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Discrete Mathematics 306 (2006) 1906–1920

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

The Cauchy numbers

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Received 13 October 2005; received in revised form 24 January 2006; accepted 28 March 2006
Available online 16 June 2006

Abstract

We study many properties of Cauchy numbers in terms of generating functions and Riordan arrays and find several new identities relating these numbers with Stirling, Bernoulli and harmonic numbers. We also reconsider the Laplace summation formula showing some applications involving the Cauchy numbers.

Keywords: Cauchy numbers; Generating functions; Riordan arrays; Laplace summation formula

1. Introduction

Recently, and rather surprisingly, we had the occasion to meet Cauchy numbers, sequences of numbers which are not particularly famous, but nonetheless have received attention in the literature. A colleague asked us if we knew something about the integral

$$\int_0^x \binom{\xi}{n} \mathrm{d}\xi, \tag{1.1}$$

in particular when $x = 1$, and a note has appeared on the American Mathematical Monthly (see [13]). Specifically, in this note the author re-discovers the *Laplace summation formula* in the particular case of the harmonic numbers. The expansion he finds involves a sequence of coefficients, and the referee rightly observes that they are related to the Stirling numbers of the first kind. Actually, these coefficients are just a version of the Cauchy numbers, which are usually defined by means of the integral (1.1).

Because of that, we thought that it might be appropriate to reconsider these Cauchy numbers and the Laplace summation formula, in which they appear. We met the Cauchy numbers for the first time in the book of Comtet [2], where they are introduced in Exercise 13 on p. 293. Successively, we encountered them in connection with the Stirling numbers of the first kind, and this relates them to Combinatorics. Another aspect of these numbers which might be interesting in Combinatorics is the fact that they appear in the quoted Laplace summation formula. This is analogous to Euler McLaurin summation formula, but uses Cauchy numbers and the difference operator instead of the Bernoulli numbers and differentiation. Usually, Euler–McLaurin formula is considered more useful than Laplace formula and therefore many textbooks do not even quote this last formula, a variant of which, however, is called *Gregory's formula*

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0012-365X/$ - see front matter
doi:10.1016/j.disc.2006.03.065

and is used to approximate integrals. Because of that, it is better known in the theory of difference–differential equations than in Combinatorics.

The main contributions of the present paper can be summarized into two points:

- we find a number of equations, expressed in terms of finite or infinite converging sums, which are new and relate Cauchy numbers to many other quantities of combinatorial interest;
- we use the so-called *method of coefficients* (see, e.g., the book by Greene and Knuth [3]), which is becoming more and more important, even if it is mostly applied in an informal way. It consists in a systematic use of the *generating function* and of the *coefficient of* operators; in particular, we will also use the concept of a *Riordan array*, which is a direct derivation of the method of coefficients (see [6]).

We massively use properties of the generating function operator $\mathscr{G}_t(f_k)_{k\in N} = \mathscr{G}(f_k) = f(t)$ which associates to a sequence $(f_k)_{k\in N}$ its ordinary generating function, the formal power series $f(t)=\sum_{k=0}^{\infty} f_k t^k$, or $\mathscr{E}_t(f_k)_{k\in N}=\mathscr{E}(f_k)=\widehat{f}(t)$ in case of an exponential generating function $\widehat{f}(t)=\sum_{k=0}^{\infty} f_k t^k/k!$. Its inverse operator is the "coefficient of operator" which, given a formal power series $f(t)=\sum_{k=0}^{\infty} f_k t^k$, extracts the coefficient of t^k: $[t^k]f(t)=f_k$, for every $k\in N$. The properties of these operators are given in a semi-formal way in Wilf's book [12]; for a more formal approach the reader is referred to [6].

The concept of Riordan arrays has been introduced by Shapiro et al. [8]. A Riordan array is an infinite, lower triangular array $D=\{d_{n,k}\}_{n,k\in N}$ defined by a couple of formal power series: $D=\mathscr{R}(d_{n,k})=(d(t),h(t))$, such that

$$d_{n,k}=[t^n]d(t)(th(t))^k \quad \forall n\in\mathbb{N}.$$

The two main properties of Riordan arrays (at least from the point of view of this paper) are:

(1) a particular sequence, called the *A-sequence* for the Riordan array, $A=(a_j)_{j\in N}$ exists such that every element in the array (not belonging to row 0 or column 0) is given by the linear combination: $d_{n+1,k+1}=\sum_{j=0}^{\infty} a_j d_{n,k+j}$; this sum is actually finite. The A-sequence does not depend on n or k, but only depends on $h(t)$; we have $h(t)=A(th(t))$, if $A(t)$ is the generating function of the A-sequence;

(2) the following *summation property* holds:

$$\sum_{k=0}^{n} d_{n,k} f_k=[t^n]d(t)f(th(t)),$$

where $f(t)$ is the generating function of any sequence $(f_k)_{k\in N}$. This reduces the evaluation of the sum to the extraction of a coefficient from a generating function. This approach has been extensively used in [9]; various and interesting examples can also be found in [10,11].

The most important example of a Riordan array is the Pascal triangle, defined by

$$\mathscr{R}\left(\binom{n}{k}\right)=\left(\frac{1}{1-t},\frac{1}{1-t}\right).$$

In that case, the A-sequence is $(1,1)$, corresponding to the basic recurrence for the binomial coefficients, and the summation property becomes

$$\sum_{k=0}^{n}\binom{n}{k}f_k=[t^n]\frac{1}{1-t}f\left(\frac{t}{1-t}\right)=[t^n]\frac{1}{1-t}\left[f(y)\,\middle|\,y=\frac{t}{1-t}\right],$$

which is also known as *Euler transformation*. The notation $[f(y)|y=g(t)]$, which just denotes substitution $f(g(t))$, will be used as an on-line version of $f(y)|_{y=g(t)}$.

In the present paper we are mainly interested in two particular Riordan arrays, associated to the Stirling numbers of both kinds. For the (signless) Stirling numbers of the first kind $\left[{n \atop k}\right]$ we have

$$\mathscr{R}\left(\frac{k!}{n!}\left[{n \atop k}\right]\right)=\left(1,\frac{1}{t}\ln\frac{1}{1-t}\right), \tag{1.2}$$

so that we have

$$\sum_{k=0}^{n} \left[{n \atop k} \right] f_k = n! \sum_{k=0}^{n} \frac{k!}{n!} \left[{n \atop k} \right] \frac{f_k}{k!} = n![t^n] \left[\widehat{f}(y) \middle| y = \ln \frac{1}{1-t} \right] = n![t^n] \widehat{f} \left(\ln \frac{1}{1-t} \right).$$

In an analogous way, for the Stirling numbers of the second kind $\left\{ {n \atop k} \right\}$ we have

$$\mathscr{R} \left(\frac{k!}{n!} \left\{ {n \atop k} \right\} \right) = \left(1, \frac{\mathrm{e}^t - 1}{t} \right), \tag{1.3}$$

and the summation property becomes

$$\sum_{k=0}^{n} \left\{ {n \atop k} \right\} f_k = n! \sum_{k=0}^{n} \frac{k!}{n!} \left\{ {n \atop k} \right\} \frac{f_k}{k!} = n![t^n][\widehat{f}(y) | y = \mathrm{e}^t - 1] = n![t^n] \widehat{f}(\mathrm{e}^t - 1).$$

The paper is organized in the following way. In Section 2 we introduce Cauchy numbers of the first and second kind and prove several properties by using their generating functions. In Section 3 we apply the concept of Riordan arrays to prove other properties of Cauchy numbers related to particular infinite matrices. Finally, in Section 4, we reconsider the Laplace summation formula (see [1,7]) and show some formulas related to its application and involving the Cauchy numbers.

2. The Cauchy numbers

According to Comtet [2], *Cauchy numbers* can be distinguished into two kinds, and are defined as the value of a definite integral; the Cauchy numbers *of the first kind* are $\mathscr{C}_n = \int_0^1 x^{\underline{n}} \, \mathrm{d}x$, where $x^{\underline{n}} = x(x-1)\cdots(x-n+1)$ is the *falling factorial*, and the Cauchy numbers *of the second kind* are $\widehat{\mathscr{C}}_n = \int_0^1 x^{\overline{n}} \, \mathrm{d}x$, where $x^{\overline{n}} = x(x+1)\cdots(x+n-1)$ is the *rising factorial*. For simplicity sake, for the Cauchy numbers we will use the equivalent definitions

$$\frac{\mathscr{C}_n}{n!} = \int_0^1 \binom{x}{n} \, \mathrm{d}x, \quad \frac{\widehat{\mathscr{C}}_n}{n!} = \int_0^1 \binom{-x}{n} \, \mathrm{d}x = (-1)^n \int_0^1 \binom{x+n-1}{n} \, \mathrm{d}x.$$

Sometimes, as in Jagerman [5], the numbers $\mathscr{C}_n/n!$ are called *Laplace numbers*. The Stirling numbers of the first kind are defined in terms of the falling factorial, and we have

$$\binom{x}{n} = \frac{1}{n!} \sum_{k=0}^{n} \left[{n \atop k} \right] (-1)^{n-k} x^k = \frac{x^{\underline{n}}}{n!}$$

and therefore

$$\frac{\mathscr{C}_n}{n!} = \int_0^1 \binom{x}{n} \, \mathrm{d}x = \frac{1}{n!} \int_0^1 \sum_{k=0}^{n} \left[{n \atop k} \right] (-1)^{n-k} x^k = \frac{1}{n!} \sum_{k=0}^{n} \left[{n \atop k} \right] \frac{(-1)^{n-k}}{k+1}.$$

The sum can be developed by means of the Riordan array method; in fact, we can use the Riordan array associated to the Stirling numbers of the first kind and the generating function:

$$\mathscr{G} \left(\frac{(-1)^{k+1}}{(k+1)!} \right) = \frac{\mathrm{e}^{-t} - 1}{t}.$$

Therefore we have

$$\frac{\mathscr{C}_n}{n!} = (-1)^{n-1} \sum_{k=0}^{n} \frac{k!}{n!} \left[{n \atop k} \right] \frac{(-1)^{k+1}}{(k+1)!} = (-1)^{n-1}[t^n] \left[\frac{\mathrm{e}^{-y} - 1}{y} \middle| y = \ln \frac{1}{1-t} \right].$$

By performing the substitution, we find

$$\frac{\mathscr{C}_n}{n!} = (-1)^{n-1}[t^n]\frac{1-t-1}{\ln(1/(1-t))} = [t^n]\frac{t}{\ln(1+t)}$$

and therefore we have the exponential generating function for the Cauchy numbers: $\mathscr{E}(\mathscr{C}_n) = \mathscr{G}(\mathscr{C}_n/n!) = t/\ln(1+t)$. This formula is given by Comtet, and our proof makes use of the more recent Riordan array concept. An easy result is now obtained:

Theorem 2.1. *The Cauchy numbers are definitely alternating in sign and the infinite sum of their values is*

$$\sum_{n=0}^{\infty} \frac{\mathscr{C}_n}{n!} = \frac{1}{\ln 2}.$$

Proof. It is sufficient to observe that the radius of convergence of the exponential generating function is 1 with a dominant logarithmic singularity at $t = -1$. The value of the sum is obtained by setting $t = 1$ in the same function. □

In order to obtain a recurrence relation for the Cauchy numbers, we observe that their exponential generating function is the inverse of the ordinary generating function for the sequence: $((-1)^n/(n+1))_{n\in N}$, that is

$$\frac{t}{\ln(1+t)} \cdot \frac{\ln(1+t)}{t} = 1.$$

Theorem 2.2. *The Cauchy numbers can be computed by the following recurrence relation*:

$$\frac{\mathscr{C}_n}{n!} = \sum_{k=0}^{n-1} \frac{\mathscr{C}_k}{k!} \frac{(-1)^{n-k+1}}{n-k+1}, \qquad n \geqslant 1.$$

Proof. By the convolution above, when we pass to the coefficients, we find

$$\begin{aligned}[t^n]1 = \delta_{n,0} &= [t^n]\frac{t}{\ln(1+t)} \cdot \frac{\ln(1+t)}{t} \\ &= \sum_{k=0}^{n} \left([t^k]\,\frac{t}{\ln(1+t)}\right) \cdot \left([t^{n-k}]\,\frac{\ln(1+t)}{t}\right) = \sum_{k=0}^{n} \frac{\mathscr{C}_k}{k!} \frac{(-1)^{n-k+1}}{n-k+1}.\end{aligned}$$

When $n = 0$ we get $\mathscr{C}_0 = 1$; for $n > 0$ we isolate the term with $k = n$ and obtain the desired expression. □

The formula can be checked if we expand the generating function

$$\frac{t}{\ln(1+t)} = 1 + \frac{1}{2}t - \frac{1}{12}t^2 + \frac{1}{24}t^3 - \frac{19}{720}t^4 + \frac{3}{160}t^5 - \cdots$$

and we have the first values

n	0	1	2	3	4	5	6	7
$\mathscr{C}_n$	1	1/2	–1/6	1/4	–19/30	9/4	–863/84	1375/24

The nature of the exponential generating function suggests that the Cauchy numbers are also related to the Stirling numbers of the second kind; in fact we have:

Theorem 2.3. *The following identity*

$$\sum_{k=0}^{n} \left\{ {n \atop k} \right\} \mathscr{C}_k = \frac{1}{n+1}$$

relates the Cauchy numbers of the first kind and the Stirling numbers of the second kind.

Proof. By using the Riordan array associated to the Stirling numbers of the second kind, we find

$$\sum_{k=0}^{n} \left\{ {n \atop k} \right\} \mathscr{C}_k = n! \sum_{k=0}^{n} \frac{k!}{n!} \left\{ {n \atop k} \right\} \frac{\mathscr{C}_k}{k!} = n![t^n] \left[\frac{y}{\ln(1+y)} \,\middle|\, y = \mathrm{e}^t - 1 \right]$$
$$= n![t^n] \frac{\mathrm{e}^t - 1}{\ln(1 + \mathrm{e}^t - 1)} = n![t^n] \frac{\mathrm{e}^t - 1}{t} = n! \frac{1}{(n+1)!} = \frac{1}{n+1}. \qquad \square$$

By isolating the term $k = n$ we could obtain a new recurrence relation for the Cauchy numbers; however, this is computationally more complex than the previous one because the Stirling numbers are present. At any rate, the identity is relevant for its own.

For the Cauchy numbers of the second kind we obtain very similar results:

$$\frac{\widehat{\mathscr{C}}_n}{n!} = (-1)^n \int_0^1 \frac{1}{n!} \sum_{k=0}^{n} \left[{n \atop k} \right] x^k \,\mathrm{d}x = (-1)^n \sum_{k=0}^{n} \frac{k!}{n!} \left[{n \atop k} \right] \frac{1}{(k+1)!}$$
$$= (-1)^n [t^n] \left[\frac{\mathrm{e}^y - 1}{y} \,\middle|\, y = \ln \frac{1}{1-t} \right] = [t^n] \frac{t}{(1+t)\ln(1+t)};$$

hence we have

$$\sum_{n=0}^{\infty} \frac{\widehat{\mathscr{C}}_n}{n!} = \frac{1}{2 \ln 2}.$$

The generating function just found tells us that $\widehat{\mathscr{C}}_n / n!$ is the partial, alternating sum of the Cauchy numbers of the first kind. More precisely, we prove the following identities:

Theorem 2.4. *The following equations relate the Cauchy numbers of both kinds*:

$$\frac{\widehat{\mathscr{C}}_n}{n!} = (-1)^n \sum_{k=0}^{n} (-1)^k \frac{\mathscr{C}_k}{k!} = (-1)^n \left(1 - \sum_{k=1}^{n} \left| \frac{\mathscr{C}_k}{k!} \right| \right), \tag{2.1}$$

$$\frac{\mathscr{C}_n}{n!} = \frac{\widehat{\mathscr{C}}_n}{n!} + \frac{\widehat{\mathscr{C}}_{n-1}}{(n-1)!} \quad \text{or} \quad \mathscr{C}_n = \widehat{\mathscr{C}}_n + n\widehat{\mathscr{C}}_{n-1}, \tag{2.2}$$

$$\sum_{k=0}^{n} \binom{n}{k} \mathscr{C}_k \widehat{\mathscr{C}}_{n-k} = (1-n)\mathscr{C}_n \quad \text{or} \quad \sum_{k=0}^{n-1} \binom{n}{k} \mathscr{C}_k \widehat{\mathscr{C}}_{n-k} = -n\mathscr{C}_n. \tag{2.3}$$

Proof. In general we have: $[t^n] f(t)/(1+t) = \sum_{k=0}^{n} (-1)^{n-k} f_k$, and this proves Eq. (2.1). Eq. (2.2) follows from the fact that the Cauchy numbers of the first kind are alternating in sign. The other relation in the same row is obtained by writing

$$\frac{t}{\ln(1+t)} = (1+t) \frac{t}{(1+t)\ln(1+t)}$$

and extracting the coefficient of t^n from both sides. Finally, the proof of formulas (2.3) is a bit more complex. In fact we have

$$\frac{\mathscr{C}_n}{n!} = [t^n] \frac{t}{\ln(1+t)} = \frac{1}{n} [t^{n-1}] \frac{\mathrm{d}}{\mathrm{d}t} \frac{t}{\ln(1+t)}$$
$$= \frac{1}{n} [t^n] \frac{t}{\ln(1+t)} - \frac{1}{n} [t^n] \frac{t}{(1+t)\ln(1+t)} \frac{t}{\ln(1+t)}$$
$$= \frac{1}{n} \frac{\mathscr{C}_n}{n!} - \frac{1}{n} \sum_{k=0}^{n} \frac{\mathscr{C}_k}{k!} \frac{\widehat{\mathscr{C}}_{n-k}}{(n-k)!}.$$

From this equation the relation on the left follows by multiplying everything by $n!$. The equation on the right is obtained by isolating the term with $k=n$. $\square$

By following the pattern of the last part in the proof of the previous theorem, we obtain a recurrence relation for the Cauchy numbers of the second kind:

Theorem 2.5. *The recurrence*

$$(n+1)\widehat{\mathscr{C}}_n = (-1)^n n! \sum_{k=0}^{n-1} \left| \frac{\widehat{\mathscr{C}}_k}{k!} \right| - \sum_{k=1}^{n-1} \binom{n}{k} \widehat{\mathscr{C}}_k \widehat{\mathscr{C}}_{n-k}$$

holds true for the Cauchy numbers of the second kind.

Proof. Let us proceed in the following way:

$$\frac{\widehat{\mathscr{C}}_n}{n!} = [t^n] \frac{t}{(1+t)\ln(1+t)} = \frac{1}{n}[t^{n-1}] \frac{\mathrm{d}}{\mathrm{d}t} \frac{t}{(1+t)\ln(1+t)}.$$

By performing differentiation, we get

$$\begin{aligned} n\frac{\widehat{\mathscr{C}}_n}{n!} &= [t^{n-1}] \left(\frac{1}{(1+t)\ln(1+t)} - \frac{t}{(1+t)^2\ln(1+t)} - \frac{t}{((1+t)\ln(1+t))^2} \right) \\ &= [t^n] \frac{1}{1+t} \frac{t}{(1+t)\ln(1+t)} - [t^n] \left(\frac{t}{(1+t)\ln(1+t)} \right)^2 \\ &= \sum_{k=0}^{n} \frac{(-1)^{n-k}\widehat{\mathscr{C}}_k}{k!} - \sum_{k=0}^{n} \frac{\widehat{\mathscr{C}}_k}{k!} \frac{\widehat{\mathscr{C}}_{n-k}}{(n-k)!}. \end{aligned}$$

From this, the desired relation follows immediately when we isolate the terms containing $\widehat{\mathscr{C}}_n$. $\square$

By developing the generating function we have

$$\widehat{\mathscr{C}}(t) = \frac{t}{(1+t)\ln(1+t)} = 1 - \frac{1}{2}t + \frac{5}{12}t^2 - \frac{3}{8}t^3 + \frac{251}{720}t^4 - \frac{475}{1440}t^5 + \cdots$$

and the first $\widehat{\mathscr{C}}_n$ values are

n	0	1	2	3	4	5	6	7
$\widehat{\mathscr{C}}_n$	1	–1/2	5/6	–9/4	251/30	–475/12	19087/84	–36799/24

These values allow us to check the previous formulas and the two following identities, analogous to the identities found for the Cauchy numbers of the first kind:

Theorem 2.6. *For the Cauchy numbers of the second kind we have*

$$\frac{\widehat{\mathscr{C}}_n}{n!} = \sum_{k=0}^{n-1} \frac{\widehat{\mathscr{C}}_k}{k!} \frac{(-1)^{n-k}}{(n-k)(n-k+1)}, \quad n \geqslant 1,$$

$$\sum_{k=0}^{n} \left\{ {n \atop k} \right\} \widehat{\mathscr{C}}_k = \frac{(-1)^n}{n+1}.$$

Proof. The first equation is a recurrence relation for the Cauchy numbers of the second kind and is proved as the analogous formula for the Cauchy numbers of the first kind. In a similar way, by using the Riordan array related to the

Stirling numbers of the second kind, we have

$$\sum_{k=0}^{n} \left\{ {n \atop k} \right\} \widehat{\mathscr{C}}_k = n! \sum_{k=0}^{n} \frac{k!}{n!} \left\{ {n \atop k} \right\} \frac{\widehat{\mathscr{C}}_k}{k!} = n![t^n] \left[\frac{y}{(1+y)\ln(1+y)} \middle| \, y = \mathrm{e}^t - 1 \right]$$
$$= n![t^n] \frac{\mathrm{e}^t - 1}{t\mathrm{e}^t} = n![t^n] \left(\frac{1}{t} - \frac{\mathrm{e}^{-t}}{t} \right) = -n! \frac{(-1)^{n+1}}{(n+1)!} = \frac{(-1)^n}{n+1}$$

as desired. □

The behavior of the Cauchy numbers when combined with binomial coefficients is rather surprising:

Theorem 2.7. *The following identity holds*:

$$\sum_{k=0}^{n} \binom{n}{k} \frac{\widehat{\mathscr{C}}_k}{k!} = (-1)^n \frac{\widehat{\mathscr{C}}_n}{n!}.$$

Therefore, for every even integer n we have:

$$\sum_{k=0}^{n-1} \binom{n}{k} \frac{\widehat{\mathscr{C}}_k}{k!} = 0$$

and for every n

$$\sum_{k=1}^{n} \binom{n-1}{k-1} \frac{\mathscr{C}_k}{k!} = (-1)^n \frac{\widehat{\mathscr{C}}_n}{n!}.$$

Proof. By using the Riordan array method and the Euler transformation we find

$$\sum_{k=0}^{n} \binom{n}{k} \frac{\widehat{\mathscr{C}}_k}{k!} = [t^n] \frac{1}{1-t} \left[\frac{y}{(1+y)\ln(1+y)} \middle| \, y = \frac{t}{1-t} \right]$$
$$= [t^n] \frac{t}{(1-t)^2} \Big/ \left(\left(1 + \frac{t}{1-t} \right) \ln \left(1 + \frac{t}{1-t} \right) \right) = [t^n] \frac{-t}{(1-t)\ln(1-t)} = (-1)^n \frac{\widehat{\mathscr{C}}_n}{n!}.$$

Finally, by passing to generating functions

$$\sum_{k=0}^{n} \binom{n}{k} \frac{\widehat{\mathscr{C}}_k}{k!} = [t^n](1+t)^n \cdot \frac{t}{(1+t)\ln(1+t)} = [t^n](1+t)^{n-1} \cdot \frac{t}{\ln(1+t)} = \sum_{k=1}^{n} \binom{n-1}{k-1} \frac{\mathscr{C}_k}{k!}. \qquad \square$$

Let us now introduce another sequence of numbers strictly related to the Cauchy numbers of the second kind. We start with the exponential generating function

$$\frac{\mathscr{L}_n}{n!} = [t^n] \ln \left(\frac{1}{t} \ln \frac{1}{1-t} \right)$$

and observe the series expansion

$$\mathscr{L}(t) = \ln \left(\frac{1}{t} \ln \frac{1}{1-t} \right) = \frac{1}{2}t + \frac{5}{24}t^2 + \frac{1}{8}t^3 + \frac{251}{2880}t^4 + \frac{19}{288}t^5 + \cdots;$$

from this series, the first $\mathscr{L}_n$ values are easily deduced:

n	0	1	2	3	4	5	6	7
$\mathscr{L}_n$	0	1/2	5/12	3/4	251/120	95/12	19087/504	5257/24

If we compare these numbers against the Cauchy numbers of the second kind we observe a simple pattern. Formally, we can proceed in the following way: by applying the rule $[t^n]f(t) = (1/n)[t^{n-1}]f'(t)$:

$$\begin{aligned}[t^n]\ln\left(\frac{1}{t}\ln\frac{1}{1-t}\right) &= \frac{1}{n}[t^{n-1}]\left(-\frac{1}{t^2}\ln\frac{1}{1-t}+\frac{1}{t(1-t)}\right)t\Big/\ln\frac{1}{1-t}\\ &= \frac{1}{n}[t^{n-1}]\frac{1}{t}\left(\frac{t}{(1-t)\ln(1/(1-t))}-1\right) = \frac{(-1)^n}{n}[t^n]\frac{t}{(1+t)\ln(1+t)} = \frac{(-1)^n}{n}\frac{\widehat{\mathscr{C}}_n}{n!}.\end{aligned}$$

In practice, we have proved:

Theorem 2.8. *The Cauchy numbers of the second kind are given by*: $\widehat{\mathscr{C}}_n = (-1)^n n\mathscr{L}_n$, for every $n \neq 0$.

These $\mathscr{L}_n$ numbers are also related to the Bernoulli numbers $B_n = n![t^n]t/(\mathrm{e}^t - 1)$; in fact we have:

Theorem 2.9. *The following identity holds true for every* $n > 1$:

$$\sum_{k=0}^{n}\left\{{n \atop k}\right\}(-1)^k\mathscr{L}_k = \sum_{k=1}^{n}\left\{{n \atop k}\right\}\frac{\widehat{\mathscr{C}}_k}{k} = -\frac{B_n}{n}.$$

Proof. First of all we observe

$$\mathscr{E}((-1)^k\mathscr{L}_k) = \ln\frac{\ln(1+t)}{t}$$

and proceed by means of the Riordan array method ($n \neq 0$):

$$\begin{aligned}\sum_{k=0}^{n}\left\{{n \atop k}\right\}(-1)^k\mathscr{L}_k &= n!\sum_{k=0}^{n}\frac{k!}{n!}\left\{{n \atop k}\right\}\frac{(-1)^k\mathscr{L}_k}{k!} = n![t^n]\left[\ln\frac{\ln(1+y)}{y}\,\middle|\, y = \mathrm{e}^t - 1\right]\\ &= n![t^n]\ln\frac{t}{\mathrm{e}^t-1} = \frac{n!}{n}[t^{n-1}]\frac{\mathrm{e}^t-1}{t}\frac{\mathrm{e}^t-1-t\mathrm{e}^t}{(\mathrm{e}^t-1)^2} = \frac{n!}{n}[t^{n-1}]\left(\frac{1}{t}-\frac{\mathrm{e}^t}{\mathrm{e}^t-1}\right)\\ &= \frac{n!}{n}[t^n]1 - \frac{n!}{n}[t^{n-1}]\frac{\mathrm{e}^t-1+1}{\mathrm{e}^t-1} = \frac{n!}{n}[t^n](1-t) - \frac{n!}{n}[t^n]\frac{t}{\mathrm{e}^t-1}\\ &= -\delta_{n,1} - \frac{B_n}{n}\end{aligned}$$

as desired. □

This formula can be “inverted” to express the Cauchy numbers of the second kind in terms of the Bernoulli numbers:

Theorem 2.10. *The following identity holds true*:

$$\sum_{k=1}^{n}\left[{n \atop k}\right]\frac{B_k}{k} = -\mathscr{L}_n = \frac{(-1)^{n+1}\widehat{\mathscr{C}}_n}{n}.$$

Proof. Let us observe that

$$\mathscr{G}\left(\frac{B_k}{k\cdot k!}\right) = \mathscr{E}\left(\frac{B_k}{k}\right) = \ln\frac{1-\mathrm{e}^{-t}}{t};$$

by applying the Riordan array method we obtain

$$\begin{aligned}\sum_{k=1}^{n}\left[{n \atop k}\right]\frac{B_k}{k} &= n!\sum_{k=1}^{n}\frac{k!}{n!}\left[{n \atop k}\right]\frac{B_k}{k\cdot k!} = n![t^n]\left[\ln\frac{1-\mathrm{e}^{-y}}{y}\,\middle|\, y = \ln\frac{1}{1-t}\right]\\ &= -n![t^n]\left(\ln\left(\frac{1}{t}\ln\frac{1}{1-t}\right)\right) = -\mathscr{L}_n.\end{aligned}$$

The last identity follows from the relation between the $\mathscr{L}$ numbers and the Cauchy numbers of the second kind. □

Theorem 2.10 implies the following identity:

$$\sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(-1)^k B_k}{k} = (n-1)! - \mathscr{L}_n = \frac{(-1)^{n+1}\widehat{\mathscr{C}}_n + n!}{n}.$$

3. Riordan arrays

As we have seen, the Riordan array associated to the Stirling numbers of the second kind corresponds to Eq. (1.3); the A-sequence can be found by solving the functional equation $h(t) = A(th(t))$; if we set $y = \mathrm{e}^t - 1$ $(=th(t))$, we obviously have $t = \ln(1+y)$, and by substituting this value in $h(t)$ we eventually find the generating function for the A-sequence:

$$A(y) = \frac{y}{\ln(1+y)} = \mathscr{C}(y).$$

By applying the rule of the A-sequence (see property (1) of Riordan arrays in the Introduction), we obtain a sum involving the Cauchy numbers and the Stirling numbers of the second kind:

$$\frac{(k+1)!}{(n+1)!} \begin{Bmatrix} n+1 \\ k+1 \end{Bmatrix} = \sum_{j=0}^{\infty} \frac{(k+j)!}{n!} \begin{Bmatrix} n \\ k+j \end{Bmatrix} \frac{\mathscr{C}_j}{j!}.$$

This formula can be easily checked, and for example for $n = 5$ and $k = 2$ the right-hand side becomes

$$\begin{aligned} &\frac{2!}{5!}\begin{Bmatrix} 5 \\ 2 \end{Bmatrix} + \frac{1}{2}\frac{3!}{5!}\begin{Bmatrix} 5 \\ 3 \end{Bmatrix} - \frac{1}{12}\frac{4!}{5!}\begin{Bmatrix} 5 \\ 4 \end{Bmatrix} + \frac{1}{24}\frac{5!}{5!}\begin{Bmatrix} 5 \\ 5 \end{Bmatrix} \\ &\quad = \frac{15}{60} + \frac{1}{2}\frac{25}{20} - \frac{1}{12}\frac{10}{5} + \frac{1}{24} = \frac{6+15-4+1}{24} = \frac{18}{24} = \frac{3}{4}, \end{aligned}$$

which corresponds to $\frac{3!}{6!}\begin{Bmatrix} 6 \\ 3 \end{Bmatrix} = \frac{90}{120} = \frac{3}{4}$. With some obvious simplification we have proved:

Theorem 3.1. *The identity* (*where the sum is actually finite*)

$$\frac{k+1}{n+1} \begin{Bmatrix} n+1 \\ k+1 \end{Bmatrix} = \sum_{j=0}^{\infty} \binom{k+j}{k} \begin{Bmatrix} n \\ k+j \end{Bmatrix} \mathscr{C}_j$$

relates the Stirling numbers of the second kind and Cauchy numbers of the first kind.

Another Riordan array related to the Cauchy numbers arises when we consider integrals of the binomial coefficients in the range $[0 \ldots k]$ $(k \in N)$ instead of the range $[0 \ldots 1]$. First of all, we have:

Theorem 3.2. *The value of a binomial coefficient integral between* 0 *and k is given by the following formula*:

$$\int_0^k \binom{x}{n} \,\mathrm{d}x = [t^n]\frac{(1+t)^k - 1}{\ln(1+t)}. \tag{3.1}$$

Proof. We follow the pattern used in the case $k = 1$:

$$\int_0^k \binom{x}{n} \,\mathrm{d}x = (-1)^{n+1} \sum_{j=0}^{n} \frac{j!}{n!} \begin{bmatrix} n \\ j \end{bmatrix} \frac{(-k)^{j+1}}{(j+1)!}. \tag{3.2}$$

The last factor is the development of e^{-kt} and by using the Riordan array associated to the Stirling numbers of the first kind we obtain the desired result. $\square$

By observing the generating function, we find that the integral (3.1) can be developed in terms of the Cauchy numbers; for example

$$\int_0^3 \binom{x}{n}\,\mathrm{d}x = [t^n]\frac{3t+3t^2+t^3}{\ln(1+t)} = 3\frac{\mathscr{C}_n}{n!} + 3\frac{\mathscr{C}_{n-1}}{(n-1)!} + \frac{\mathscr{C}_{n-2}}{(n-2)!};$$

on the other hand, Eq. (3.2) shows an alternative development in terms of the Stirling numbers of the first kind, and therefore we can state the following result:

Corollary 3.3. *The following relation holds true*:

$$\int_0^k \binom{x}{n}\,\mathrm{d}x = \sum_{j=1}^{k}\binom{k}{j}\frac{\mathscr{C}_{n-j+1}}{(n-j+1)!} = \frac{(-1)^{n+1}}{n!}\sum_{j=0}^{n}\left[{n \atop j}\right]\frac{(-k)^{j+1}}{j+1}.$$

From a computational point of view, one can choose the more convenient formula according to the fact that k is smaller than n, or vice versa.

Formula (3.1) suggests to introduce the Riordan array

$$M = \left(\frac{t}{\ln(1+t)}, (1+t)\right)$$

the generic element of which is

$$M_{n,k} = [t^n]\frac{t}{\ln(1+t)}(t(1+t))^k = [t^{n-k}]\frac{t(1+t)^k}{\ln(1+t)}.$$

This implies:

Lemma 3.4. *The elements of the Riordan array M are given by*

$$M_{n,k} = \int_0^k \binom{x}{n-k-1}\,\mathrm{d}x + \frac{\mathscr{C}_{n-k}}{(n-k)!}.$$

Proof. Let us consider the following identity:

$$\frac{((1+t)^k-1)t^{k+1}}{\ln(1+t)} = \frac{t(t(1+t))^k}{\ln(1+t)} - \frac{t^{k+1}}{\ln(1+t)}.$$

When we extract the coefficient of t^n from the left-hand side, by (3.1) we get the integral of a binomial coefficient; by extracting the same coefficient from the right-hand side, we obtain $M_{n,k} - \mathscr{C}_{n-k}/(n-k)!$, and from this the desired identity follows immediately. □

Obviously, the previous formula can be read in such a way that the integral is computed by means of the elements in M. In order to obtain a simple way to compute these elements, we can look for the A-sequence of M:

Theorem 3.5. *The A-sequence for the Riordan array M is given by*

$$A(t) = \frac{1+\sqrt{1+4t}}{2} = 1 + t - t^2 + 2t^3 - 5t^4 + 14t^5 - 42t^6 + 132t^7 + \cdots.$$

Therefore, the array M can be build by observing that column 0 *is constituted by the Cauchy numbers and all the other elements are given by*

$$M_{n+1,k+1} = a_0 M_{n,k} + a_1 M_{n,k+1} + a_2 M_{n,k+2} + a_3 M_{n,k+3} + \cdots,$$

where $a_0 = 1$ and $\{a_1, a_2, a_3, \ldots\}$ are the Catalan numbers with alternating signs.

Proof. We use the fundamental relation $h(t) = A(th(t))$ for the A-sequence of the Riordan array $(d(t), h(t))$; in this case $h(t) = 1 + t$ and by solving in t the equation $y = t(1 + t)$ we find $t = (-1 + \sqrt{1 + 4y})/2$, because it should be $t(0) = 0$. By substituting this value in $h(t)$, we find $A(t)$. The rest is a general property of Riordan arrays, and we have $a_0 = 1$ and $a_j = (-1)^j C_{j-1}$, for every $j > 0$, where $C_j = \binom{2j}{j}/(j + 1)$ is the jth Catalan number. □

Another reason for considering the array M is given by the following result. First of all, let us introduce a definition: if $r \in \mathbb{Z}$, we call *Cauchy number of the r-th kind* the numbers $\mathscr{C}_n^{[r]}$, whose exponential generating function is

$$\mathscr{C}^{[r]}(t) = \frac{t(1+t)^{1-r}}{\ln(1+t)}.$$

In this way, we have $\mathscr{C}(t) = \mathscr{C}^{[1]}(t)$ and $\widehat{\mathscr{C}}(t) = \mathscr{C}^{[2]}(t)$ as expected, and for non-positive values of r we find the columns of the Riordan array M (except for a shift of $r - 1$ positions). The behavior of these numbers when considered with binomial coefficients is as follows:

Theorem 3.6. *For every $r \in \mathbb{Z}$ the relation*

$$\sum_{k=0}^{n} \binom{n}{k} \frac{\mathscr{C}_k^{[r]}}{k!} = (-1)^n \frac{\mathscr{C}_n^{[4-r]}}{n!}$$

holds true.

Proof. We apply the Euler transformation to the generating functions:

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k} \frac{\mathscr{C}_k^{[r]}}{k!} &= [t^n]\frac{1}{1-t}\left[\frac{y(1+y)^{1-r}}{\ln(1+y)}\middle| y = \frac{t}{1-t}\right] \\
&= [t^n]\frac{t}{(1-t)^2}\left(1 + \frac{t}{1-t}\right)^{1-r} \Big/ \ln\left(1 + \frac{t}{1-t}\right) \\
&= [t^n]\frac{t(1-t)^{r-3}}{\ln(1/(1-t))} = (-1)^n [t^n]\frac{t(1+t)^{r-3}}{\ln(1+t)}.
\end{aligned}$$

This is the desired relation. □

In particular, the Cauchy numbers of the first kind generate the Cauchy numbers of the third kind (and vice versa), while the Cauchy numbers of the second kind are "invariant", as we have already shown in Theorem 2.7. This leads us to introduce another Riordan array:

$$N = \left(\frac{t}{\ln(1+t)}, \frac{1}{1+t}\right).$$

Again, column 0 is composed of the Cauchy numbers of the first kind; the other elements are obtained in a very simple way. In fact we have:

Theorem 3.7. *The A-sequence of the Riordan array N is*

$$A(t) = 1 - t,$$

so that every element $N_{n+1,k+1}$ is obtained by the simple rule

$$N_{n+1,k+1} = N_{n,k} - N_{n,k+1}.$$

Proof. By using the general relation $h(t) = A(th(t))$ and setting $y = th(t) = t/(1 + t)$, we find $t = 1/(1 - y)$, and therefore $A(y) = 1 - y$. □

It is interesting now to find the inverse Riordan array N^{-1}; according to the theory, we have $N^{-1}=(\overline{d}(t),\overline{h}(t))$, where

$$\overline{d}(t)=[d(y)^{-1}\mid y=th(t)]\quad\text{and}\quad\overline{h}(t)=[h(y)^{-1}\mid y=th(t)].$$

This immediately gives

$$N^{-1}=\left(\frac{(1-t)\ln(1/(1-t))}{t},\frac{1}{1-t}\right)$$

and therefore:

- column 0 is composed of the numbers 1 and $-1/n(n+1)$, for every $n>0$;
- column 1 is given by the numbers $1/n$;
- column 2 is composed of the harmonic numbers;

and so on, every element being the partial sum of the elements in the previous column. By using Zave's identity (see [14]), we have

$$N^{-1}_{n,k}=(H_{n-1}-H_{k-2})\binom{n-1}{k-2},\quad n>1,\ k\geqslant 2.$$

4. The Laplace summation formula

One of the most important application of the Cauchy numbers is the so-called *Laplace summation formula*; it is analogous to the Euler–McLaurin formula, but uses the difference operator Δ instead of the differentiation operator $D=\mathrm{d}/\mathrm{d}t$. A formal and classical method to obtain the formula is as follows. Taylor's theorem at $t=0$, in the form

$$f(t+1)=\frac{I}{0!}f(t)+\frac{D}{1!}f(t)+\frac{D^2}{2!}f(t)+\frac{D^3}{3!}f(t)+\cdots$$

can be considered as a way to express the shift operator: $Ef(t)=f(t+1)$, in terms of the differentiation operator, that is $E=\mathrm{e}^D$. By writing $E=1+(E-1)=1+\Delta$, where Δ is the *forward difference operator* $\Delta f(t)=f(t+1)-f(t)=(E-1)f(t)$, we find

$$1+\Delta=\mathrm{e}^D\quad\text{or}\quad D=\ln(1+\Delta).$$

The *integration operator* $\int=\int f(t)\,\mathrm{d}t$ can be expressed in terms of Δ by inverting the differentiation operator, that is

$$\int=D^{-1}=\frac{1}{\ln(1+\Delta)}=\Delta^{-1}\frac{\Delta}{\ln(1+\Delta)}.$$

Here we recognize the exponential generating function of the Cauchy numbers and therefore

$$\int=\Delta^{-1}\sum_{k=0}^{\infty}\frac{\mathscr{C}_k}{k!}\Delta^k.$$

This formula can be explicitly written as

$$\int=\Delta^{-1}+\frac{1}{2}-\frac{1}{12}\Delta+\frac{1}{24}\Delta^2-\frac{19}{720}\Delta^3+\frac{3}{160}\Delta^4-\cdots$$

and is known as the Laplace summation formula, which (as the Euler–McLaurin formula) was devised to perform approximate integration. We, however, are more interested in the formula to obtain indefinite and definite sums:

$$\Delta^{-1}=\int-\frac{1}{2}+\frac{1}{12}\Delta-\frac{1}{24}\Delta^2+\frac{19}{720}\Delta^3-\frac{3}{160}\Delta^4+\cdots,$$

where Δ^{-1} is the *indefinite summation operator*, the inverse of the difference operator. Passing to definite summation we have

$$\sum_{k=a}^{b-1} f(k) = \int_a^b f(t)\,\mathrm{d}t - \frac{1}{2}[f(t)]_a^b + \frac{1}{12}[\Delta f(t)]_a^b - \frac{1}{24}[\Delta^2 f(t)]_a^b + \cdots.$$

Usually, the Euler–McLaurin formula is considered more useful than Laplace formula. First of all, the difference operator is more complex than differentiation, except in a few cases, and consequently the resulting expressions are simpler for the Euler–McLaurin case. Furthermore, Bernoulli numbers are zero for odd indices $n \geqslant 3$, and this simplifies formulas and improves accuracy. Because of that, Laplace formula is not even quoted in many texts: the interested reader is referred to the books of Milne-Thomson [7] and Boole [1]. Strictly related to Laplace's is Gregory's formula, for which a more recent quotation is the book of Henrici [4].

However, some applications of the Laplace summation formula are worth considering, and here we wish to give some examples of this fact and to emphasize the role of Cauchy numbers.

Let us begin with the classical case of *harmonic numbers* $H_n = \sum_{k=1}^{n} 1/k$; here the relevant function is $f(t) = 1/t$, the differences of which are

$$\Delta^n \frac{1}{t} = \frac{(-1)^n n!}{t(t+1)\cdots(t+n)};$$

therefore we have

$$\begin{aligned}\sum_{k=1}^{n-1} \frac{1}{k} &= \int_1^n \frac{\mathrm{d}t}{t} - \frac{1}{2}\left[\frac{1}{t}\right]_1^n + \frac{1}{12}\left[\frac{-1}{t(t+1)}\right]_1^n - \frac{1}{24}\left[\frac{2}{t(t+1)(t+2)}\right]_1^n + \cdots \\ &= \ln n + \frac{1}{2n} + \frac{1}{2} - \frac{1}{12n(n+1)} + \frac{1}{24} - \frac{1}{12n(n+1)(n+2)} + \frac{1}{72} - \cdots.\end{aligned}$$

By collecting all the constants and adding $1/n$ to both sides, we get

$$H_n = \ln n + \gamma + \frac{1}{2n} - \frac{1}{12n(n+1)} - \frac{1}{12n(n+1)(n+2)} - \frac{19}{720n(n+1)(n+2)(n+3)} - \cdots.$$

Some interesting points are worth to be noted in this formula:

1. the signs are all negative from the forth term onwards; consequently, the formula should converge to H_n, in contrast with the analogous formula obtained by the Euler–McLaurin method, which definitely oscillates because of the presence of the Bernoulli numbers;
2. by comparing the Laplace and the Euler–McLaurin formulas, we obtain

   $$\ln n + \frac{1}{2n} - \frac{1}{12n^2} < H_n < \ln n + \frac{1}{2n} - \frac{1}{12n(n+1)},$$

   a rather stringent limitation for harmonic numbers;
3. by looking at the constants appearing in the development and corresponding to the value $t = 1$, we obtain a formula for γ involving the Cauchy numbers, that is

   $$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{\mathscr{C}_k}{k \cdot k!} = \gamma;$$

   the terms are all positive, so the formula is convergent; the convergence is rather slow, but is effective, this is also in contrast with the analogous formula involving the Bernoulli numbers.

The other classical case, i.e., the Stirling approximation for $n!$, is a negative example for applying the Laplace formula; in fact, the successive differences of $\ln(t)$ are much more complicated than the successive differentiations.

The constant involved in that case is $\sigma = \ln\sqrt{2\pi} \approx 0.9189388$; the Laplace formula gives

$$\sigma = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k \mathscr{C}_k}{k!} \Phi(k-1),$$

where $\Phi(j)$ is the jth difference $\Delta^j \ln(t)$, computed at $t = 1$. In fact, we find

$$\Phi(p) = [\Delta^p \ln(t) | t = 1] = \ln \prod_{k=1}^{p+1} k^{(-1)^k \binom{p}{k-1}}.$$

For example, when $p = 4$ we have

$$\Phi(4) = [\Delta^4 \ln(t) | t = 1] = \ln \frac{2^4 \cdot 4^4}{1 \cdot 3^6 \cdot 5} = \ln \frac{4096}{3645}$$

and the sum begins

$$\sigma = 1 - \tfrac{1}{12} \ln 2 - \tfrac{1}{24} \ln \tfrac{4}{3} - \tfrac{19}{720} \ln \tfrac{32}{27} - \tfrac{3}{160} \ln \tfrac{4096}{3645} - \cdots.$$

The series converges, but its expression is not particularly appealing. A more interesting example is given by the sum $\sum_{k=1}^{n} 1/k^2$:

$$\sum_{k=1}^{n-1} \frac{1}{k^2} = \int_1^n \frac{dt}{t^2} - \frac{1}{2}\left[\frac{1}{t^2}\right]_1^n + \frac{1}{12}\left[\frac{1}{(t+1)^2} - \frac{1}{t^2}\right]_1^n - \frac{1}{24}\left[\frac{1}{(t+2)^2} - \frac{2}{(t+1)^2} + \frac{1}{t^2}\right]_1^n + \cdots,$$

$$\sum_{k=1}^{n-1} \frac{1}{k^2} = K - \frac{1}{n} + \frac{1}{2n^2} - \frac{2n+1}{12n^2(n+1)^2} - \cdots,$$

where the constant K should be $\pi^2/6$ and actually is

$$K = 1 + \tfrac{1}{2} + \tfrac{1}{12}\left(1 - \tfrac{1}{4}\right) + \tfrac{1}{24}\left(1 - \tfrac{2}{4} + \tfrac{1}{9}\right) + \tfrac{19}{720}\left(1 - \tfrac{3}{4} + \tfrac{3}{9} - \tfrac{1}{16}\right) + \cdots.$$

The expressions within parentheses are sums, which can be closed by means of the Riordan array method:

Lemma 4.1. *The sums involved in the computation of the constant K are*

$$S_n = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{(k+1)^2} = \frac{H_{n+1}}{n+1}.$$

Proof. First of all we observe the ordinary generating function

$$\mathscr{G}\left(\frac{1}{k^2}\right) = \int_0^t \ln\left(\frac{1}{1-\xi}\right) \frac{d\xi}{\xi};$$

this function should be shifted and transformed by $t \to -t$ to get

$$\mathscr{G}\left(\frac{(-1)^k}{(k+1)^2}\right) = \frac{1}{t}\left(\int_0^t \ln\left(\frac{1}{1-\xi}\right) \frac{d\xi}{\xi} - 1\right).$$

By using the Euler transformation, we now have

$$\begin{aligned} S_n &= [t^n] \frac{1}{1-t} \frac{1-t}{t} \left(\int_0^t \ln\left(1 + \frac{\xi}{1-\xi}\right) \frac{1-\xi}{\xi} \frac{d\xi}{(1-\xi)^2} - 1\right) \\ &= [t^{n+1}] \left(\int_0^t \ln\left(\frac{1}{1-\xi}\right)\left(\frac{1}{\xi} + \frac{1}{1-\xi}\right) d\xi - 1\right) = \frac{1}{(n+1)^2} + [t^{n+1}] \frac{1}{2}\left(\ln \frac{1}{1-t}\right)^2. \end{aligned}$$

The last coefficient can be extracted easily:

$$\frac{1}{2}[t^{n+1}]\left(\ln\frac{1}{1-t}\right)^2=\frac{1}{2(n+1)}[t^n]\frac{\mathrm{d}}{\mathrm{d}t}\left(\ln\frac{1}{1-t}\right)^2=\frac{1}{2(n+1)}[t^n]\frac{2}{1-t}\ln\frac{1}{1-t}=\frac{H_n}{n+1}.$$

In conclusion

$$S_n=\frac{1}{(n+1)^2}+\frac{H_n}{n+1}=\frac{H_{n+1}}{n+1}$$

as desired. □

By substituting this value into the expression for K, we obtain the immediate corollary:

Corollary 4.2. *The following expression for the constant K holds true*:

$$K=\frac{\pi^2}{6}=1+\sum_{k=1}^{\infty}\frac{\mathscr{C}_k}{k!}\,\frac{(-1)^{k-1}H_k}{k}.$$

This is another relevant identity related to the Cauchy numbers. Again, the terms are all positive and therefore the series should converge, even if the convergence is extremely slow.

Acknowledgments

We wish to thank the anonymous referees for their useful suggestions.

References

[1] G. Boole, An Investigation of the Law of Thought, Dover, New York, 1958 (reproduction of 1984 edition).
[2] L. Comtet, Advanced Combinatorics, Reidel, Dordrecht, 1974.
[3] D.H. Greene, D.E. Knuth, Mathematics for the Analysis of Algorithms, Birkäuser, Boston, 1982.
[4] P. Henrici, Applied and Computational Complex Analysis, I, Wiley, New York, 1988.
[5] D.L. Jagerman, Difference Equations with Applications to Queues, Marcel Dekker, New York, 2000.
[6] D. Merlini, R. Sprugnoli, M.C. Verri, The method of coefficients, Amer. Math. Monthly, accepted for publication.
[7] L.M. Milne-Thomson, The Calculus of Finite Differences, Macmillan and Co., London, 1951.
[8] L.W. Shapiro, S. Getu, W.-J. Woan, L. Woodson, The Riordan group, Discrete Appl. Math. 34 (1991) 229–239.
[9] R. Sprugnoli, Riordan arrays and combinatorial sums, Discrete Math. 132 (1994) 267–290.
[10] T. Wang, X. Zhao, Some identities related to reciprocal functions, Discrete Math. 265 (2003) 323–335.
[11] T. Wang, X. Zhao, S. Ding, Some summation rules related to Riordan arrays, Discrete Math. 281 (2004) 295–307.
[12] H.S. Wilf, Generating Functionology, Academic Press, Boston, 1990.
[13] L. Yingying, On Euler's constant—calculating sums by integrals, Amer. Math. Monthly 109 (2002) 845–850.
[14] D.A. Zave, A series expansion involving the harmonic numbers, Inform. Process. Lett. 5 (1976) 75–77.