Hindawi Publishing Corporation
Journal of Discrete Mathematics
Volume 2013, Article ID 373927, 10 pages
http://dx.doi.org/10.1155/2013/373927

Research Article

Sums of Products of Cauchy Numbers, Including Poly-Cauchy Numbers

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Received 24 July 2012; Accepted 24 October 2012

Academic Editor: Gi Sang Cheon

We investigate sums of products of Cauchy numbers including poly-Cauchy numbers: $T_m^{(k)}(n) = \sum_{i_1+\cdots+i_m=n, i_1,\dots,i_m\ge 0}\binom{n}{i_1,\dots,i_m} c_{i_1}\cdots c_{i_{m-1}} c_{i_m}^{(k)}$ $(m \ge 1, n \ge 0)$. A relation among these sums $T_m^{(k)}(n)$ shown in the paper and explicit expressions of sums of two and three products (the case of $m = 2$ and that of $m = 3$ described in the paper) are given. We also study the other three types of sums of products related to the Cauchy numbers of both kinds and the poly-Cauchy numbers of both kinds.

1. Introduction

The Cauchy numbers (of the first kind) c_n are defined by the integral of the falling factorial:

$$c_n = \int_0^1 x(x-1)\cdots(x-n+1)\,dx = n!\int_0^1 \binom{x}{n}dx \tag{1}$$

(see [1, Chapter VII]). The numbers $c_n/n!$ are sometimes called the Bernoulli numbers of the second kind (see e.g., [2, 3]). Such numbers have been studied by several authors [4–8] because they are related to various special combinatorial numbers, including Stirling numbers of both kinds, Bernoulli numbers, and harmonic numbers. It is interesting to see that the Cauchy numbers of the first kind c_n have the similar properties and expressions to the Bernoulli numbers B_n. For example, the generating function of the Cauchy numbers of the first kind c_n is expressed in terms of the logarithmic function:

$$\frac{x}{\ln(1+x)} = \sum_{n=0}^{\infty} c_n \frac{x^n}{n!}, \tag{2}$$

(see [1, 6]), and the generating function of Bernoulli numbers B_n is expressed in terms of the exponential function:

$$\frac{x}{e^x-1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}, \tag{3}$$

(see [1]) or

$$\frac{x}{1-e^{-x}} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \tag{4}$$

(see [9]). In addition, Cauchy numbers of the first kind c_n can be written explicitly as

$$c_n = (-1)^n \sum_{m=0}^{n} \left[{n \atop m}\right] \frac{(-1)^m}{m+1} \tag{5}$$

(see [1, Chapter VII], [6, page 1908]), where $\left[{n \atop m}\right]$ are the (unsigned) Stirling numbers of the first kind, arising as coefficients of the rising factorial

$$x(x+1)\cdots(x+n-1) = \sum_{m=0}^{n} \left[{n \atop m}\right] x^m \tag{6}$$

(see e.g., [10]). Bernoulli numbers B_n (in the latter definition) can be also written explicitly as

$$B_n = (-1)^n \sum_{m=0}^{n} \left\{{n \atop m}\right\} \frac{(-1)^m m!}{m+1}, \tag{7}$$

where $\left\{{n \atop m}\right\}$ are the Stirling numbers of the second kind, determined by

$$\left\{{n \atop m}\right\} = \frac{1}{m!}\sum_{j=0}^{m}(-1)^j \binom{m}{j}(m-j)^n \tag{8}$$

(see, e.g., [10]). Recently, Liu et al. [5] established some recurrence relations about Cauchy numbers of the first kind as analogous results about Bernoulli numbers by Agoh and Dilcher [11].

In 1997 Kaneko [9] introduced the poly-Bernoulli numbers $B_n^{(k)}$ $(n \ge 0, k \ge 1)$ by the generating function

$$\frac{\mathrm{Li}_k\left(1-e^{-x}\right)}{1-e^{-x}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{x^n}{n!}, \tag{9}$$

where

$$\mathrm{Li}_k(z) = \sum_{m=0}^{\infty} \frac{z^m}{m^k} \tag{10}$$

is the kth polylogarithm function. When $k=1$, $B_n^{(1)} = B_n$ is the classical Bernoulli number with $B_1^{(1)} = 1/2$. On the other hand, the author [12] introduced the poly-Cauchy numbers (of the first kind) $c_n^{(k)}$ as a generalization of the Cauchy numbers and an analogue of the poly-Bernoulli numbers by the following:

$$\begin{aligned} c_n^{(k)} = \underbrace{\int_0^1 \cdots \int_0^1}_{k} &\left(x_1 x_2 \cdots x_k\right)\left(x_1 x_2 \cdots x_k - 1\right) \cdots \\ &\left(x_1 x_2 \cdots x_k - n + 1\right) dx_1 dx_2 \cdots dx_k. \end{aligned} \tag{11}$$

In addition, the generating function of poly-Cauchy numbers is given by

$$\mathrm{Lif}_k(\ln(1+x)) = \sum_{n=0}^{\infty} c_n^{(k)} \frac{x^n}{n!}, \tag{12}$$

where

$$\mathrm{Lif}_k(z) := \sum_{m=0}^{\infty} \frac{z^m}{m!(m+1)^k} \tag{13}$$

is the kth *polylogarithm factorial* function, which is also introduced by the author [12, 13]. If $k=1$, then $c_n^{(1)} = c_n$ is the classical Cauchy number.

The following identity on sums of two products of Bernoulli numbers is known as Euler's formula:

$$\sum_{i=0}^{n} \binom{n}{i} B_i B_{n-i} = -nB_{n-1} - (n-1) B_n \quad (n \ge 0). \tag{14}$$

The corresponding formula for Cauchy numbers was discovered in [8]:

$$\sum_{i=0}^{n} \binom{n}{i} c_i c_{n-i} = -(n-1) c_n - n(n-2) c_{n-1} \quad (n \ge 0). \tag{15}$$

In this paper, we shall give more analogous results by investigating a general type of sums of products of Cauchy numbers including poly-Cauchy numbers:

$$\sum_{\substack{i_1+\cdots+i_m=n\\ i_1,\ldots,i_m\ge 0}} \binom{n}{i_1,\ldots,i_m} \underbrace{c_{i_1}\cdots c_{i_{m-1}}}_{m-1} c_{i_m}^{(k)} \quad (m \ge 1, n \ge 0), \tag{16}$$

whose Bernoulli version is discussed in [14]. A relation among these sums and explicit expressions of sums of two and three products are also given.

2. Main Results

We shall consider the sums of products of Cauchy numbers including poly-Cauchy numbers. Kamano [14] investigated the following types of sums of products:

$$\begin{aligned} S_m^{(k)}(n) := \sum_{\substack{i_1+\cdots+i_m=n\\ i_1,\ldots,i_m\ge 0}} \binom{n}{i_1,\ldots,i_m} \underbrace{B_{i_1}\cdots B_{i_{m-1}}}_{m-1} B_{i_m}^{(k)} \\ (m \ge 1, n \ge 0), \end{aligned} \tag{17}$$

where Bernoulli numbers B_n are defined by the generating function (3) and poly-Bernoulli numbers $B_n^{(k)}$ are defined by the generating function (9) and $\mathrm{Li}_k(z)$ is the kth polylogarithm function defined in (10). It is shown [14] that

$$\begin{aligned} &\sum_{l=0}^{m} (-1)^{m-l} \begin{bmatrix} m+1 \\ l+1 \end{bmatrix} S_{m+1}^{(k-l)}(n) \\ &= \begin{cases} \dfrac{n!}{(n-m)!} \displaystyle\sum_{l=0}^{m} \begin{bmatrix} m \\ l \end{bmatrix} B_{n-m+l}^{(k)}, & (n \ge m), \\ 0, & (0 \le n \le m-1). \end{cases} \end{aligned} \tag{18}$$

Consider an analogous type of sums of products of Cauchy numbers including poly-Cauchy numbers:

$$\begin{aligned} T_m^{(k)}(n) := \sum_{\substack{i_1+\cdots+i_m=n\\ i_1,\ldots,i_m\ge 0}} \binom{n}{i_1,\ldots,i_m} \underbrace{c_{i_1}\cdots c_{i_{m-1}}}_{m-1} c_{i_m}^{(k)} \\ (m \ge 1, n \ge 0). \end{aligned} \tag{19}$$

Then we show the following result.

Theorem 1. *For an integer k and a nonnegative integer m, one has*

$$\begin{aligned} &\sum_{l=0}^{m} (-1)^{m-l} \begin{bmatrix} m+1 \\ l+1 \end{bmatrix} T_{m+1}^{(k-l)}(n) \\ &= \begin{cases} \displaystyle\sum_{l=0}^{m} \sum_{i=0}^{n-m} \frac{n!}{i!} \binom{l}{n-m-i} \begin{Bmatrix} m \\ l \end{Bmatrix} c_{l+i}^{(k)} & (n \ge m), \\ 0 & (0 \le n \le m-1). \end{cases} \end{aligned} \tag{20}$$

Note that the generating function of $T_m^{(k)}$ is given by

$$\left(\frac{x}{\ln(1+x)}\right)^{m-1} \mathrm{Lif}_k(\ln(1+x)) = \sum_{n=0}^{\infty} T_m^{(k)}(n) \frac{x^n}{n!}. \tag{21}$$

Put

$$G_k(x) := \mathrm{Lif}_k(\ln(1+x)) = \sum_{n=0}^{\infty} c_n^{(k)} \frac{x^n}{n!}. \tag{22}$$

Since

$$\begin{aligned} &\mathrm{Lif}_1(z) = \frac{e^z-1}{z}, \qquad \mathrm{Lif}_0(z) = e^z, \\ &\mathrm{Lif}_{-1}(z) = (z+1) e^z, \end{aligned} \tag{23}$$

we have

$$G_1(x) = \frac{x}{\ln(1+x)}, \qquad G_0(x) = 1+x,$$
$$G_{-1}(x) = (1+x)(\ln(1+x)+1). \tag{24}$$

Since

$$x^m \frac{d^l}{dx^l} G_k(x) = \sum_{i=0}^{\infty} c_{l+i}^{(k)} \frac{x^{m+i}}{i!} \quad (m, l \geq 0, k \geq 1), \tag{25}$$

the coefficient of x^n in

$$x^m \frac{d^l}{dx^l} G_k(x) \tag{26}$$

is equal to

$$\begin{aligned} &\frac{c_{n-m+l}^{(k)}}{(n-m)!} \quad (n \geq m), \\ &0 \quad (0 \leq n \leq m-1). \end{aligned} \tag{27}$$

We need the following lemma in order to prove Theorem 1.

Lemma 2. *For an integer k and a positive integer m, one has*

$$\begin{aligned} &\left(\sum_{i=1}^{m} \left\{ {m \atop i} \right\} \frac{1}{(1+x)^{m-i}} \frac{d^i}{dx^i}\right) G_k(x) \\ &= \frac{1}{(1+x)^m (\ln(1+x))^m} \sum_{l=0}^{m} (-1)^{m-l} \left[{m+1 \atop l+1} \right] G_{k-l}(x). \end{aligned} \tag{28}$$

Proof of Lemma 2. Since

$$\begin{aligned} \frac{d}{dx} \mathrm{Lif}_k(x) &= \frac{1}{x} \sum_{m=0}^{\infty} \frac{m x^m}{m!(m+1)^k} \\ &= \frac{1}{x} \sum_{m=0}^{\infty} \left(\frac{x^m}{m!(m+1)^{k-1}} - \frac{x^m}{m!(m+1)^k} \right) \\ &= \frac{\mathrm{Lif}_{k-1}(x) - \mathrm{Lif}_k(x)}{x}, \end{aligned} \tag{29}$$

we have

$$\frac{d}{dx} G_k(x) = \frac{G_{k-1}(x) - G_k(x)}{(1+x)\ln(1+x)}. \tag{30}$$

By induction, we can show that for $m \geq 1$

$$\frac{d^m}{dx^m} G_k(x) = \sum_{\nu=1}^{m} \frac{(-1)^{m-\nu}}{(1+x)^m (\log(1+x))^\nu} \left[{m \atop \nu} \right] g_{\nu+1}(x), \tag{31}$$

where

$$g_{\nu+1}(x) := \sum_{l=0}^{\nu} (-1)^{\nu-l} \left[{\nu+1 \atop l+1} \right] G_{k-l}(x) \quad (\nu = 1, 2, \ldots, m). \tag{32}$$

Thus, by using the inversion relationship

$$\sum_{j=\nu}^{m} (-1)^{j-\nu} \left\{ {m \atop j} \right\} \left[{j \atop \nu} \right] = \begin{cases} 1 & (\nu = m); \\ 0 & (\nu = 1, 2, \ldots, m-1) \end{cases} \tag{33}$$

(see e.g., [10, Chapter 6]), the left-hand side of the identity in the previous lemma is equal to

$$\begin{aligned} &\sum_{j=1}^{m} \left\{ {m \atop j} \right\} \frac{1}{(1+x)^{m-j}} \sum_{\nu=1}^{j} \frac{(-1)^{j-\nu}}{(1+x)^j (\ln(1+x))^\nu} \left[{j \atop \nu} \right] g_{\nu+1}(x) \\ &= \frac{1}{(1+x)^m} \sum_{\nu=1}^{m} \frac{g_{\nu+1}(x)}{(\ln(1+x))^\nu} \sum_{j=\nu}^{m} (-1)^{j-\nu} \left\{ {m \atop j} \right\} \left[{j \atop \nu} \right] \\ &= \frac{g_{m+1}(x)}{(1+x)^m (\ln(1+x))^m}, \end{aligned} \tag{34}$$

which is the right-hand side of the desired identity. □

Now, by the generating function (21), the identity (25), and Lemma 2,

$$\begin{aligned} &\sum_{n=0}^{\infty} \left(\sum_{l=0}^{m} (-1)^{m-l} \left[{m+1 \atop l+1} \right] T_{m+1}^{(k-l)}(n) \right) \frac{x^n}{n!} \\ &= \left(\frac{x}{\ln(1+x)} \right)^m \sum_{l=0}^{m} (-1)^{m-l} \left[{m+1 \atop l+1} \right] G_{k-l}(x) \\ &= x^m \left(\sum_{i=1}^{m} (1+x)^i \left\{ {m \atop i} \right\} \frac{d^i}{dx^i} \right) G_k(x) \\ &= \sum_{l=0}^{m} (1+x)^l \left\{ {m \atop l} \right\} \sum_{i=0}^{\infty} c_{l+i}^{(k)} \frac{x^{m+i}}{i!} \\ &= \sum_{l=0}^{m} \sum_{\kappa=0}^{l} \binom{l}{\kappa} x^\kappa \left\{ {m \atop l} \right\} \sum_{i=0}^{\infty} c_{l+i}^{(k)} \frac{x^{m+i}}{i!} \\ &= \sum_{l=0}^{m} \sum_{n=0}^{\infty} \sum_{i=0}^{n} \binom{l}{n-i} \left\{ {m \atop l} \right\} \frac{c_{l+i}^{(k)}}{i!} x^{m+n} \\ &= \sum_{n=m}^{\infty} \left(\sum_{l=0}^{m} \sum_{i=0}^{n-m} \frac{n!}{i!} \binom{l}{n-m-i} \left\{ {m \atop l} \right\} c_{l+i}^{(k)} \right) \frac{x^n}{n!}. \end{aligned} \tag{35}$$

Note that $\binom{l}{\nu} = 0$ $(l < \nu)$ and $\left\{ {m \atop 0} \right\} = 0$ $(m \geq 1)$. Therefore,

$$\begin{aligned} &\sum_{l=0}^{m} (-1)^{m-l} \left[{m+1 \atop l+1} \right] T_{m+1}^{(k-l)}(n) \\ &= \begin{cases} \displaystyle\sum_{l=0}^{m} \sum_{i=0}^{n-m} \frac{n!}{i!} \binom{l}{n-m-i} \left\{ {m \atop l} \right\} c_{l+i}^{(k)} & (n \geq m); \\ 0 & (0 \leq n \leq m-1). \end{cases} \end{aligned} \tag{36}$$

If we put $m = 1$ in Theorem 1, we get an analogous formula to Euler's formula (14) for sums of products of Cauchy number and a poly-Cauchy number.

Corollary 3. *One has*

$$\sum_{i=0}^{n}\binom{n}{i}c_i\left(c_{n-i}^{(k-1)}-c_{n-i}^{(k)}\right)=n(n-1)c_{n-1}^{(k)}+nc_n^{(k)}\quad(n\geq 0). \tag{37}$$

2.1. Explicit Formula for $T_2^{(k)}(n)$. Theorem 1 gives only relations among sums of products $T_m^{(k)}(n)$. For $m=2$, an explicit formula for

$$T_2^{(k)}(n)=\sum_{i=0}^{n}\binom{n}{i}c_ic_{n-i}^{(k)} \tag{38}$$

is given.

Theorem 4. *For $n\geq 0$ and $k\geq 1$ one has*

$$\begin{aligned}T_2^{(0)}(n)&=c_n^{(1)}(-1)\\&=(-1)^n\sum_{i=0}^{n}\sum_{m=i}^{n}(-1)^m\begin{bmatrix}n\\m\end{bmatrix}\binom{m}{i}\frac{1}{m-i+1},\end{aligned} \tag{39}$$

$$T_2^{(k)}(n)=c_n^{(1)}(-1)-n\sum_{j=1}^{k}\left(c_n^{(j)}+(n-1)c_{n-1}^{(j)}\right), \tag{40}$$

$$T_2^{(-k)}(n)=c_n^{(1)}(-1)+n\sum_{j=0}^{k-1}\left(c_n^{(-j)}+(n-1)c_{n-1}^{(-j)}\right), \tag{41}$$

where $c_n^{(1)}(-1)=c_n+nc_{n-1}$.

Proof. Consider

$$\begin{aligned}\sum_{n=0}^{\infty}T_2^{(0)}(n)\frac{x^n}{n!}&=\frac{x}{\ln(1+x)}\mathrm{Lif}_0(\ln(1+x))\\&=\frac{(1+x)x}{\ln(1+x)}=\sum_{n=0}^{\infty}c_n^{(1)}(-1)\frac{x^n}{n!},\end{aligned} \tag{42}$$

where $c_n^{(k)}(z)$ are poly-Cauchy polynomials of the first kind, defined by the generating function

$$\frac{\mathrm{Lif}_k(\ln(1+x))}{(1+x)^z}=\sum_{n=0}^{\infty}c_n^{(k)}(z)\frac{x^n}{n!}. \tag{43}$$

$c_n^{(k)}(z)$ are expressed explicitly in terms of the Stirling numbers of the first kind [13, Theorem 1]:

$$c_n^{(k)}(z)=\sum_{m=0}^{n}\begin{bmatrix}n\\m\end{bmatrix}(-1)^{n-m}\sum_{i=0}^{m}\binom{m}{i}\frac{(-z)^i}{(m-i+1)^k}. \tag{44}$$

Hence, the identity (39) holds because

$$\begin{aligned}T_2^{(0)}(n)=c_n^{(1)}(-1)&=\sum_{m=0}^{n}\begin{bmatrix}n\\m\end{bmatrix}(-1)^{n-m}\sum_{i=0}^{m}\binom{m}{i}\frac{1}{m-i+1}\\&=(-1)^n\sum_{i=0}^{n}\sum_{m=i}^{n}(-1)^m\begin{bmatrix}n\\m\end{bmatrix}\binom{m}{i}\frac{1}{m-i+1}.\end{aligned} \tag{45}$$

Next, by (30) and $G_0(x)=1+x$ we have

$$\begin{aligned}\sum_{j=1}^{k}\frac{d}{dx}G_j(x)&=\sum_{j=1}^{k}\frac{G_{j-1}(x)-G_j(x)}{(1+x)\ln(1+x)}\\&=\frac{1}{\ln(1+x)}-\frac{G_k(x)}{(1+x)\ln(1+x)}.\end{aligned} \tag{46}$$

Hence,

$$\begin{aligned}\sum_{n=0}^{\infty}T_2^{(k)}(n)\frac{x^n}{n!}&=\frac{x}{\ln(1+x)}G_k(x)\\&=\frac{x(1+x)}{\ln(1+x)}-x(1+x)\sum_{j=1}^{k}\frac{d}{dx}G_j(x)\\&=\sum_{n=0}^{\infty}c_n^{(1)}(-1)\frac{x^n}{n!}-x(1+x)\sum_{j=1}^{k}\sum_{n=0}^{\infty}c_{n+1}^{(j)}\frac{x^n}{n!}\\&=\sum_{n=0}^{\infty}c_n^{(1)}(-1)\frac{x^n}{n!}\\&\quad-\sum_{n=0}^{\infty}\sum_{j=1}^{k}\left(nc_n^{(j)}+n(n-1)c_{n-1}^{(j)}\right)\frac{x^n}{n!}.\end{aligned} \tag{47}$$

Therefore, we get the identity (40).

Finally, by

$$\frac{d}{dx}\mathrm{Lif}_{-k}(x)=\frac{\mathrm{Lif}_{-k-1}(x)-\mathrm{Lif}_{-k}(x)}{x}, \tag{48}$$

we have

$$\begin{aligned}\sum_{j=0}^{k-1}\frac{d}{dx}G_{-j}(x)&=\sum_{j=0}^{k-1}\frac{G_{-j-1}(x)-G_{-j}(x)}{(1+x)\ln(1+x)}\\&=\frac{G_{-k}(x)}{(1+x)\ln(1+x)}-\frac{1}{\ln(1+x)}.\end{aligned} \tag{49}$$

Hence,

$$\begin{aligned}\sum_{n=0}^{\infty}T_2^{(-k)}(n)\frac{x^n}{n!}&=\frac{x}{\ln(1+x)}G_{-k}(x)\\&=\frac{x(1+x)}{\ln(1+x)}+x(1+x)\sum_{j=0}^{k-1}\frac{d}{dx}G_{-j}(x)\\&=\sum_{n=0}^{\infty}c_n^{(1)}(-1)\frac{x^n}{n!}+x(1+x)\sum_{j=0}^{k-1}\sum_{n=0}^{\infty}c_{n+1}^{(-j)}\frac{x^n}{n!}\\&=\sum_{n=0}^{\infty}c_n^{(1)}(-1)\frac{x^n}{n!}\\&\quad+\sum_{n=0}^{\infty}\sum_{j=0}^{k-1}\left(nc_n^{(-j)}+n(n-1)c_{n-1}^{(-j)}\right)\frac{x^n}{n!}.\end{aligned} \tag{50}$$

Therefore, we get the identity (41). □

Table 1

n	0	1	2	3	4	5	6	7	8	9
c_n	1	1/2	−1/6	1/4	−19/30	9/4	−863/84	1375/24	−33953/90	57281/20
$c_n^{(1)}(-1)$	1	3/2	5/6	−1/4	11/30	−11/12	271/84	−117/8	7297/90	−2125/4

Putting $k = 1$ in (40), we have the following identity, which is also found in [8]. This is also an analogous formula to Euler's formula (14).

Corollary 5. *One has (see also Table 1)*

$$\sum_{i=0}^{n} \binom{n}{i} c_i c_{n-i} = -(n-1) c_n - n(n-2) c_{n-1} \quad (n \geq 0). \quad (51)$$

Proof. Since

$$\begin{aligned} \sum_{n=0}^{\infty} T_2^{(0)}(n) \frac{x^n}{n!} &= \frac{(1+x)x}{\ln(1+x)} = (1+x) \sum_{n=0}^{\infty} c_n \frac{x^n}{n!} \\ &= \sum_{n=0}^{\infty} (c_n + n c_{n-1}) \frac{x^n}{n!}, \end{aligned} \quad (52)$$

we have $c_n^{(1)}(-1) = c_n + n c_{n-1}$. Hence,

$$\begin{aligned} \sum_{i=0}^{n} \binom{n}{i} c_i c_{n-i} &= T_2^{(1)}(n) \\ &= c_n^{(1)}(-1) - n(c_n + (n-1)c_{n-1}) \\ &= -(n-1)c_n - n(n-2)c_{n-1}. \end{aligned} \quad (53)$$

□

2.2. Explicit Formulae for $T_3^{(k)}(n)$. For $m = 3$, an explicit formula for

$$T_3^{(k)}(n) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} \frac{n!}{i!\,j!\,(n-i-j)!} c_i c_j c_{n-i-j}^{(k)} \quad (54)$$

is also given.

Theorem 6. *For $n \geq 0$ and $k \geq 1$ one has*

$$\begin{aligned} T_3^{(0)}(n) &= -(n-1)c_n - 2n(n-2)c_{n-1} \\ &\quad - n(n-1)(n-3)c_{n-2}, \end{aligned} \quad (55)$$

$$\begin{aligned} T_3^{(k)}(n) &= T_3^{(0)}(n) - \left(1 - 2^{-k}\right) n \left(c_{n-1} + (n-1)c_{n-2}\right) \\ &\quad + n(n-1) \sum_{j=1}^{k} \left(1 - 2^{j-k-1}\right) \\ &\quad \times \left((n-2)^2 c_{n-2}^{(j)} + (2n-3) c_{n-1}^{(j)} + c_n^{(j)}\right), \end{aligned} \quad (56)$$

$$\begin{aligned} T_3^{(-k)}(n) &= T_3^{(0)}(n) + \left(2^k - 1\right) n \left(c_{n-1} + (n-1)c_{n-2}\right) \\ &\quad + n(n-1) \sum_{j=0}^{k-2} \left(2^{k-j-1} - 1\right) \\ &\quad \times \left((n-2)^2 c_{n-2}^{(-j)} + (2n-3) c_{n-1}^{(-j)} + c_n^{(-j)}\right). \end{aligned} \quad (57)$$

Proof. Consider

$$\begin{aligned} \sum_{n=0}^{\infty} T_3^{(0)}(n) \frac{x^n}{n!} &= \frac{x^2(1+x)}{(\ln(1+x))^2} = (1+x) \sum_{n=0}^{\infty} T_2^{(1)}(n) \frac{x^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(T_2^{(1)}(n) + n T_2^{(1)}(n-1)\right) \frac{x^n}{n!}. \end{aligned} \quad (58)$$

Hence, by Corollary 5

$$\begin{aligned} T_3^{(0)}(n) &= T_2^{(1)}(n) + n T_2^{(1)}(n-1) \\ &= -(n-1)c_n - n(n-2)c_{n-1} \\ &\quad + n\left(-(n-2)c_{n-1} - (n-1)(n-3)c_{n-2}\right) \\ &= -(n-1)c_n - 2n(n-2)c_{n-1} - n(n-1)(n-3)c_{n-2}, \end{aligned} \quad (59)$$

which is the identity (55).

By Lemma 2 with $m = 2$

$$\begin{aligned} &\left(\frac{d^2}{dx^2} + \frac{1}{1+x}\frac{d}{dx}\right) G_k(x) \\ &\quad = \frac{\left(2G_k(x) - G_{k-1}(x)\right) - \left(2G_{k-1}(x) - G_{k-2}(x)\right)}{(1+x)^2(\ln(1+x))^2}. \end{aligned} \quad (60)$$

Thus,

$$\begin{aligned} &\sum_{j=1}^{l} \left(\frac{d^2}{dx^2} + \frac{1}{1+x}\frac{d}{dx}\right) G_j(x) \\ &\quad = \frac{2G_l(x) - G_{l-1}(x)}{(1+x)^2(\ln(1+x))^2} - \frac{2G_0(x) - G_{-1}(x)}{(1+x)^2(\ln(1+x))^2}. \end{aligned} \quad (61)$$

By multiplying both sides by 2^{l-1} and summing over l from 1 to k, we obtain

$$\begin{aligned}
&2^k \frac{G_k(x)}{(1+x)^2(\ln(1+x))^2} \\
&\quad= \frac{G_0(x)}{(1+x)^2(\ln(1+x))^2} + \left(\sum_{l=1}^{k} 2^{l-1}\right) \frac{2G_0(x) - G_{-1}(x)}{(1+x)^2(\ln(1+x))^2} \\
&\qquad+ \sum_{l=1}^{k} 2^{l-1} \sum_{j=1}^{l} \left(\frac{d^2}{dx^2} + \frac{1}{1+x}\frac{d}{dx}\right) G_j(x) \\
&\quad= \frac{\left(2^{k+1}-1\right) G_0(x) - \left(2^k - 1\right) G_{-1}(x)}{(1+x)^2(\ln(1+x))^2} \\
&\qquad+ \sum_{j=1}^{k} \left(\sum_{l=j}^{k} 2^{l-1}\right) \left(\frac{d^2}{dx^2} + \frac{1}{1+x}\frac{d}{dx}\right) G_j(x).
\end{aligned} \tag{62}$$

Hence, we have

$$\begin{aligned}
&2^k \left(\frac{x}{\ln(1+x)}\right)^2 G_k(x) \\
&\quad= \left(2^{k+1}-1\right) \frac{(1+x)x^2}{(\ln(1+x))^2} \\
&\qquad- \left(2^k-1\right) \frac{(1+x)x^2(\ln(1+x)+1)}{(\ln(1+x))^2} \\
&\qquad+ x^2(1+x)^2 \sum_{j=1}^{k} \left(2^k - 2^{j-1}\right) \left(\frac{d^2}{dx^2} + \frac{1}{1+x}\frac{d}{dx}\right) G_j(x).
\end{aligned} \tag{63}$$

By comparing the coefficients of $x^n/n!$ in both sides,

$$\begin{aligned}
2^k T_3^{(k)}(n) &= \left(2^{k+1}-1\right) T_3^{(0)}(n) - \left(2^k-1\right) n T_2^{(0)}(n-1) \\
&\quad- \left(2^k-1\right) T_3^{(0)}(n) + \sum_{j=1}^{k} \left(2^k - 2^{j-1}\right) \\
&\quad\times \left(n(n-1) c_n^{(j)} + 2n(n-1)(n-2) c_{n-1}^{(j)}\right. \\
&\qquad+ n(n-1)(n-2)(n-3) c_{n-2}^{(j)} + n(n-1) c_{n-1}^{(j)} \\
&\qquad\left.+ n(n-1)(n-2) c_{n-2}^{(j)}\right) \\
&= 2^k T_3^{(0)}(n) - \left(2^k-1\right) n T_2^{(0)}(n-1) \\
&\quad+ \sum_{j=1}^{k} \left(2^k - 2^{j-1}\right) n(n-1) \\
&\quad\times \left((n-2)^2 c_{n-2}^{(j)} + (2n-3) c_{n-1}^{(j)} + c_n^{(j)}\right).
\end{aligned} \tag{64}$$

Dividing both sides by 2^k, we have the identity (56).

Finally, since

$$\begin{aligned}
&\sum_{j=0}^{l-2} \left(\frac{d^2}{dx^2} + \frac{1}{1+x}\frac{d}{dx}\right) G_{-j}(x) \\
&\quad= \sum_{j=0}^{l-2} \frac{\left(2G_{-j}(x) - G_{-j-1}\right) - \left(2G_{-j-1}(x) - G_{-j-2}\right)}{(1+x)^2(\ln(1+x))^2} \\
&\quad= \sum_{j=0}^{l-2} \left(\frac{2G_0(x) - G_{-1}}{(1+x)^2(\ln(1+x))^2} - \frac{2G_{-l+1}(x) - G_{-l}}{(1+x)^2(\ln(1+x))^2}\right),
\end{aligned} \tag{65}$$

we obtain

$$\begin{aligned}
&2^{-k} \frac{G_{-k}(x)}{(1+x)^2(\ln(1+x))^2} \\
&\quad= \frac{\left(1-2^{-k}\right) G_{-1}(x) - \left(1 - 2^{-k+1}\right) G_0(x)}{(1+x)^2(\ln(1+x))^2} \\
&\qquad+ \sum_{j=0}^{k-2} \left(\sum_{l=j+2}^{k} 2^{-l}\right) \left(\frac{d^2}{dx^2} + \frac{1}{1+x}\frac{d}{dx}\right) G_{-j}(x).
\end{aligned} \tag{66}$$

Hence, we have

$$\begin{aligned}
&2^{-k} \left(\frac{x}{\ln(1+x)}\right)^2 G_{-k}(x) \\
&\quad= \left(1-2^{-k}\right) \frac{(1+x)x^2}{\ln(1+x)} + 2^{-k} \frac{(1+x)x^2}{(\ln(1+x))^2} \\
&\qquad+ x^2(1+x)^2 \sum_{j=0}^{k-2} \left(2^{-j-1} - 2^{-k}\right) \\
&\qquad\times \left(\frac{d^2}{dx^2} + \frac{1}{1+x}\frac{d}{dx}\right) G_{-j}(x),
\end{aligned} \tag{67}$$

yielding the identity (57). □

2.3. Poly-Cauchy Numbers of the Second Kind. Similarly, define $\widehat{T}_m^{(k)}(n)$ by

$$\widehat{T}_m^{(k)}(n) := \sum_{\substack{i_1+\cdots+i_m=n \\ i_1,\ldots,i_m \ge 0}} \binom{n}{i_1,\ldots,i_m} \underbrace{\widehat{c}_{i_1} \cdots \widehat{c}_{i_{m-1}}}_{m-1} \widehat{c}_{i_m}^{(k)} \quad (m \ge 1, n \ge 0), \tag{68}$$

where $\widehat{c}_n^{(k)}$ is poly-Cauchy number of the second kind [12], whose generating function is given by

$$\mathrm{Lif}_k(-\ln(1+x)) = \sum_{n=0}^{\infty} \widehat{c}_n^{(k)} \frac{x^n}{n!}. \tag{69}$$

Table 2

n	0	1	2	3	4	5	6	7
$\widehat{c}_n$	1	$-1/2$	$5/6$	$-9/4$	$251/30$	$-475/12$	$19087/84$	$-36799/24$
$\widehat{c}_n^{(1)}(-1)$	1	$-3/2$	$23/6$	$-55/4$	$1901/30$	$-4277/12$	$198721/84$	$-144747/8$

$\widehat{c}_n^{(k)}$ can be also defined by

$$\widehat{c}_n^{(k)} = \underbrace{\int_0^1 \cdots \int_0^1}_{k} (-x_1x_2\cdots x_k)(-x_1x_2\cdots x_k - 1)\cdots (-x_1x_2\cdots x_k - n + 1)\,dx_1dx_2\cdots dx_k \tag{70}$$

(see [12]). In this sense, $c_n^{(k)}$ is called poly-Cauchy number of the first kind. When $k = 1$, $\widehat{c}_n = \widehat{c}_n^{(1)}$ is the classical Cauchy number of the second kind, whose generating function is given by

$$\frac{x}{(1+x)\ln(1+x)} = \sum_{n=0}^{\infty} \widehat{c}_n \frac{x^n}{n!}. \tag{71}$$

By using the corresponding lemma to Lemma 2, where $G_k(x)$ is replaced by $\widehat{G}_k(x) = \mathrm{Lif}_k(-\ln(1+x))$, we can obtain the following result.

Theorem 7. *For an integer k and a nonnegative integer m, one has*

$$\sum_{l=0}^{m} (-1)^{m-l} \begin{bmatrix} m+1 \\ l+1 \end{bmatrix} \widehat{T}_{m+1}^{(k-1)}(n) = \begin{cases} (-1)^{n-m} \displaystyle\sum_{l=0}^{m-1} \sum_{i=0}^{n-m} (-1)^i \frac{n!}{i!} \binom{n-l-i-1}{n-m-i} \begin{Bmatrix} m \\ l \end{Bmatrix} \widehat{c}_{l+i}^{(k)} \\ + \dfrac{n!}{(n-m)!} \widehat{c}_n^{(k)} \quad (n \ge m); \\ 0 \quad (0 \le n \le m-1). \end{cases} \tag{72}$$

Putting $m = 1$ in Theorem 7, one has the following.

Corollary 8.

$$\sum_{i=0}^{n} \binom{n}{i} \widehat{c}_i \left(\widehat{c}_{n-i}^{(k-1)} - \widehat{c}_{n-i}^{(k)}\right) = n\widehat{c}_n^{(k)} \quad (n \ge 0). \tag{73}$$

Consider the case $m = 2$. Note that the generating function of poly-Cauchy polynomial of the second kind $\widehat{c}_n^{(k)}(z)$ [13] is given by

$$(1+x)^z \mathrm{Lif}_k(-\ln(1+x)) = \sum_{n=0}^{\infty} \widehat{c}_n^{(k)}(z) \frac{x^n}{n!}. \tag{74}$$

$\widehat{c}_n^{(k)}(z)$ are expressed explicitly in terms of the Stirling numbers of the first kind [13, Theorem 4]

$$\widehat{c}_n^{(k)}(z) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} (-1)^n \sum_{i=0}^{m} \binom{m}{i} \frac{(-z)^i}{(m-i+1)^k}. \tag{75}$$

Hence,

$$\begin{aligned} \widehat{c}_n^{(1)}(-1) &= \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} (-1)^n \sum_{i=0}^{m} \binom{m}{i} \frac{1}{m-i+1} \\ &= (-1)^n \sum_{i=0}^{n} \sum_{m=i}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \binom{m}{i} \frac{1}{m-i+1}. \end{aligned} \tag{76}$$

On the other hand,

$$\begin{aligned} &\sum_{n=0}^{\infty} \widehat{c}_n^{(1)}(-1) \frac{x^n}{n!} \\ &= \frac{x}{(1+x)^2 \ln(1+x)} \\ &= \frac{1}{1+x} \sum_{n=0}^{\infty} \widehat{c}_n \frac{x^n}{n!} \\ &= \left(\sum_{\mu=0}^{\infty} (-x)^{\mu}\right) \left(\sum_{\nu=0}^{\infty} \widehat{c}_{\nu} \frac{x^{\nu}}{\nu!}\right) \\ &= \sum_{n=0}^{\infty} (-1)^n \sum_{i=0}^{n} \frac{n!}{i!} (-1)^i \widehat{c}_i \frac{x^n}{n!}. \end{aligned} \tag{77}$$

Thus,

$$\widehat{c}_n^{(1)}(-1) = (-1)^n \sum_{i=0}^{n} \frac{n!}{i!} (-1)^i \widehat{c}_i \tag{78}$$

(see Table 2).

Theorem 9. *For $n \ge 0$ and $k \ge 1$ one has*

$$\widehat{T}_2^{(0)}(n) = \widehat{c}_n^{(1)}(-1), \tag{79}$$

$$\widehat{T}_2^{(k)}(n) = \widehat{c}_n^{(1)}(-1) - n \sum_{j=1}^{k} \widehat{c}_n^{(j)}, \tag{80}$$

$$\widehat{T}_2^{(-k)}(n) = \widehat{c}_n^{(1)}(-1) + n \sum_{j=0}^{k-1} \widehat{c}_n^{(-j)}. \tag{81}$$

Putting $k = 1$ in (80), we have the following identity. This is also an analogous formula to Euler's formula (14).

Corollary 10. *One has*

$$\sum_{i=0}^{n} \binom{n}{i} \widehat{c}_i \widehat{c}_{n-i} = \widehat{c}_n^{(1)}(-1) - n\widehat{c}_n \quad (n \ge 0). \tag{82}$$

If $m = 3$, then $\widehat{T}_3^{(k)}$ can be expressed as follows.

Theorem 11. *For $n \ge 0$ and $k \ge 1$ one has*

$$\begin{aligned}
\widehat{T}_3^{(0)}(n) &= (-1)^n \sum_{l=0}^{n} (-1)^l \frac{n!}{l!} \widehat{T}_2^{(1)}(l) \\
&= (-1)^n \sum_{l=0}^{n} (-1)^l \frac{n!}{l!} \left(\widehat{c}_l^{(1)}(-1) - l\widehat{c}_l \right), \\
\widehat{T}_3^{(k)}(n) &= \widehat{T}_3^{(0)}(n) + \left(1 - 2^{-k}\right) n\widehat{c}_{n-1}^{(1)}(-2) \\
&\quad + n(n-1) \sum_{j=1}^{k} \left(1 - 2^{j-k-1}\right) \\
&\quad \times \left(\widehat{c}_n^{(j)} + (-1)^n (n-2)! \sum_{l=0}^{n-2} (-1)^l \frac{\widehat{c}_{l+1}^{(j)}}{l!} \right), \\
\widehat{T}_3^{(-k)}(n) &= \widehat{T}_3^{(0)}(n) - \left(2^k - 1\right) n\widehat{c}_{n-1}^{(1)}(-2) \\
&\quad + n(n-1) \sum_{j=0}^{k-2} \left(2^{k-j-1} - 1\right) \\
&\quad \times \left(\widehat{c}_n^{(-j)} + (-1)^n (n-2)! \sum_{l=0}^{n-2} (-1)^l \frac{\widehat{c}_{l+1}^{(-j)}}{l!} \right).
\end{aligned} \tag{83}$$

Remark 12. Note that

$$n\widehat{c}_{n-1}^{(1)}(-2) = (-1)^{n-1} n! \sum_{l=0}^{n-1} (n-l)(-1)^l \frac{\widehat{c}_l}{l!}. \tag{84}$$

2.4. Two Kinds of Poly-Cauchy Numbers. Define $U_m^{(k)}(n)$ by

$$U_m^{(k)}(n) := \sum_{\substack{i_1+\cdots+i_m=n \\ i_1,\dots,i_m \ge 0}} \binom{n}{i_1,\dots,i_m} \underbrace{c_{i_1} \cdots c_{i_{m-1}}}_{m-1} \widehat{c}_{i_m}^{(k)} \quad (m \ge 1, n \ge 0). \tag{85}$$

Then we obtain the following.

Theorem 13. *For an integer k and a nonnegative integer m, one has*

$$\sum_{l=0}^{m} (-1)^{m-l} \begin{bmatrix} m+1 \\ l+1 \end{bmatrix} U_{m+1}^{(k-l)}(n) = \begin{cases} \displaystyle\sum_{l=0}^{m} \sum_{i=0}^{n-m} \frac{n!}{i!} \binom{l}{n-m-i} \begin{Bmatrix} m \\ l \end{Bmatrix} \widehat{c}_{l+i}^{(k)} & (n \ge m); \\ 0 & (0 \le n \le m-1). \end{cases} \tag{86}$$

Putting $m = 1$ in Theorem 13, we have the following.

Corollary 14.

$$\sum_{i=0}^{n} \binom{n}{i} c_i \left(\widehat{c}_{n-i}^{(k-1)} - \widehat{c}_{n-i}^{(k)} \right) = n \left((n-1) \widehat{c}_{n-1}^{(k)} + \widehat{c}_n^{(k)} \right) \quad (n \ge 0). \tag{87}$$

If $m = 2$, then $U_2^{(k)}$ can be expressed explicitly.

Theorem 15. *For $n \ge 0$ and $k \ge 1$ one has*

$$U_2^{(0)}(n) = \widehat{c}_n, \tag{88}$$

$$U_2^{(k)}(n) = \widehat{c}_n - n \sum_{j=1}^{k} \left(\widehat{c}_n^{(j)} + (n-1) \widehat{c}_{n-1}^{(j)} \right), \tag{89}$$

$$U_2^{(-k)}(n) = \widehat{c}_n + n \sum_{j=0}^{k-1} \left(\widehat{c}_n^{(-j)} + (n-1) \widehat{c}_{n-1}^{(-j)} \right). \tag{90}$$

Putting $k = 1$ in (89), we have the alternative identity (2.3) in [6, Theorem 2.4] because $c_n = \widehat{c}_n + n\widehat{c}_{n-1}$ by (2.2) in [6, Theorem 2.4].

Corollary 16. *One has*

$$\sum_{i=0}^{n} \binom{n}{i} c_i \widehat{c}_{n-i} = -(n-1)\left(\widehat{c}_n + n\widehat{c}_{n-1}\right) \quad (n \ge 1). \tag{91}$$

If $m = 3$, then $U_3^{(k)}$ can be expressed as follows.

Theorem 17. *For $n \ge 0$ and $k \ge 1$ one has*

$$\begin{aligned}
U_3^{(0)}(n) &= -(n-1)\left(\widehat{c}_n + n\widehat{c}_{n-1}\right) \\
&= -(n-1) c_n,
\end{aligned} \tag{92}$$

$$\begin{aligned}
U_3^{(k)}(n) &= U_3^{(0)}(n) + \left(1 - 2^{-k}\right) n\widehat{c}_{n-1} \\
&\quad + n(n-1) \sum_{j=1}^{k} \left(1 - 2^{j-k-1}\right) \\
&\quad \times \left((n-2)^2 \widehat{c}_{n-2}^{(j)} + (2n-3) \widehat{c}_{n-1}^{(j)} + \widehat{c}_n^{(j)} \right),
\end{aligned} \tag{93}$$

$$\begin{aligned}
U_3^{(-k)}(n) &= U_3^{(0)}(n) - \left(2^k - 1\right) n\widehat{c}_{n-1} \\
&\quad + n(n-1) \sum_{j=0}^{k-2} \left(2^{k-j-1} - 1\right) \\
&\quad \times \left((n-2)^2 \widehat{c}_{n-2}^{(-j)} + (2n-3) \widehat{c}_{n-1}^{(-j)} + \widehat{c}_n^{(-j)} \right).
\end{aligned} \tag{94}$$

Define $V_m^{(k)}(n)$ by

$$V_m^{(k)}(n) := \sum_{\substack{i_1+\cdots+i_m=n \\ i_1,\dots,i_m \ge 0}} \binom{n}{i_1,\dots,i_m} \underbrace{\widehat{c}_{i_1} \cdots \widehat{c}_{i_{m-1}}}_{m-1} c_{i_m}^{(k)} \quad (m \ge 1, n \ge 0). \tag{95}$$

Then we obtain the following.

Theorem 18. *For an integer k and a nonnegative integer m, one has*

$$\sum_{l=0}^{m}(-1)^{m-l}\begin{bmatrix} m+1 \\ l+1 \end{bmatrix} V_{m+1}^{(k-l)}(n) = \begin{cases} (-1)^{n-m}\displaystyle\sum_{l=0}^{m-1}\sum_{i=0}^{n-m}(-1)^{i}\frac{n!}{i!}\binom{n-l-i-1}{n-m-i}\left\{ \begin{matrix} m \\ l \end{matrix} \right\} c_{l+i}^{(k)} \\ +\dfrac{n!}{(n-m)!}c_n^{(k)} \quad (n \ge m); \\ 0 \quad (0 \le n \le m-1). \end{cases} \tag{96}$$

Putting $m = 1$ in Theorem 18, we have the following.

Corollary 19.

$$\sum_{i=0}^{n}\binom{n}{i}\widehat{c}_i\left(c_{n-i}^{(k-1)} - c_{n-i}^{(k)}\right) = nc_n^{(k)} \quad (n \ge 0). \tag{97}$$

If $m = 2$, then $V_2^{(k)}$ can be expressed explicitly.

Theorem 20. *For $n \ge 0$ and $k \ge 1$ one has*

$$V_2^{(0)}(n) = c_n, \tag{98}$$

$$V_2^{(k)}(n) = c_n - n\sum_{j=1}^{k}c_n^{(j)}, \tag{99}$$

$$V_2^{(-k)}(n) = c_n + n\sum_{j=0}^{k-1}c_n^{(-j)}. \tag{100}$$

Putting $k = 1$ in (99), we have the identity (2.3) in [6, Theorem 2.4].

Corollary 21. *One has*

$$\sum_{i=0}^{n}\binom{n}{i}\widehat{c}_i c_{n-i} = -(n-1)c_n \quad (n \ge 1). \tag{101}$$

If $m = 3$, then $V_3^{(k)}$ can be expressed as follows.

Theorem 22. *For $n \ge 0$ and $k \ge 1$ one has*

$$V_3^{(0)}(n) = -(n-1)c_n,$$

$$\begin{aligned} V_3^{(k)}(n) &= V_3^{(0)}(n) - \left(1-2^{-k}\right)n\widehat{c}_{n-1} \\ &\quad + n(n-1)\sum_{j=1}^{k}\left(1-2^{j-k-1}\right) \\ &\quad \times \left(c_n^{(j)} + (-1)^n(n-2)!\sum_{l=0}^{n-2}(-1)^l\frac{c_{l+1}^{(j)}}{l!}\right), \\ V_3^{(-k)}(n) &= V_3^{(0)}(n) + \left(2^k-1\right)n\widehat{c}_{n-1} \\ &\quad + n(n-1)\sum_{j=0}^{k-2}\left(2^{k-j-1}-1\right) \\ &\quad \times \left(c_n^{(-j)} + (-1)^n(n-2)!\sum_{l=0}^{n-2}(-1)^l\frac{c_{l+1}^{(-j)}}{l!}\right). \end{aligned} \tag{102}$$

3. Further Study

Kamano [14] mentioned that explicit formulae of $S_m^{(k)}(n)$ for $m \ge 4$ seemed to be complicated to describe. We will give explicit formulae of $T_m^{(k)}(n)$ for any $m \ge 2$ later anywhere else. In addition, one may consider the sums of products of $(m-k)$ Cauchy numbers and k poly-Cauchy numbers. It would be an interesting work to establish the explicit expressions of such summations.

Acknowledgments

This work was supported in part by the Grant-in-Aid for Scientific research (C) (no. 22540005), the Japan Society for the Promotion of Science.

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