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DERIVATIVES OF CATALAN RELATED SUMS

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ABSTRACT. We investigate the representation of sums of the derivative of the reciprocal of Catalan type numbers in integral form. We show that for various parameter values the sums maybe expressed in closed form. Finally we give bounds for the sums under investigation, in terms of the parameters.

Key words and phrases: Catalan numbers, Binomial coefficients, Estimates, Integral identities.

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1. INTRODUCTION

We define $C_n(j)$, the Catalan related numbers, for $j \in \mathbb{N} \cup \{0\}$, $\mathbb{N} = \{1, 2, 3, ...\}$ as

$$C_n(j) = \begin{cases} \frac{1}{n+1}\binom{2n+j}{n} = \frac{1}{2n+j+1}\binom{2n+j+1}{n+1}, & \text{for } n = 1, 2, 3, ... \\ 1, & \text{for } n = 0, \end{cases} \tag{1.1}$$

and in particular, from (1.1), the Catalan numbers $C_n = C_n(0)$ are defined by

$$C_n = \begin{cases} \frac{1}{n+1}\binom{2n}{n} = \frac{1}{2n+1}\binom{2n+1}{n}, & \text{for } n = 1, 2, 3, ... \\ 1, & \text{for } n = 0. \end{cases}.$$

Catalan numbers have many representations, see Adamchik [1], in particular Penson and Sixdeniers [3], by employing some ideas of the Mellin transform gave the integral representation,

$$C_n = \frac{1}{2\pi}\int_0^4 t^{n-1/2}\sqrt{4-t}\,dt. \tag{1.2}$$

By the change of variable $tw^2 = 4 - t$, in (1.2) we find that

$$C_n = \frac{2^{2n+2}}{\pi}\int_0^\infty \frac{w^2}{(1+w^2)^{n+2}}\,dw,$$

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which, see Bailey et.al. [2], is closely related to cb_n, the central binomial coefficient

$$cb_n = \binom{2n}{n} = \frac{2^{2n+1}}{\pi}\int_0^\infty \frac{1}{(1+w^2)^{n+1}}\,dw.$$

Moreover, it can be seen that $cb_n = (n+1)\,C_n$.

Lemma 1.1. *For $j \geq 0$ and $n \in \mathbb{N}$, let the Catalan related numbers*

$$C_n(j) = \begin{cases} \frac{1}{n+1}\binom{2n+j}{n} = \frac{1}{2n+j+1}\binom{2n+j+1}{n+1}, & \textit{for } n = 1,2,3,... \\ 1, & \textit{for } n = 0 \end{cases}$$

be an analytic function in j then

$$\begin{aligned} \frac{d}{dj}\left(\frac{1}{C_n(j)}\right) &= \left(\frac{1}{C_n(j)}\right)' \\ &= -\frac{n+1}{\binom{2n+j}{n}}\sum_{r=1}^{n}\frac{1}{r+j+n} \\ &= -\frac{n+1}{\binom{2n+j}{n}}\left[\Psi(1+j+2n) - \Psi(1+j+n)\right], \end{aligned} \tag{1.3}$$

where the Psi(or digamma function)

$$\Psi(z) = \frac{d}{dz}\ln(\Gamma(z)) = \frac{(\Gamma(z))'}{\Gamma(z)}$$

and the Gamma function

$$\Gamma(w) = \int_0^\infty t^{w-1}e^{-t}\,dt,$$

for $\Re(w) > 0$.

Proof. For the first derivative of $\frac{1}{C_n(j)}$ with respect to j, let for integer n,

$$\begin{aligned} \frac{1}{C_n(j)} &= \frac{n+1}{\binom{2n+j}{n}} \\ &= \frac{(n+1)\,\Gamma(n+1)\,\Gamma(n+j+1)}{\Gamma(2n+j+1)} \\ &= \frac{(n+1)\,\Gamma(n+1)}{\prod_{r=1}^{n}(n+r+j)}. \end{aligned}$$

Taking the logs on both sides we have

$$\ln\left[\frac{1}{C_n(j)}\right] = \ln(n+1) + \ln\left[\Gamma(n+1)\right] - \ln\left[\sum_{r=1}^{n}(n+r+j)\right]$$

and differentiating with respect to j we obtain the result (1.3).

It may be seen that for $j = 0$, we obtain

$$\left(\frac{1}{C_n(j)}\right)'\Bigg]_{j=0} = -\frac{n+1}{\binom{2n}{n}}\sum_{r=1}^{n}\frac{1}{r+n} = -\frac{n+1}{\binom{2n}{n}}\left[H_{2n+1}^{(1)} - H_{n+1}^{(1)}\right],$$

where the n^{th} Harmonic number

$$H_n^{(1)} = H_n = \int_{t=0}^{1} \frac{1-t^n}{1-t} dt = \sum_{r=1}^{n} \frac{1}{r} = \gamma + \Psi(n+1),$$

and γ denotes the Euler-Mascheroni constant, defined by

$$\gamma = \lim_{n\to\infty} \left(\sum_{r=1}^{n} \frac{1}{r} - \log(n) \right) = -\Psi(1) \approx 0.5772156649015328606065 12.........$$

An extension of the n^{th} harmonic numbers is introduced and studied by Sandor [4]. As an aside it is possible to consider higher derivatives of (1.3). □

Some numbers of (1.3), without the negative sign are

$$\begin{bmatrix} n & \left(\frac{1}{C_n(j)}\right)' \\ 1 & \frac{2}{(j+2)^2} \\ 2 & \frac{3!(2j+7)}{(j+3)^2(j+4)^2} \\ 3 & \frac{4!\left(3j^2+30j+74\right)}{(j+4)^2(j+5)^2(j+6)^2} \\ 4 & \frac{5!\left(2j^3+39j^2+251j+533\right)}{(j+5)^2(j+6)^2(j+7)^2(j+8)^2} \\ 5 & \frac{6!\left(5j^4+160j^3+1905j^2+10000j+19524\right)}{(j+6)^2(j+7)^2(j+8)^2(j+9)^2(j+10)^2} \\ 6 & \frac{7!\left(6j^5+285j^4+5380j^3+50445j^2+234908j+434568\right)}{(j+7)^2(j+8)^2(j+9)^2(j+10)^2(j+11)^2(j+12)^2} \\ \dots & \dots\dots\dots\dots\dots\dots\dots\dots\dots \end{bmatrix}$$

In the next theorems we give integral representations for various summation expressions of Catalan type numbers with parameters. In some particular cases we give closed form values of the sums and then determine upper and lower bounds in terms of the given parameters.

The following theorem is proved.

2. CATALAN RELATED SUMS

Theorem 2.1. *Let the Catalan related numbers, with parameter* $j = 0, 1, 2, 3, 4, \dots,$ $C_n(j) = \frac{1}{n+1}\binom{2n+j}{n}$ *and* $|t| < 4$, *then*

$$\begin{aligned} S_j(t) &= \sum_{n=1}^{\infty} \frac{t^n}{C_n(j)} \sum_{r=1}^{n} \frac{1}{r+j+n} \\ &= \sum_{n=1}^{\infty} \frac{t^n(n+1)}{\binom{2n+j}{n}} \sum_{r=1}^{n} \frac{1}{r+j+n} \\ &= \sum_{n=1}^{\infty} \frac{t^n(n+1)}{\binom{2n+j}{n}} \left[\Psi(1+j+2n) - \Psi(1+j+n)\right] \\ &= -2t \int_0^1 \frac{(1-x)^{j+1}\log(1-x)}{(1-tx(1-x))^3}\, dx. \end{aligned} \tag{2.1}$$

Proof. Consider

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{t^n}{C_n(j)} &= \sum_{n=1}^{\infty}\frac{t^n(n+1)}{\binom{2n+j}{n}} \\ &= \sum_{n=1}^{\infty}\frac{t^n(n+1)\Gamma(n+1)\Gamma(n+j+1)}{\Gamma(2n+j+1)}.\end{aligned}$$

Now by the use of the Gamma property $\Gamma(n+1) = n\Gamma(n)$ we have

$$\sum_{n=1}^{\infty}\frac{t^n n\ (n+1)\Gamma(n)\Gamma(n+j+1)}{\Gamma(2n+j+1)} = \sum_{n=1}^{\infty} t^n n(n+1) B(n, n+j+1),$$

where

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)} = \int_0^1 (1-y)^{\alpha-1} y^{\beta-1} dy = \int_0^1 (1-y)^{\beta-1} y^{\alpha-1} dy$$

for $\alpha > 0$ and $\beta > 0$ is the classical Beta function, therefore

$$\begin{aligned}\sum_{n=1}^{\infty} t^n n(n+1) B(n, n+j+1) &= \sum_{n=1}^{\infty} t^n n(n+1)\int_0^1 (1-x)^{n+j} x^{n-1} dx \\ &= \int_0^1 \frac{(1-x)^j}{x}\sum_{n=1}^{\infty} n(n+1)(tx(1-x))^n\ dx\end{aligned}$$

by interchanging sum and integral. Now applying Lemma 1.1 we obtain

$$S_j(t) = -2t\int_0^1 \frac{(1-x)^{j+1}\log(1-x)}{(1-tx(1-x))^3}\,dx,$$

which is the result (2.1). □

Other integral representations involving binomial coefficients and Harmonic numbers can be seen in [5], [6], [7] and [8].

Remark 1. For $j \in \mathbb{N}$ we obtain for $t = 2$

$$\begin{aligned}S_j(2) &= \sum_{n=1}^{\infty}\frac{2^n(n+1)}{\binom{2n+j}{n}}\sum_{r=1}^{n}\frac{1}{r+j+n} \\ &= -4\int_0^1 \frac{(1-x)^{j+1}\log(1-x)}{(1-2x(1-x))^3}\,dx \\ &= \alpha_1 G + \alpha_2\zeta(2) + \alpha_3\pi\ln(2) + \alpha_4\pi + \alpha_5,\end{aligned}$$

where the Catalan constant

$$G = \int_0^{\frac{\pi}{4}} \ln(\cot(x))\,dx = .0.91596559...,$$

and $\zeta(z)$ is the Zeta function. Some specific cases of $S_j(2)$ are

$$\begin{bmatrix} j & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 \\ 0 & 3 & 0 & \frac{3}{4} & \frac{1}{4} & 0 \\ 6 & \frac{3}{8} & -\frac{27}{32} & \frac{3}{32} & -\frac{7}{32} & \frac{7}{4} \\ 13 & \frac{35}{32} & -\frac{27}{256} & \frac{35}{128} & -\frac{3}{32} & -\frac{6920563}{6350400} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \end{bmatrix}$$

Remark 2. For $j \in \mathbb{N}$ we obtain for $t = -\frac{1}{2}$

$$\begin{aligned} S_j\left(-\frac{1}{2}\right) &= \sum_{n=1}^{\infty} \frac{\left(-\frac{1}{2}\right)^n (n+1)}{\binom{2n+j}{n}} \sum_{r=1}^{n} \frac{1}{r+j+n} \\ &= \int_0^1 \frac{(1-x)^{j+1} \log(1-x)}{\left(1+\frac{1}{2}x(1-x)\right)^3}\, dx \\ &= \beta_1 \zeta(2) + \beta_2 (\ln(2))^2 + \beta_3 \ln(2) + \beta_4. \end{aligned}$$

Some specific cases of $S_j\left(-\frac{1}{2}\right)$ are

$$\begin{bmatrix} j & \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ 0 & -\frac{8}{81} & \frac{4}{81} & -\frac{2}{27} & 0 \\ 2 & \frac{8}{81} & -\frac{8}{81} & -\frac{4}{27} & -\frac{2}{9} \\ 8 & -\frac{31412}{81} & \frac{31744}{81} & \frac{6596}{27} & \frac{5045}{18} \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}.$$

Next we shall give upper and lower bounds for the series $S_j(t)$ given by (2.1).

Theorem 2.2. *For* $j = 0, 1, 2, 3, \ldots$, *and* $0 < t < 4$

$$\begin{aligned} \frac{2t}{(j+2)^2} &< S_j(t) \\ &\leq \begin{cases} \frac{2t}{(j+2)^2}\left(\frac{4}{4-t}\right)^3, \\ 2t\sqrt{\frac{2}{(2j+3)^3}}\left[\frac{2t^4-41t^3+342t^2-1490t+3860}{5(4-t)^5} + \frac{1008\arcsin\left(\frac{\sqrt{t}}{2}\right)}{\sqrt{t(4-t)^{11}}}\right]^{\frac{1}{2}}. \end{cases} \end{aligned} \tag{2.2}$$

Proof. Consider the integral inequality

$$\int_{x_0}^{x_1} |f(x)\, g(x)|\, dx \leq \sup_{x\in[x_0,x_1]} |f(x)| \int_{x_0}^{x_1} |g(x)|\, dx,$$

and from the integral (2.1) we can identify

$$\int_{x_0}^{x_1} |g(x)|\, dx = \int_0^1 (1-x)^{j+1} \ln(1-x)\, dx = \frac{1}{(j+2)^2}. \tag{2.3}$$

Similarly

$$f(x)=\frac{2t}{(1-tx(1-x))^{3}} \tag{2.4}$$

is monotonic on $x\in[0,1]$ with $\lim\limits_{x\to 0} f(x)=2t,\ \lim\limits_{x\to 1} f(x)=0$, hence

$$\sup_{x\in[0,1]} f(x)=2t\left(\frac{4}{4-t}\right)^{3}.$$

The series (2.1) is one of positive terms and its lower bound is given by the first term, hence combining these results we obtain the first part of the inequality (2.2). For the second part of the inequality (2.2), consider the Euclidean norm where for $\alpha=\beta=2,\ \frac{1}{\alpha}+\frac{1}{\beta}=1$, and for $f(x)$ and $g(x)$ defined by (2.4) and (2.3) respectively we have that $|f(x)|^{2}$ and $|g(x)|^{2}$ are integrable functions defined on $x\in[0,1]$. From (2.1) and by Hölder's integral inequality, which is a special case of the Cauchy-Buniakowsky-Schwarz inequality,

$$S_j(t)\le\left(\int_0^1|f(x)|^2\,dx\right)^{\frac{1}{2}}\left(\int_0^1|g(x)|^2\,dx\right)^{\frac{1}{2}},$$

where

$$\left(\int_0^1|g(x)|^2\,dx\right)^{\frac{1}{2}}=\left(\int_0^1\left|(1-x)^{j+1}\ln(1-x)\right|^2dx\right)^{\frac{1}{2}}=\sqrt{\frac{2}{(2j+3)^3}}$$

and

$$\left(\int_0^1|f(x)|^2\,dx\right)^{\frac{1}{2}}=2t\left[\frac{2t^4-41t^3+342t^2-1490t+3860}{5(4-t)^5}+\frac{1008\arcsin\left(\frac{\sqrt{t}}{2}\right)}{\sqrt{t(4-t)^{11}}}\right]^{\frac{1}{2}},$$

so that

$$S_j(t)\le 2t\sqrt{\frac{2}{(2j+3)^3}}\left[\frac{2t^4-41t^3+342t^2-1490t+3860}{5(4-t)^5}+\frac{1008\arcsin\left(\frac{\sqrt{t}}{2}\right)}{\sqrt{t(4-t)^{11}}}\right]^{\frac{1}{2}}$$

and the second part of (2.2) follows. □

Remark 3. For a particular value of t and $j\ge 0$ we can plot the exact value of $S_j(t)$, (2.1) against the upper bounds from (2.2)

$$A_j(t)=\frac{2t}{(j+2)^2}\left(\frac{4}{4-t}\right)^3 \tag{2.5}$$

and

$$B_j(t)=2t\sqrt{\frac{2}{(2j+3)^3}}\left[\frac{2t^4-41t^3+342t^2-1490t+3860}{5(4-t)^5}+\frac{1008\arcsin\left(\frac{\sqrt{t}}{2}\right)}{\sqrt{t(4-t)^{11}}}\right]^{\frac{1}{2}}. \tag{2.6}$$

Remark 4. For $t=1.5$ we have the graph, Figure 2.1, showing that $B_j(1.5)$ is a better estimator of $S_j(1.5)$, (the lower curve in Figure 2.1), than $A_j(1.5)$ up to about $j=4.5$. The exact value of j for a given value of t can be exactly calculated from (2.5) and (2.6).

The graph in Figure 2.1 suggests that the sum in (2.1) may be convex. We prove the convexity of (2.1) in the following theorem.

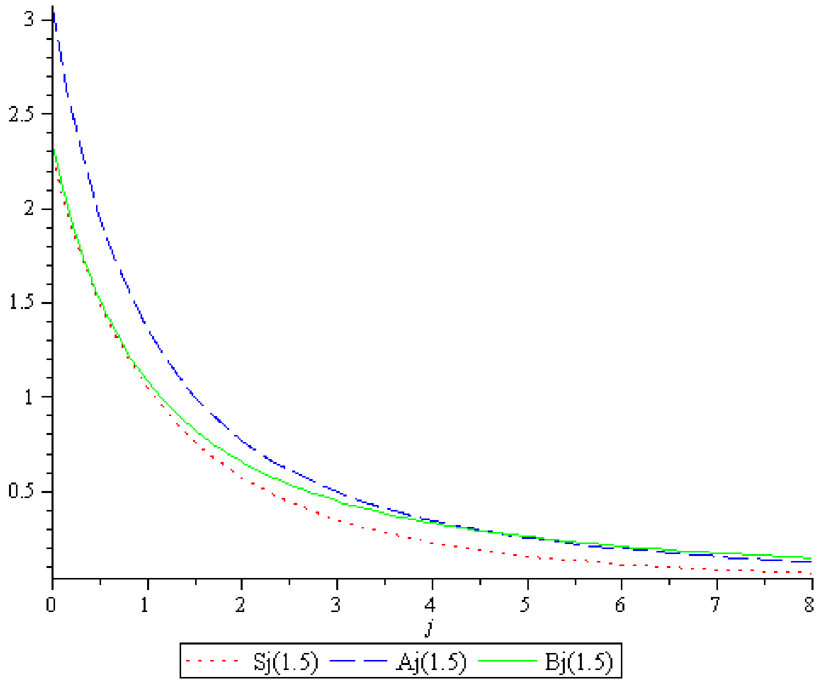

Figure 2.1: A plot of $S_j(1.5)$, $A_j(1.5)$ and $B_j(1.5)$ showing the crossover point at about $j = 4.5$.

Theorem 2.3. *For $j \geq 0$ and $0 < t < 4$ the function $j \mapsto S_j(t)$, as given in Theorem 2.1 is strictly decreasing and convex with respect to the parameter $j \in [0, \infty)$ for every $x \in [0, 1]$.*

Proof. Let

$$g_j(x, t) = \frac{(1-x)^{j+1} \ln(1-x)}{(1 - tx(1-x))^3}$$

be an integrable function for $x \in [0, 1]$ and put

$$S_j(t) = -2t \int_0^1 g_j(x, t)\, dx,$$

so that

$$S_0(t) = -2t \int_0^1 g_0(x, t)\, dx = -2t \int_0^1 \frac{(1-x)\ln(1-x)}{(1 - tx(1-x))^3} dx.$$

Applying the Leibniz rule for differentiation under the integral sign, we have that

$$\begin{aligned} S_j'(t) &= \int_0^1 \frac{\partial}{\partial j} g_j(x, t)\, dx \\ &= -2t \int_0^1 \frac{(1-x)^{j+1} (\ln(1-x))^2}{(1 - tx(1-x))^3} dx. \end{aligned}$$

Since $j \geq 0$ and $0 < t < 4$

$$\frac{(1-x)^{j+1} (\ln(1-x))^2}{(1 - tx(1-x))^3} > 0 \quad \text{for } x \in (0, 1),$$

then $S_j'(t) < 0$, so that the sum in (2.1), is a strictly decreasing sum with respect to the parameter j for $x \in [0, 1]$.

Now

$$
\begin{aligned}
S_j''(t) &= \int_0^1 \frac{\partial^2}{\partial^2 j} g_j(x,t)\,dx \\
&= -2t\int_0^1 \frac{(1-x)^{j+1}(\ln(1-x))^3}{(1-tx(1-x))^3}dx,
\end{aligned}
$$

and since

$$
\frac{(1-x)^{j+1}(\ln(1-x))^3}{(1-tx(1-x))^3} < 0,
$$

then $S_j''(t) > 0$ so that (2.1) is a convex function for $x \in [0,1]$. □

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