

# HIGHER-ORDER CAUCHY NUMBERS AND POLYNOMIALS

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ABSTRACT. Recently, Komatsu introduced the concept of poly-Cauchy numbers and polynomials which generalize Cauchy numbers and polynomials. In this paper, we consider the new concept of higher-order Cauchy numbers and polynomials which generalize Cauchy numbers and polynomials in different direction and investigate some properties of those new class of numbers and polynomials. From our investigation, we derive some identities involving higher-order Cauchy numbers and polynomials, which generalize some relations between two kinds of Cauchy polynomials and some identities for Cauchy numbers and Stirling numbers.

## 1. INTRODUCTION

In the book of Comtet[1], two kinds of Cauchy numbers are introduced: The first kind is given by

$$C_n = \int_0^1 (x)_n dx, \quad (n \in \mathbb{Z}_{\geq 0}) \quad (1.1)$$

and the second kind is given by

$$\widehat{C}_n = \int_0^1 (-x)_n dx, \quad (n \in \mathbb{Z}_{\geq 0}), \quad (1.2)$$

where  $(x)_n = x(x-1)\dots(x-n+1)$ .

In [2,6,7], Komatsu introduced two kinds of poly-Cauchy numbers: The poly-Cauchy numbers of the first kind  $\mathbb{C}_n^{(k)}$  as a generalization of the Cauchy numbers are given by

$$\mathbb{C}_n^{(k)} = \int_0^1 \cdots \int_0^1 (x_1 x_2 \cdots x_k)_n dx_1 dx_2 \cdots dx_k, \quad (1.3)$$

and the poly Cauchy numbers of the second kind  $\widehat{\mathbb{C}}_n^{(k)}$  are given by

$$\widehat{\mathbb{C}}_n^{(k)} = \int_0^1 \cdots \int_0^1 (-x_1 x_2 \cdots x_k)_n dx_1 \cdots dx_k, \quad (n \in \mathbb{Z}_{\geq 0}, k \in \mathbb{N}). \quad (1.4)$$

The (signed) Stirling number of the first kind is defined by

$$(x)_n = \sum_{l=0}^n S_1(n, l) x^l, \quad (n \in \mathbb{Z}_{\geq 0}). \quad (1.5)$$

From (1.5), we have

$$(\log(1+t))^n = n! \sum_{l=n}^{\infty} S_1(l, n) \frac{t^l}{l!}. \quad (1.6)$$

The Stirling number of the second kind is defined by the generating function to be

$$(e^t - 1)^n = n! \sum_{l=n}^{\infty} S_2(l, n) \frac{t^l}{l!}, \quad (\text{see [3, 4, 9]}). \quad (1.7)$$

From (1.1) and (1.5), we note that

$$C_n = \sum_{m=0}^n S_1(n, m) \frac{1}{m+1} = (-1)^n \sum_{m=0}^n \begin{bmatrix} n \\ m \end{bmatrix} \frac{(-1)^m}{m+1}, \quad (\text{see [1, 10]}), \quad (1.8)$$

where  $\begin{bmatrix} n \\ m \end{bmatrix}$  are the (unsigned) Stirling number of the first kind, arising as coefficients of the rising factorial

$$x^{(n)} = x(x+1) \cdots (x+n-1) = \sum_{m=0}^n \begin{bmatrix} n \\ m \end{bmatrix} x^m, \quad (\text{see [1, 8, 10]}).$$

An explicit formula for  $\mathbb{C}_n^{(k)}$  is given by

$$\mathbb{C}_n^{(k)} = (-1)^n \sum_{m=0}^n \begin{bmatrix} n \\ m \end{bmatrix} \frac{(-1)^m}{(m+1)^k}, \quad (n \geq 0, k \geq 1),$$

and

$$\widehat{\mathbb{C}}_n^{(k)} = (-1)^n \sum_{m=0}^n \begin{bmatrix} n \\ m \end{bmatrix} \frac{1}{(m+1)^k}, \quad (n \geq 0, k \geq 1), \quad (\text{see [4, 5, 6]}).$$

The poly-Cauchy polynomials of the first kind  $\mathbb{C}_n^{(k)}(z)$  are defined by

$$\mathbb{C}_n^{(k)}(z) = \int_0^1 \cdots \int_0^1 (x_1 x_2 \cdots x_k - z)_m dx_1 \cdots dx_k \quad (1.9)$$

and are expressed explicitly in terms of Stirling numbers of the first kind:

$$\mathbb{C}_n^{(k)}(z) = \sum_{m=0}^n \begin{bmatrix} n \\ m \end{bmatrix} (-1)^{n-m} \sum_{i=0}^m \binom{m}{i} \frac{(-z)^i}{(m-i+1)^k}, \quad (\text{see [4, 5, 7]}). \quad (1.10)$$

The poly-Cauchy polynomials of the second kind  $\widehat{\mathbb{C}}_n^{(k)}(z)$  are defined by

$$\widehat{\mathbb{C}}_n^{(k)}(z) = \int_0^1 \cdots \int_0^1 (-x_1 \cdots x_k + z)_m dx_1 \cdots dx_k, \quad (1.11)$$

and are expressed explicitly in terms of Stirling numbers of the second kind:

$$\widehat{\mathbb{C}}_n^{(k)}(z) = \sum_{m=0}^n \begin{bmatrix} n \\ m \end{bmatrix} (-1)^n \sum_{i=0}^m \binom{m}{i} \frac{(-z)^i}{(m-i+1)^k}, \quad (\text{see [2, 6, 7]}). \quad (1.12)$$

For  $\alpha \in \mathbb{N}$ , as is well known, the Bernoulli polynomials of order  $\alpha$  are defined by the generating function to be

$$\left( \frac{t}{e^t - 1} \right)^\alpha e^{xt} = \underbrace{\left( \frac{t}{e^t - 1} \right) \times \cdots \times \left( \frac{t}{e^t - 1} \right)}_{\alpha\text{-times}} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x) \frac{t^n}{n!}. \quad (1.13)$$

When  $x = 0$ ,  $B_n^{(\alpha)} = B_n^{(\alpha)}(0)$  are the Bernoulli numbers of order  $\alpha$ . (see [3,4,9]). In this paper, we consider the new concept of higher-order Cauchy numbers and polynomials which generalize Cauchy numbers and polynomials and investigate some properties of those new class of numbers and polynomials. From our investigation, we derive some identities involving higher-order Cauchy numbers and polynomials, which generalize some relations between two kinds of Cauchy polynomials and some identities for Cauchy numbers and Stirling numbers.

Finally, we introduce some identities of higher-order Cauchy polynomials arising from umbral calculus.

## 2. HIGHER-ORDER CAUCHY POLYNOMIALS

For  $k \in \mathbb{N}$ , let us consider the Cauchy numbers of the first kind of order  $k$  as follows:

$$C_n^{(k)} = \int_0^1 \cdots \int_0^1 (x_1 + x_2 + \cdots + x_k)_n dx_1 \cdots dx_k, \quad (2.1)$$

where  $n \in \mathbb{Z}_{\geq 0}$  and  $k \in \mathbb{N}$ .

Then, from (2.1), we can derive the generating function of  $C_n^{(k)}$  as follows:

$$\begin{aligned} \sum_{n=0}^{\infty} C_n^{(k)} \frac{t^n}{n!} &= \int_0^1 \cdots \int_0^1 \sum_{n=0}^{\infty} \binom{x_1 + \cdots + x_k}{n} t^n dx_1 \cdots dx_k \\ &= \int_0^1 \cdots \int_0^1 (1+t)^{x_1 + \cdots + x_k} dx_1 \cdots dx_k. \end{aligned} \quad (2.2)$$

It is easy to show that

$$\left( \frac{(1+t)^x}{\log(1+t)} \right)' = (1+t)^x \quad (2.3)$$

Thus, by (2.3), we get

$$\int_0^1 (1+t)^x dx = \frac{t}{\log(1+t)}. \quad (2.4)$$

From (2.2) and (2.4), we have

$$\begin{aligned} \sum_{n=0}^{\infty} C_n^{(k)} \frac{t^n}{n!} &= \int_0^1 \cdots \int_0^1 (1+t)^{x_1 + \cdots + x_k} dx_1 \cdots dx_k \\ &= \left( \frac{t}{\log(1+t)} \right)^k. \end{aligned} \quad (2.5)$$

It is known that

$$\left( \frac{t}{\log(1+t)} \right)^n (1+t)^{x-1} = \sum_{k=0}^{\infty} B_k^{(k-n+1)}(x) \frac{t^k}{k!}, \quad (\text{see [1]}). \quad (2.6)$$

Therefore, by (2.5) and (2.6), we obtain the following theorem.

**Theorem 2.1.** For  $n \geq 0$ , we have

$$C_n^{(k)} = B_n^{(n-k+1)}(1).$$

From (1.1), we have

$$\sum_{n=0}^{\infty} C_n \frac{t^n}{n!} = \int_0^1 \sum_{n=0}^{\infty} \binom{x}{n} t^n dx = \int_0^1 (1+t)^x dx = \frac{t}{\log(1+t)}. \quad (2.7)$$

Thus, by (2.5) and (2.7), we get

$$\sum_{n=0}^{\infty} C_n^{(k)} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left( \sum_{l_1+\dots+l_k=n} \binom{n}{l_1, \dots, l_k} C_{l_1} \cdots C_{l_k} \right) \frac{t^n}{n!}. \quad (2.8)$$

From (2.1), we note that

$$\begin{aligned} C_n^{(k)} &= \int_0^1 \cdots \int_0^1 (x_1 + \cdots + x_k)_n dx_1 \cdots dx_k \\ &= \sum_{l=0}^n S_1(n, l) \int_0^1 \cdots \int_0^1 (x_1 + \cdots + x_k)^l dx_1 \cdots dx_k \\ &= \sum_{l=0}^n \sum_{l_1+\dots+l_k=l} S_1(n, l) \binom{l}{l_1, \dots, l_k} \int_0^1 \cdots \int_0^1 x_1^{l_1} x_2^{l_2} \cdots x_k^{l_k} dx_1 \cdots dx_k \\ &= \sum_{l=0}^n \sum_{l_1+\dots+l_k=l} \binom{l}{l_1, \dots, l_k} S_1(n, l) \frac{1}{(l_1+1) \cdots (l_k+1)}. \end{aligned} \quad (2.9)$$

Therefore, by (2.8) and (2.9), we obtain the following theorem.

**Theorem 2.2.** *For  $n \geq 0$ , we have*

$$\begin{aligned} C_n^{(k)} &= \sum_{l_1+\dots+l_k=n} \binom{n}{l_1, \dots, l_k} C_{l_1} \cdots C_{l_k} \\ &= \sum_{l=0}^n \sum_{l_1+\dots+l_k=l} \binom{l}{l_1, \dots, l_k} S_1(n, l) \frac{1}{(l_1+1) \cdots (l_k+1)}. \end{aligned}$$

From (2.5), we can derive the following equations.

$$\begin{aligned} \sum_{n=0}^{\infty} C_n^{(k)} \frac{(e^t - 1)^n}{n!} &= \frac{1}{t^k} (e^t - 1)^k = \sum_{n=0}^{\infty} S_2(n+k, k) \frac{k!}{(n+k)!} t^n \\ &= \sum_{n=0}^{\infty} \frac{n!k!}{(n+k)!} S_2(n+k, k) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{S_2(n+k, k)}{\binom{n+k}{n}} \frac{t^n}{n!}, \end{aligned} \quad (2.10)$$

and

$$\begin{aligned} \sum_{n=0}^{\infty} C_n^{(k)} \frac{1}{n!} (e^t - 1)^n &= \sum_{n=0}^{\infty} C_n^{(k)} \sum_{m=n}^{\infty} S_2(m, n) \frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty} \left( \sum_{n=0}^m C_n^{(k)} S_2(m, n) \right) \frac{t^m}{m!}. \end{aligned} \quad (2.11)$$

Therefore, by (2.10) and (2.11), we obtain the following theorem.

**Theorem 2.3.** For  $m \in \mathbb{Z}_{\geq 0}$ ,  $k \in \mathbb{N}$ , we have

$$\begin{aligned} S_2(m+k, k) &= \binom{m+k}{m} \sum_{n=0}^m C_n^{(k)} S_2(m, n) \\ &= \binom{m+k}{m} \sum_{n=0}^m B_n^{(n-k+1)}(1) S_2(m, n). \end{aligned}$$

Now, we consider the higher-order Cauchy polynomials of the first kind as follows:

$$C_n^{(k)}(x) = \int_0^1 \cdots \int_0^1 (x_1 + \cdots + x_k - x)_n dx_1 \cdots dx_k. \quad (2.12)$$

Then, by (2.12), we get

$$\begin{aligned} C_n^{(k)}(x) &= \sum_{l=0}^n S_1(n, l) \int_0^1 \cdots \int_0^1 (x_1 + x_2 \cdots + x_k - x)^l dx_1 \cdots dx_k \\ &= \sum_{l=0}^n S_1(n, l) \sum_{j=0}^l \binom{l}{j} (-x)^{l-j} \int_0^1 \cdots \int_0^1 (x_1 + \cdots + x_k)^j dx_1 \cdots dx_k \\ &= \sum_{l=0}^n \sum_{j=0}^l \sum_{j_1 + \cdots + j_k = j} \binom{j}{j_1, \dots, j_k} \binom{l}{j} S_1(n, l) (-x)^{l-j} \frac{1}{(j_1 + 1) \cdots (j_k + 1)}. \end{aligned} \quad (2.13)$$

From (2.12), we can derive the generating function of  $C_n^{(k)}(x)$  as follows:

$$\begin{aligned} \sum_{n=0}^{\infty} C_n^{(k)}(x) \frac{t^n}{n!} &= \int_0^1 \cdots \int_0^1 \sum_{n=0}^{\infty} \binom{x_1 + \cdots + x_k - x}{n} t^n dx_1 \cdots dx_k \\ &= \int_0^1 \cdots \int_0^1 (1+t)^{x_1 + \cdots + x_k - x} dx_1 \cdots dx_k \\ &= \left( \frac{t}{\log(1+t)} \right)^k (1+t)^{-x}. \end{aligned} \quad (2.14)$$

It is known that

$$\left( \frac{t}{\log(1+t)} \right)^k (1+t)^x = \sum_{n=0}^{\infty} B_n^{(n-k+1)}(x+1) \frac{t^n}{n!}. \quad (2.15)$$

By (2.14) and (2.15), we get

$$C_n^{(k)}(x) = B_n^{(n-k+1)}(1-x).$$

Therefore, by (2.13) and (2.15), we obtain the following theorem.

**Theorem 2.4.** For  $n \in \mathbb{Z}_{\geq 0}$ ,  $k \in \mathbb{N}$ , we have

$$\begin{aligned} C_n^{(k)}(x) &= B_n^{(n-k+1)}(1-x) \\ &= \sum_{l=0}^n \sum_{j=0}^l \sum_{j_1 + \cdots + j_k = j} \binom{j}{j_1, \dots, j_k} \binom{l}{j} S_1(n, l) \frac{(-x)^{l-j}}{(j_1 + 1) \cdots (j_k + 1)}. \end{aligned}$$

By (2.14), we see that

$$\begin{aligned}
\sum_{n=0}^{\infty} C_n^{(k)}(x) \frac{(e^t - 1)^n}{n!} &= e^{-tx} \left( \frac{e^t - 1}{t} \right)^k \\
&= e^{-tx} \sum_{n=0}^{\infty} S_2(n+k, k) \frac{n!k!}{(n+k)!} \frac{t^n}{n!} \\
&= \left( \sum_{l=0}^{\infty} \frac{(-x)^l}{l!} t^l \right) \left( \sum_{n=0}^{\infty} \frac{S_2(n+k, k) t^n}{\binom{n+k}{n} n!} \right) \\
&= \sum_{m=0}^{\infty} \left\{ \sum_{n=0}^m \frac{\binom{m}{n}}{\binom{n+k}{n}} S_2(n+k, k) (-x)^{m-n} \right\} \frac{t^m}{m!},
\end{aligned} \tag{2.16}$$

and

$$\begin{aligned}
\sum_{n=0}^{\infty} C_n^{(k)}(x) \frac{(e^t - 1)^n}{n!} &= \sum_{n=0}^{\infty} C_n^{(k)}(x) \sum_{m=n}^{\infty} S_2(m, n) \frac{t^m}{m!} \\
&= \sum_{m=0}^{\infty} \left\{ \sum_{n=0}^m C_n^{(k)}(x) S_2(m, n) \right\} \frac{t^m}{m!}.
\end{aligned} \tag{2.17}$$

Therefore, by (2.16) and (2.17), we obtain the following theorem.

**Theorem 2.5.** *For  $m \in \mathbb{Z}_{\geq 0}$ ,  $k \in \mathbb{N}$ , we have*

$$\sum_{n=0}^m \frac{\binom{m}{n}}{\binom{n+k}{n}} S_2(n+k, k) (-x)^{m-n} = \sum_{n=0}^m C_n^{(k)}(x) S_2(m, n).$$

We now define the Cauchy numbers of the second kind of order  $k$  as follows:

$$\widehat{C}_n^{(k)} = \int_0^1 \cdots \int_0^1 (-x_1 + \cdots + x_k)_n dx_1 \cdots dx_k. \tag{2.18}$$

From (2.18), we can derive the generating function of  $\widehat{C}_n^{(k)}$  as follows:

$$\begin{aligned}
\sum_{n=0}^{\infty} \widehat{C}_n^{(k)} \frac{t^n}{n!} &= \int_0^1 \cdots \int_0^1 \sum_{n=0}^{\infty} \binom{-x_1 - \cdots - x_k}{n} t^n dx_1 \cdots dx_k \\
&= \int_0^1 \cdots \int_0^1 (1+t)^{-x_1 - \cdots - x_k} dx_1 \cdots dx_k \\
&= \left( \frac{t}{(1+t) \log(1+t)} \right)^k.
\end{aligned} \tag{2.19}$$

Thus, by (2.19), we get

$$\begin{aligned}
\sum_{m=0}^{\infty} \widehat{C}_m^{(k)} \frac{(e^t - 1)^m}{m!} &= \left( \frac{e^t - 1}{te^t} \right)^k \\
&= \left( \sum_{l=0}^{\infty} \frac{(-k)^l}{l!} t^l \right) \left( \sum_{m=0}^{\infty} \frac{S_2(k+m, k) k!}{(k+m)!} t^m \right) \\
&= \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^n \frac{\binom{n}{m}}{\binom{k+m}{m}} (-k)^{n-m} S_2(k+m, k) \right\} \frac{t^n}{n!},
\end{aligned} \tag{2.20}$$

and

$$\begin{aligned} \sum_{m=0}^{\infty} \widehat{C}_m^{(k)} \frac{(e^t - 1)^m}{m!} &= \sum_{m=0}^{\infty} \widehat{C}_m^{(k)} \sum_{n=m}^{\infty} S_2(n, m) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left( \sum_{m=0}^n \widehat{C}_m^{(k)} S_2(n, m) \right) \frac{t^n}{n!}. \end{aligned} \quad (2.21)$$

Therefore, by (2.20) and (2.21), we obtain the following theorem.

**Theorem 2.6.** *For  $n \geq 0$ ,  $k \in \mathbb{N}$ , we have*

$$\sum_{m=0}^n \frac{\binom{n}{k+m}}{\binom{n}{m}} S_2(k+m, k) (-k)^{n-m} = \sum_{m=0}^n \widehat{C}_m^{(k)} S_2(m, n).$$

We also consider the higher-order Cauchy polynomials of the second kind as follows:

$$\widehat{C}_n^{(k)}(x) = \int_0^1 \cdots \int_0^1 (x - (x_1 + \cdots + x_k))_n dx_1 \cdots dx_k. \quad (2.22)$$

By (2.22), we get

$$\begin{aligned} \widehat{C}_n^{(k)}(x) &= \sum_{l=0}^n S_1(n, l) \int_0^1 \cdots \int_0^1 (-(x_1 + \cdots + x_k) + x)^l dx_1 \cdots dx_k \\ &= \sum_{l=0}^n S_1(n, l) \sum_{i=0}^l \binom{l}{i} x^{l-i} (-1)^i \int_0^1 \cdots \int_0^1 (x_1 + \cdots + x_k)^i dx_1 \cdots dx_k \\ &= \sum_{l=0}^n \sum_{i=0}^l S_1(n, l) \binom{l}{i} x^{l-i} (-1)^i \sum_{j_1 + \cdots + j_k = i} \binom{i}{j_1, \dots, j_k} \frac{1}{(j_1 + 1) \cdots (j_k + 1)} \\ &= \sum_{l=0}^n \sum_{i=0}^l \sum_{j_1 + \cdots + j_k = i} \binom{i}{j_1, \dots, j_k} \binom{l}{i} S_1(n, l) x^{l-i} (-1)^i \frac{1}{(j_1 + 1) \cdots (j_k + 1)}. \end{aligned} \quad (2.23)$$

Let us consider the generating function of the higher-order Cauchy polynomials of the second kind as follow:

$$\begin{aligned} \sum_{n=0}^{\infty} \widehat{C}_n^{(k)}(x) \frac{t^n}{n!} &= \int_0^1 \cdots \int_0^1 \sum_{n=0}^{\infty} \binom{x - (x_1 + \cdots + x_k)}{n} t^n dx_1 \cdots dx_k \\ &= \int_0^1 \cdots \int_0^1 (1+t)^{-x_1 - \cdots - x_k + x} dx_1 \cdots dx_k \\ &= \left( \frac{t}{(1+t) \log(1+t)} \right)^k (1+t)^x. \end{aligned} \quad (2.24)$$

It is not difficult to show that

$$\begin{aligned}
\sum_{n=0}^{\infty} \widehat{C}_n^{(k)}(x) \frac{t^n}{n!} &= \left( \frac{t}{(1+t) \log(1+t)} \right)^k (1+t)^x \\
&= \sum_{n=0}^{\infty} B_n^{(n-k+1)}(x-k+1) \frac{t^n}{n!}.
\end{aligned} \tag{2.25}$$

Therefore, by (2.23) and (2.25), we obtain the following theorem.

**Theorem 2.7.** *For  $n \geq 0$ ,  $k \in \mathbb{N}$ , we have*

$$\begin{aligned}
\widehat{C}_n^{(k)}(x) &= B_n^{(n-k+1)}(x-k+1) \\
&= \sum_{l=0}^n \sum_{i=0}^l \sum_{j_1+\dots+j_k=i} \binom{i}{j_1, \dots, j_k} \binom{l}{i} S_1(n, l) x^{l-i} \frac{(-1)^i}{(j_1+1) \cdots (j_k+1)}.
\end{aligned}$$

From (2.25), we note that

$$\begin{aligned}
\sum_{n=0}^{\infty} \widehat{C}_n^{(k)}(x) \frac{(e^t - 1)^n}{n!} &= \sum_{n=0}^{\infty} \widehat{C}_n^{(k)}(x) \sum_{m=n}^{\infty} S_2(m, n) \frac{t^m}{m!} \\
&= \sum_{m=0}^{\infty} \left( \sum_{n=0}^m \widehat{C}_n^{(k)}(x) S_2(m, n) \right) \frac{t^m}{m!},
\end{aligned} \tag{2.26}$$

and

$$\begin{aligned}
\sum_{n=0}^{\infty} \widehat{C}_n^{(k)}(x) \frac{(e^t - 1)^n}{n!} &= \left( \frac{e^t - 1}{t} \right)^k e^{t(x-k)} \\
&= \left( \sum_{l=0}^{\infty} \frac{k!}{(l+k)!} S_2(l+k, k) t^l \right) \left( \sum_{n=0}^{\infty} (x-k)^n \frac{t^n}{n!} \right) \\
&= \sum_{m=0}^{\infty} \left( \sum_{n=0}^m \frac{S_2(n+k, k) k!}{(n+k)!} (x-k)^{m-n} \frac{m!}{(m-n)!} \right) \frac{t^m}{m!} \\
&= \sum_{m=0}^{\infty} \left( \sum_{n=0}^m S_2(n+k, k) \frac{\binom{m}{n}}{\binom{m}{n+k}} (x-k)^{m-n} \right) \frac{t^m}{m!}
\end{aligned} \tag{2.27}$$

Therefore, by (2.26) and (2.27), we obtain the following theorem.

**Theorem 2.8.** *For  $m \in \mathbb{Z}_{\geq 0}$ ,  $k \in \mathbb{N}$ , we have*

$$\sum_{n=0}^m \widehat{C}_n^{(k)}(x) S_2(m, n) = \sum_{n=0}^m S_2(n+k, k) \frac{\binom{m}{n}}{\binom{m}{n+k}} (x-k)^{m-n}.$$

Now, we observe that



$$\begin{aligned}
(-1)^n \frac{C_n^{(k)}(x)}{n!} &= (-1)^n \int_0^1 \cdots \int_0^1 \binom{x_1 + \cdots + x_k - x}{n} dx_1 \cdots dx_k \\
&= \int_0^1 \cdots \int_0^1 \binom{-(x_1 + \cdots + x_k) + x + n - 1}{n} dx_1 \cdots dx_k \\
&= \sum_{m=0}^n \int_0^1 \cdots \int_0^1 \binom{-(x_1 + \cdots + x_k) + x}{m} \binom{n-1}{n-m} dx_1 \cdots dx_k \\
&= \sum_{m=0}^n \binom{n-1}{n-m} \frac{1}{m!} m! \int_0^1 \cdots \int_0^1 \binom{-(x_1 + \cdots + x_k) + x}{m} dx_1 \cdots dx_k \\
&= \sum_{m=0}^n \binom{n-1}{n-m} \frac{1}{m!} \widehat{C}_m^{(k)}(x) = \sum_{m=1}^n \binom{n-1}{n-m} \frac{1}{m!} \widehat{C}_m^{(k)}(x).
\end{aligned} \tag{2.28}$$

Therefore, by (2.28), we obtain the following theorem.

**Theorem 2.9.** *For  $n, k \in \mathbb{N}$ , we have*

$$(-1)^n \frac{C_n^{(k)}(x)}{n!} = \sum_{m=1}^n \binom{n-1}{n-m} \frac{1}{m!} \widehat{C}_m^{(k)}(x).$$

By the same method of (3.7), we get

$$\begin{aligned}
(-1)^n \frac{\widehat{C}_n^{(k)}(x)}{n!} &= (-1)^n \int_0^1 \cdots \int_0^1 \binom{-(x_1 + \cdots + x_k) + x}{n} dx_1 \cdots dx_k \\
&= \int_0^1 \cdots \int_0^1 \binom{x_1 + \cdots + x_k - x + n - 1}{n} dx_1 \cdots dx_k \\
&= \sum_{m=0}^n \binom{n-1}{n-m} \int_0^1 \cdots \int_0^1 \binom{x_1 + \cdots + x_k - x}{m} dx_1 \cdots dx_k \\
&= \sum_{m=0}^n \binom{n-1}{n-m} \frac{1}{m!} m! \int_0^1 \cdots \int_0^1 \binom{x_1 + \cdots + x_k - x}{m} dx_1 \cdots dx_k \\
&= \sum_{m=0}^n \binom{n-1}{n-m} \frac{C_m^{(k)}(x)}{m!} = \sum_{m=1}^n \binom{n-1}{n-m} \frac{C_m^{(k)}(x)}{m!}.
\end{aligned} \tag{2.29}$$

Therefore, by (2.29), we obtain the following theorem.

**Theorem 2.10.** *For  $n, k \in \mathbb{N}$ , we have*

$$(-1)^n \frac{\widehat{C}_n^{(k)}(x)}{n!} = \sum_{m=1}^n \binom{n-1}{n-m} \frac{C_m^{(k)}(x)}{m!}.$$

### 3. SHEFFER SEQUENCES ASSOCIATED WITH HIGHER-ORDER CAUCHY NUMBERS AND POLYNOMIALS

Let  $\mathbb{P}$  be the algebra of polynomials in a single variable  $x$  over  $\mathbb{C}$  and let  $\mathbb{P}^*$  be the vector space of all linear functionals on  $\mathbb{P}$ . The action of the linear functional

$L$  on the polynomial  $p(x)$  is denoted by  $\langle L|p(x) \rangle$ , and the vector space structure on  $\mathbb{P}^*$  is defined by

$$\langle L + M|p(x) \rangle = \langle L|p(x) \rangle + \langle M|p(x) \rangle, \text{ and } \langle cL|p(x) \rangle = c\langle L|p(x) \rangle,$$

where  $c$  is any complex constant. Let  $\mathcal{F}$  denote the algebra of formal power series in a single variable  $t$ :

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \mid a_k \in \mathbb{C} \right\}. \quad (3.1)$$

The formal power series  $f(t)$  defines the linear functional on  $\mathbb{P}$  by setting

$$\langle f(t)|x^n \rangle = a_n \text{ for all } n \geq 0, \text{ (see [5,9]).} \quad (3.2)$$

By (3.1) and (3.2), we easily get

$$\langle t^k|x^n \rangle = n!\delta_{n,k}, \text{ for all } n, k \geq 0, \text{ (see [5,9]),}$$

where  $\delta_{n,k}$  is the Kronecker's symbol.

Let  $f_L(t) = \sum_{k \geq 0} \langle L|x^k \rangle \frac{t^k}{k!}$ . By (3.3), we get  $\langle f_L(t)|x^n \rangle = \langle L|x^n \rangle$ . So, the map  $L \mapsto f_L(t)$  is a vector space isomorphism from  $\mathbb{P}^*$  onto  $\mathcal{F}$ . Henceforth,  $\mathcal{F}$  is thought of as both the algebra of formal power series and the space of linear functionals. We call  $\mathcal{F}$  the umbral algebra. The umbral calculus is the study of umbral algebra. The order  $o(f(t))$  of the non-zero power series  $f(t)$  is the smallest integer  $k$  for which the coefficient of  $t^k$  does not vanish. (see [3,5,9]). If  $o(f(t)) = 1$  (respectively,  $o(f(t)) = 0$ ), then  $f(t)$  is called a delta (repectively, an invertible) series. For  $o(f(t)) = 1$  and  $o(g(t)) = 0$ , there exists a unique sequence  $S_n(x)$  of polynomials such that  $\langle g(t)f(t)^k|S_n(x) \rangle = n!\delta_{n,k}$ , where  $n, k \geq 0$ . The sequence  $S_n(x)$  is called the Sheffer sequence for  $(g(t), f(t))$ , which is denoted by  $S_n(x) \sim (g(t), f(t))$  (see [3,5,9]). For  $f(t) \in \mathcal{F}$  and  $p(x) \in \mathbb{P}$ , we have

$$\langle e^{yt}|p(x) \rangle = p(y), \quad \langle f(t)g(t)|p(x) \rangle = \langle g(t)|f(t)p(x) \rangle = \langle f(t)|g(t)p(x) \rangle, \quad (3.3)$$

and

$$f(t) = \sum_{k=0}^{\infty} \langle f(t)|x^k \rangle \frac{t^k}{k!}, \quad p(x) = \sum_{k=0}^{\infty} \langle t^k|p(x) \rangle \frac{x^k}{k!} \text{ (see [4,5,9]).} \quad (3.4)$$

From (3.4), we note that

$$\langle t^k|p(x) \rangle = p^{(k)}(0), \quad \langle 1|p^{(k)}(x) \rangle = p^{(k)}(0), \quad (3.5)$$

where  $p^{(k)}(0)$  denotes the  $k$ -th derivative of  $p(x)$  at  $x = 0$ . Thus, by (3.5), we get

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k}, \text{ for all } k \geq 0, \text{ (see [4,5,9]).} \quad (3.6)$$

Let  $S_n(x) \sim (g(t), f(t))$ . Then we have

$$\frac{1}{g(\bar{f}(t))} e^{y\bar{f}(t)} = \sum_{k=0}^{\infty} S_k(y) \frac{t^k}{k!}, \text{ for all } y \in \mathbb{C}, \quad (3.7)$$

where  $\bar{f}(t)$  is the compositional inverse of  $f(t)$  with  $f(\bar{f}(t)) = \bar{f}(f(t)) = t$ .

For  $S_n(x) \sim (g(t), f(t))$ ,  $q_n(x) \sim (h(t), l(t))$ , let

$$S_n(x) = \sum_{k=0}^n C_{n,k} q_k(x), \quad (3.8)$$

then we have

$$C_{n,k} = \frac{1}{k!} \left\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))} (l(\bar{f}(t)))^k | x^n \right\rangle, \quad (\text{see } [2, 5, 9]). \quad (3.9)$$

From (2.14), (2.25) and (3.7), we note that

$$C_n^{(k)}(x) \sim \left( \left( \frac{t}{1-e^{-t}} \right)^k, e^{-t} - 1 \right), \quad (3.10)$$

and

$$\widehat{C}_n^{(k)}(x) \sim \left( \left( \frac{te^t}{e^t - 1} \right)^k, e^t - 1 \right). \quad (3.11)$$

For  $S_n(x) \sim (g(t), f(t))$ , as is well known, we have

$$f(t)S_n(x) = nS_{n-1}(x), \quad (\text{see } [3, 5, 9]). \quad (3.12)$$

By (3.10), (3.11) and (3.12), we get

$$nC_{n-1}^{(k)}(x) = (e^{-t} - 1)C_n^{(k)}(x) = C_n^{(k)}(x-1) - C_n^{(k)}(x), \quad (3.13)$$

and

$$n\widehat{C}_{n-1}^{(k)}(x) = (e^t - 1)\widehat{C}_n^{(k)}(x) = \widehat{C}_n^{(k)}(x+1) - \widehat{C}_n^{(k)}(x). \quad (3.14)$$

Therefore, by (3.13) and (3.14), we obtain the following lemma.

**Lemma 3.1.** *For  $n \in \mathbb{Z}_{\geq 0}$ ,  $k \in \mathbb{N}$ , we have*

$$nC_{n-1}^{(k)}(x) = C_n^{(k)}(x-1) - C_n^{(k)}(x), \quad n\widehat{C}_{n-1}^{(k)}(x) = \widehat{C}_n^{(k)}(x+1) - \widehat{C}_n^{(k)}(x).$$

From (3.10), we have

$$\left( \frac{t}{1-e^{-t}} \right)^k C_n^{(k)}(x) \sim (1, e^{-t} - 1), \quad (-1)^n x^{(n)} \sim (1, e^{-t} - 1), \quad (3.15)$$

where  $x^{(n)} = x(x+1) \cdots (x+n-1)$ .

Thus, by (3.15), we get

$$\left( \frac{t}{1-e^{-t}} \right)^k C_n^{(k)}(x) = (-1)^n x^{(n)} = \sum_{l=0}^n (-1)^l S_1(n, l) x^l. \quad (3.16)$$

From (3.16), we have

$$\begin{aligned}
C_n^{(k)}(x) &= \left( \frac{1 - e^{-t}}{t} \right)^k \sum_{l=0}^n (-1)^l S_1(n, l) x^l \\
&= \sum_{l=0}^n \sum_{m=0}^l \frac{k!}{(k+m)!} (l)_m S_2(k+m, k) S_1(n, l) (-1)^{k+l+m} x^{l-m} \\
&= \sum_{l=0}^n \sum_{m=0}^l \frac{\binom{l}{m}}{\binom{k+l-m}{k}} S_2(k+l-m, k) S_1(n, l) (-1)^{k-m} x^m.
\end{aligned} \tag{3.17}$$

By (3.11), we get

$$\left( \frac{te^t}{e^t - 1} \right)^k \widehat{C}_n^{(k)}(x) \sim (1, e^t - 1), \quad (x)_n \sim (1, e^t - 1). \tag{3.18}$$

Thus, from (3.18), we have

$$\begin{aligned}
\widehat{C}_n^{(k)}(x) &= \left( \frac{e^t - 1}{te^t} \right)^k (x)_n = \left( \frac{e^t - 1}{te^t} \right)^k \sum_{l=0}^n S_1(n, l) x^l \\
&= e^{-kt} \sum_{m=0}^{\infty} \frac{k!}{(k+m)!} S_2(k+m, k) t^m \sum_{l=0}^n S_1(n, l) x^l \\
&= e^{-kt} \sum_{l=0}^n \sum_{m=0}^l \frac{k!}{(k+m)!} S_2(k+m, k) S_1(n, l) (l)_m x^{l-m} \\
&= \sum_{l=0}^n \sum_{m=0}^l \frac{\binom{l}{m}}{\binom{m+k}{m}} S_2(k+m, k) S_1(n, l) e^{-kt} x^{l-m} \\
&= \sum_{l=0}^n \sum_{m=0}^l \frac{\binom{l}{m}}{\binom{m+k}{m}} S_2(k+m, k) S_1(n, l) (x-k)^{l-m} \\
&= \sum_{l=0}^n \sum_{m=0}^l \frac{\binom{l}{m}}{\binom{k+l-m}{k}} S_2(k+l-m, k) S_1(n, l) (x-k)^m.
\end{aligned} \tag{3.19}$$

Therefore, by (3.17) and (3.19), we obtain the following theorem.

**Theorem 3.2.** For  $n \in \mathbb{Z}_{\geq 0}$ ,  $k \in \mathbb{N}$ , we have

$$C_n^{(k)}(x) = \sum_{l=0}^n \sum_{m=0}^l \frac{\binom{l}{m}}{\binom{k+l-m}{k}} S_2(k+l-m, k) S_1(n, l) (-1)^{k-m} x^m,$$

and

$$\widehat{C}_n^{(k)}(x) = \sum_{l=0}^n \sum_{m=0}^l \frac{\binom{l}{m}}{\binom{k+l-m}{k}} S_2(k+l-m, k) S_1(n, l) (x-k)^m.$$

For  $\widehat{C}_n^{(k)}(x) \sim \left( \left( \frac{te^t}{e^t - 1} \right)^k, e^t - 1 \right)$ ,  $B_n^{(\alpha)}(x) \sim \left( \left( \frac{e^t - 1}{t} \right)^\alpha, t \right)$ , ( $\alpha \in \mathbb{N}$ ), let us assume that

$$\widehat{C}_n^{(k)}(x) = \sum_{m=0}^n C_{n,m} B_m^{(\alpha)}(x). \quad (3.20)$$

Then, by (3.8), (3.9) and (3.20), we get

$$\begin{aligned} C_{n,m} &= \frac{1}{m!} \left\langle \left( \frac{t}{(1+t)\log(1+t)} \right)^k \left( \frac{t}{\log(1+t)} \right)^\alpha (\log(1+t)^m |x^n\rangle \right. \\ &= \frac{1}{m!} \left\langle \left( \frac{t}{(1+t)\log(1+t)} \right)^{k+\alpha} (1+t)^\alpha |(\log(1+t))^m x^n\rangle \right. \\ &= \sum_{l=0}^{n-m} \frac{\binom{n}{l+m}}{(l+m)!} S_1(l+m, m) \left\langle \left( \frac{t}{(1+t)\log(1+t)} \right)^{k+\alpha} (1+t)^\alpha |x^{n-l-m}\rangle \right. \\ &= \sum_{l=0}^{n-m} \binom{n}{l+m} S_1(l+m, m) \widehat{C}_{n-l-m}^{(k+\alpha)}(\alpha) \\ &= \sum_{l=0}^{n-m} \binom{n}{l} S_1(n-l, m) \widehat{C}_l^{(k+\alpha)}(\alpha). \end{aligned} \quad (3.21)$$

Therefore, by (3.20) and (3.21), we obtain the following theorem.

**Theorem 3.3.** For  $n \geq 0$ ,  $k \in \mathbb{N}$ , we have

$$\widehat{C}_n^{(k)}(x) = \sum_{m=0}^n \left\{ \sum_{l=0}^{n-m} \binom{n}{l} S_1(n-l, m) \widehat{C}_l^{(k+\alpha)}(\alpha) \right\} B_n^{(\alpha)}(x).$$

ACKNOWLEDGEMENTS. This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MOE) (No.2012R1A1A2003786 ).

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