) to show that $b_1 = b_2 = b_3 = \cdots$.

We establish (8.8). As the sequence $\langle a_n \rangle$ is periodic with period two, we can replace a_r in (8.8) with a_s, provided that $r - s$ is even. It follows from this that (8.8) certainly holds if m is even. The same argument reveals that, when m is odd, it suffices to prove (8.8) in the case $m = 1$ with $n + 2$ replacing n. Thus we have to show only that

$$\sum_{0 \le i < j \le n} a_i a_j = \sum_{0 \le i < j \le n} a_{i+n} a_{j+n}. \tag{8.10}$$

Letting $s = n - i$ and $t = n - j$, we compute

$$\sum_{0\le i<j\le n} a_{n+i}a_{n+j} = \sum_{0\le t<s\le n} a_{2n-s}a_{2n-t}$$
$$= \sum_{0\le t<s\le n} a_{2(n-s)+s}a_{2(n-t)+t}$$
$$= \sum_{0\le t<s\le n} a_s a_t,$$

as required. This justifies (8.7)–(8.10) and, in particular, establishes (8.9).

We shall need (8.9) with $n = 2$, namely, the relation

$$b_1 + b_2 = b_{4m+1} + b_{4m+2}. \tag{8.11}$$

Next, we put $m = 1$ in (8.9) and consider the cases $n = 2k$ and $n = 2k - 2$. Together, these give

$$\begin{aligned} b_{2k} &= (b_1 + \cdots + b_{2k}) - (b_1 + \cdots + b_{2k-2}) \\ &= -b_{2k+2} + b_{4k+1} + b_{4k+2} \end{aligned}$$

or, equivalently,

$$b_{2k} + b_{2k+2} = b_{4k+1} + b_{4k+2}. \tag{8.12}$$

In tandem, (8.11) and (8.12) yield

$$b_1 + b_2 = b_{2m} + b_{2m+2}.$$

This implies that the sum of two consecutive terms from the sequence $b_2, b_4, b_6, \ldots$ is constant, which means that

$$b_2 = b_6 = b_{10} = b_{14} = \cdots, \qquad b_4 = b_8 = b_{12} = b_{16} = \cdots.$$

We now return to (8.9) and take $n = 1$. This gives $b_1 = b_{3m+1}$ for all m, hence implies that $b_1 = b_4 = b_7 = b_{10}$. We now know that

$$b_1 = b_7 = b_2 = b_4 = b_6 = b_8 = \cdots = b_{2m} = \cdots = b,$$

say. With this, (8.12) for $k = 1$ gives $b_5 = b$, while (8.9) with $n = 3$ gives $b_3 = b_7$, so $b_3 = b$. We conclude that

$$b_1 = b_3 = b_5 = b_7 = b_2 = b_4 = b_6 = b_8 = \cdots = b_{2m} = \cdots = b.$$

Finally, if $n \ge 3$, (8.9) shows that

$$b_{2n+1} = (b_2 + \cdots + b_n) - (b_{n+3} + \cdots + b_{2n}),$$

which enables us to prove easily (by induction) that $b_{2n+1} = b$ for all n. The proof that $\langle b_n\rangle$ is a constant sequence is complete, and with it the proof of Theorem 7.3. ∎

9. ORTHOGONAL POLYNOMIALS. A Borel measure μ on $\mathbb{R}$ with the property that, for each positive integer k, $|x|^k$ is integrable over $\mathbb{R}$ induces a scalar product

$$(p, q) = \int_{\mathbb{R}} p(x)q(x)\, d\mu(x)$$

on the vector space of all real polynomials. A sequence p_n of real polynomials, with p_n monic and of degree n, is *μ-orthogonal* if $(p_i, p_j) = 0$ whenever $i \neq j$. Notice that here the p_n are normalized by the condition that they are monic rather than the usual condition $(p_n, p_n) = 1$.

Suppose now that $\langle p_n \rangle$ is a μ-orthogonal sequence. Then xp_{n+1} is monic and of degree $n+2$, so can be expressed in the form

$$xp_{n+1} = \lambda_0 p_0 + \cdots + \lambda_{n+1} p_{n+1} + p_{n+2}.$$

Now for $k = 0, 1, \ldots, n-1$,

$$\big(p_k, xp_{n+1}\big) = \big(xp_k, p_{n+1}\big) = 0,$$

which implies that the p_n satisfy a second-order recurrence relation

$$xp_{n+1} = \lambda_n p_n + \lambda_{n+1} p_{n+1} + p_{n+2}. \tag{9.1}$$

In fact, $\lambda_n > 0$ because $p_{n+1} - xp_n$ is of degree at most n, making it orthogonal to p_{n+1}. Thus

$$\big(p_{n+1}, p_{n+1}\big) = \big(p_{n+1}, xp_n\big) = \big(xp_{n+1}, p_n\big) = \lambda_n (p_n, p_n),$$

which forces λ_n to be positive. The converse result (namely, that any sequence $\langle p_n \rangle$ of polynomials, with p_n monic and of degree n, that satisfies a relation of the form (9.1) with $\lambda_n > 0$ is orthogonal with respect to some μ), is known as *Favard's Theorem.* We now see that the p_n are orthogonal with respect to some measure μ if and only if they satisfy a real three-term recurrence relation of the form

$$p_{n+2} = (x + a_n)p_{n+1} - b_n p_n \quad (n \geq 0), \tag{9.2}$$

where each b_n is positive. For more details, see [**3**], [**5**], and [**9**].

Our assumptions about the p_n imply that $p_0 = 1$ and $p_1 = x + a$ for some a. In order to consider primary solutions we extend (9.2) to the case $n = -1$ by putting $a_{-1} = a$, $b_{-1} = 1$, and $p_{-1} = 0$. Then

$$p_{-1} = 0, \quad p_0 = 1, \quad p_{n+2} = (x + a_n)p_{n+1} - b_n p_n \quad (n \geq -1). \tag{9.3}$$

It is clear that, if we let $\tilde{p}_n = p_{n-1}$, then

$$\tilde{p}_0 = 0, \quad \tilde{p}_1 = 1, \quad \tilde{p}_{n+2} = (x + a_{n-1})\tilde{p}_{n+1} - b_{n-1}\tilde{p}_n \quad (n \geq 0), \tag{9.4}$$

so $\langle \tilde{p}_n \rangle$ is the primary solution of the recurrence relation (9.4).

Starting with (9.3), we also define a sequence $\langle q_n \rangle$ of polynomials by

$$q_0 = 0, \quad q_1 = 1, \quad q_{n+2} = (x + a_n)q_{n+1} - b_n q_n \quad (n \geq 0). \tag{9.5}$$

In the theory of orthogonal polynomials the q_n are known as the *associated polynomials normal to* the p_n (see, for example, [**7**]), but for us they are the primary solution of the recurrence relation (9.5). Moreover, if we use the natural notation $\tilde{p}_n(A^0, B^0)$ for appropriate sequences A^0 and B^0, then, in our earlier notation, $q_n = \tilde{p}_n(A^1, B^1)$. Thus the polynomials q_n associated to the p_n are obtained by translating the sequences of coefficients in the relation precisely in the way that we have introduced in Theorem 4.1. Moreover, from such a perspective, this construction is not dependent on orthogonality, and it is equally applicable to recurrence relations whose coefficients are real numbers. In this context, Theorem 4.1 shows how to express x_{n+q+1}, for varying q, as a linear combination of two fixed terms x_n and x_{n+1}, where the coefficients in this linear combination are identified in terms of the higher order associated polynomials.

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