



Integral Representations of Catalan and Related Numbers

K.A.Penson

J.-M. Sixdeniers

Université Pierre et Marie Curie

Laboratoire de Physique Théorique des Liquides

Tour 16, 5^{ième} étage, 4, place Jussieu, 75252 Paris Cedex 05, France

Email addresses: penson@lptl.jussieu.fr and sixdeniers@lptl.jussieu.fr

Abstract

We derive integral representations for the Catalan numbers $C(n)$, shifted Catalan numbers $C(n+p)$, and the numbers $n! \cdot C(n)$ and $C(n) \cdot B(n)$, where $B(n)$ are the Bell numbers, for $n = 0, 1, \dots$. Our method is to use inverse Mellin transform. All these numbers are power moments of positive functions, and their representations turn out to be unique.

The Catalan numbers $C(n)$, $n = 0, 1, 2, \dots$, defined by

$$C(n) = \frac{\binom{2n}{n}}{n+1}, \quad (1)$$

are among the most ubiquitous sequences in enumerative combinatorics. Stanley [13] cites no less than 66 different combinatorial settings where these numbers appear. The first few Catalan numbers are

$$1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862$$

for $n = 0 \dots 9$. A plethora of information about the $C(n)$'s can be found in [11], under sequence no. [A000108](#).

In this note we derive an integral representation of $C(n)$ as the n -th power moment of a certain non-negative function $W_C(x)$ on the positive half-axis. We also study the ramifications of this representation for other integer sequences involving $C(n)$.

To this end we seek a function $W_C(x)$ such that

$$\int_0^\infty x^n W_C(x) dx = C(n) \quad (2)$$

$$= \frac{4^n \Gamma(n + 1/2)}{\sqrt{\pi} \Gamma(n + 2)} \quad , \quad n = 0, 1, \dots \quad . \quad (3)$$

Replacing n by a complex variable $s - 1$, we rewrite Eq.(3) as

$$\int_0^\infty x^{s-1} W_C(x) dx = \frac{4^{s-1} \Gamma(s - 1/2)}{\sqrt{\pi} \Gamma(s + 1)} \quad , \quad \text{Re } s > 1 \quad , \quad (4)$$

which implies that

$$W_C(x) = \mathcal{M}^{-1} \left[\frac{4^{s-1} \Gamma(s - 1/2)}{\sqrt{\pi} \Gamma(s + 1)}; x \right] \quad , \quad (5)$$

where $\mathcal{M}^{-1} [f^*(s); x] = f(x)$ is the inverse Mellin transform [12], with $f^*(s) = \mathcal{M} [f(x); s] = \int_0^\infty x^{s-1} f(x) dx$ the Mellin transform of $f(x)$. We note the following property of \mathcal{M} [12] :

$$\mathcal{M} [x^b f(ax^h); s] = \frac{1}{h} a^{-\frac{s+b}{h}} f^* \left(\frac{s+b}{h} \right) \quad , \quad b \in R, \quad h > 0, \quad (6)$$

which, when specialized to $a = \frac{1}{4}$, $b = -\frac{1}{2}$ and $h = 1$, implies that

$$\mathcal{M} \left[x^{-\frac{1}{2}} f \left(\frac{x}{4} \right); s \right] = 4^s f^*(s - 1/2)/2 \quad . \quad (7)$$

Adopting the standard notation $(y)_+^\alpha = y^\alpha$ if $y > 0$, $(y)_+^\alpha = 0$ otherwise, and using the formula 2.2(1), p.151 of [5] :

$$\mathcal{M} [(1-x)_+^{\alpha-1}; s] = \Gamma(\alpha) \frac{\Gamma(s)}{\Gamma(\alpha+s)} \quad , \quad \alpha > 0, \quad s > 0, \quad (8)$$

we can apply Eq.(7) with $f(x) = (1-x)_+^{\alpha-1}$ and $\alpha = \frac{3}{2}$. This yields

$$W_C(x) = \frac{x^{-\frac{1}{2}}}{\pi} \left(1 - \frac{x}{4} \right)_+^{\frac{1}{2}} \quad . \quad (9)$$

The function $W_C(x)$ is displayed on Fig.(1). The desired integral representation of $C(n)$ is then

$$C(n) = \int_0^4 x^n \left(\frac{\sqrt{\frac{4-x}{x}}}{2\pi} \right) dx \quad . \quad (10)$$

This is the solution of the Hausdorff moment problem on $[0, 4]$, which is always unique [1], and so the representation of Eq.(10) is also unique.

By the same token we can find the solution of

$$\int_0^\infty x^n W_{C,p}(x) dx = C(n+p), \quad n = 0, 1, 2, \dots, \quad p = 1, 2, \dots, \quad (11)$$

i.e. the unique representation of the shifted Catalan numbers $C(n+p)$, as the Hausdorff moments of

$$W_{C,p}(x) = \frac{x^{p-\frac{1}{2}}}{\pi} \left(1 - \frac{x}{4} \right)_+^{\frac{1}{2}} \quad . \quad (12)$$

The Mellin convolution property for products of Mellin transforms, in its simplest incarnation, states ([12], [5]) that if $\mathcal{M} [W_{1,2}(x); s] = \rho_{1,2}(s)$ then

$$\mathcal{M}^{-1} [\rho_1(s) \rho_2(s); x] = W_{12}(x) \equiv \int_0^\infty \frac{1}{t} W_1 \left(\frac{x}{t} \right) W_2(t) dt \quad . \quad (13)$$

Observe that $W_{1,2}(x) > 0$ implies $W_{12}(x) > 0$.

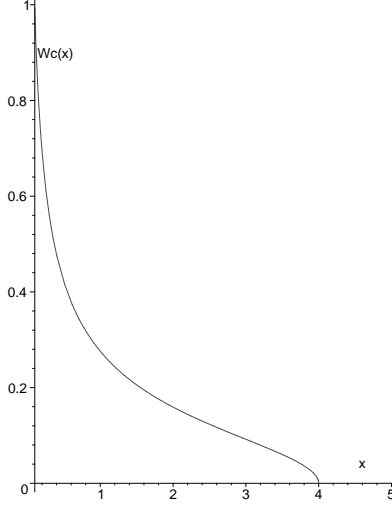


Figure 1: : The function $W_C(x)$, s. Eq.(9). This function diverges at $x = 0$.

As an application of Eq.(13) we look for an integral representation of the sequence $n! \cdot C(n)$ whose initial terms are 1, 1, 4, 30, 336, 5040, 95040, 2162160, 57657600, 1764322560, for $n = 0 \dots 9$; compare [11], no. **A001761**. Using Eq.(9) and performing the Mellin convolution in Eq.(13) with $W_1(x) = e^{-x}$ and $W_2(x) = W_C(x)$, one ends up with the following Stieltjes moment problem :

$$\int_0^\infty x^n W_{1C}(x) dx = n! \cdot C(n) = \frac{(2n)!}{(n+1)!} \quad , \quad n = 0, 1, \dots, \quad (14)$$

with the solution

$$W_{1C}(x) = \frac{1}{2\pi\sqrt{x}} \int_{\frac{x}{4}}^\infty e^{-t} \frac{\sqrt{4t-x}}{t} dt \quad (15)$$

$$= -\frac{1}{2} + \frac{1}{\sqrt{\pi x}} e^{-\frac{x}{4}} + \frac{1}{2} \operatorname{erf}\left(\frac{\sqrt{x}}{2}\right) \quad , \quad (16)$$

where $\operatorname{erf}(y)$ is the error function. The function $W_{1C}(x)$ is shown in Fig.(2). As $W_{1C}(x) > 0$, the (sufficient) Carleman condition ($\sum_{n=1}^\infty (\frac{(2n)!}{(n+1)!})^{-\frac{1}{2n}} = \infty$) (cf. Ref.[1]) indicates that the solution $W_{1C}(x)$ of Eq.(16) is also unique. Similar results are obtained by using $W_{C,p}(x)$ instead of $W_C(x)$ in Eq.(13).

Another use of Eq.(13) is illustrated by considering the sequence $C(n) \cdot B(n)$, where $B(n)$ are the Bell numbers (see [11], no. **A000110**, and [2]). The initial terms of this sequence are 1, 1, 4, 25, 210, 2184, 26796, 376233, 5920200, 102816714, for $n = 0 \dots 9$. For this last sequence see [11], no. **A064299**. The weight function whose n -th moment is equal to $B(n)$ is

$$W_B(x) = \frac{1}{e} \sum_{k=1}^\infty \frac{\delta(x-k)}{k!} \quad , \quad (17)$$

which is a consequence of Dobiński formula, $B(n) = \frac{1}{e} \sum_{k=1}^\infty \frac{k^n}{k!}$, see [2]. In Eq.(17), $\delta(y)$ is Dirac's delta function. By Mellin convolution of $W_B(x)$ with $W_C(x)$ one obtains

$$W_{BC}(x) = \frac{1}{2\pi e} \sum_{k=1}^\infty \frac{1}{kk!} \sqrt{\frac{4k-x}{x}} H\left(4 - \frac{x}{k}\right) \quad , \quad (18)$$

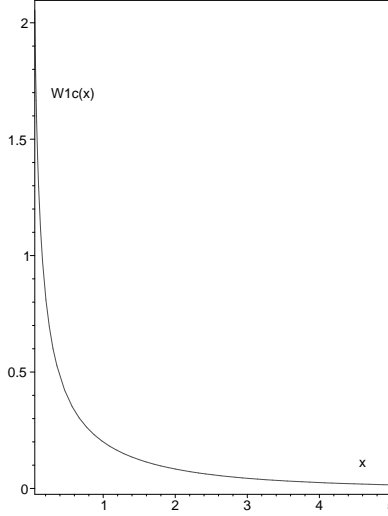


Figure 2: : The function $W_{1C}(x)$, s. Eq.(16). This function diverges at $x = 0$.

which, via Carleman’s criterion, is the only positive function such that its n -th moment is equal to $C(n) \cdot B(n)$. In Eq.(18) $H(y)$ is the Heaviside function. The function $W_{BC}(x)$ is displayed on Fig.(3).

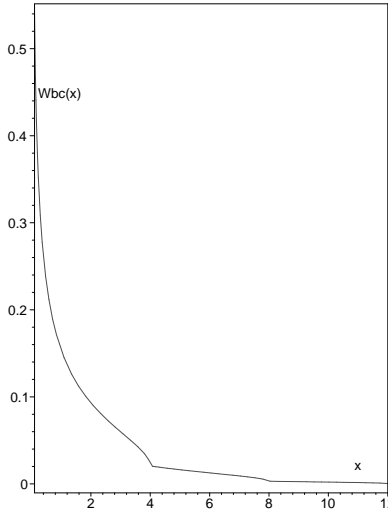


Figure 3: : The function $W_{BC}(x)$, s. Eq.(18). This function diverges at $x = 0$.

The last sequence that will concern us here is $(n!)^2 C(n) = \frac{(2n)!}{n+1}$. Its initial terms are

$$1, 1, 8, 180, 8064, 604800, 68428800, 10897286400, 2324754432000$$

for $n = 0 \dots 9$; compare [11], no. **A060593**. Proceeding as in Eqs.(3) and (4), we are looking for $W_3(x)$ satisfying

$$\int_0^\infty x^{s-1} W_3(x) dx = \frac{4^{s-1} \Gamma(s-1/2) \Gamma^2(s)}{\sqrt{\pi} \Gamma(s+1)}, \quad \text{Re } s > 1 \quad . \quad (19)$$

It appears that when studying Eq.(19) it is possible to avoid using $W_C(x)$. As the first step we observe from

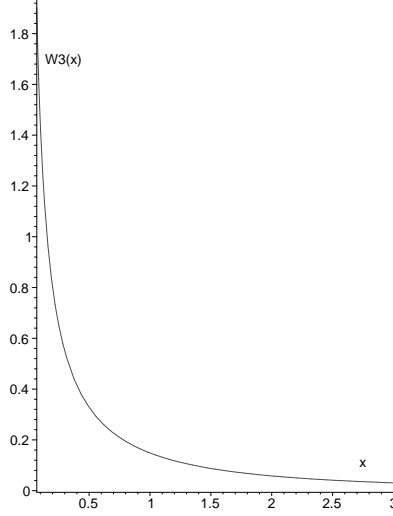


Figure 4: : The function $W_3(x)$, s. Eq.(24). This function diverges at $x = 0$.

Eq.(6) that

$$\mathcal{M}^{-1} \left[\Gamma \left(s - \frac{1}{2} \right); x \right] = \frac{e^{-x}}{\sqrt{x}} \quad . \quad (20)$$

In addition, the following relation holds :

$$\mathcal{M}^{-1} \left[\frac{\Gamma^2(s)}{\Gamma(s+1)}; x \right] = -Ei(-x) \quad , \quad (21)$$

which is the formula 8.1(1), p.182 of [5]. In Eq.(21) $Ei(y)$ is the exponential integral function. Combining Eqs.(20) and (21) in the Mellin convolution we obtain

$$\mathcal{M}^{-1} \left[\frac{\Gamma(s-1/2)\Gamma^2(s)}{\Gamma(s+1)}; x \right] = -\frac{1}{\sqrt{x}} \int_0^\infty t^{-\frac{1}{2}} t^{-\frac{x}{t}} Ei(-t) dt \quad (22)$$

$$= 2\sqrt{\frac{\pi}{x}} e^{-2\sqrt{x}} + 4\sqrt{\pi} Ei(-2\sqrt{x}) \quad , \quad x > 0 \quad . \quad (23)$$

In writing Eq.(23) we have used the formula 2.5.4.2, p.72 of [6]. Finally, we use Eq.(6) again (with $a = \frac{1}{4}$, $b = 0$ and $h = 1$) and from Eq.(19) we get the solution

$$W_3(x) = \frac{1}{\sqrt{x}} e^{-\sqrt{x}} + Ei(-\sqrt{x}) \quad , \quad (24)$$

which is plotted in Fig.(4). As $W_3(x) > 0$, by Carleman's criterion the solution is again unique.

Remark: E. P.Wigner [14] has demonstrated that Eq.(10), under a suitable parametrization, describes the distribution function of eigenvalues of an ensemble of random, symmetric, real matrices.

Integral representations of other combinatorial numbers can be found in [3]. For further applications of Mellin convolution formula Eq.(13), one may consult [5], [10], [7], [8], [4] and [9].

References

- [1] N. I. Akhiezer, *The Classical Moment Problem and Some Related Questions in Analysis*, (Oliver and Boyd, London, 1965)

- [2] L. Comtet, *Advanced Combinatorics*, (D. Reidel, Boston, 1984)
- [3] G. P. Egorychev, *Integral Representation and the Computation of Combinatorial Sums*, Translations of Mathematical Monographs, Vol. 59, (American Mathematical Society, Rhode Island, 1984)
- [4] J. R. Klauder, K. A. Penson and J. M. Sixdeniers, *Constructing coherent states through solutions of Stieltjes and Hausdorff moment problems*, Phys. Rev. **A64**, 013817 (2001)
- [5] O. I. Marichev, *Handbook of Integral Transforms of Higher Transcendental Functions, Theory and Algorithmic Tables*, (Ellis Horwood Ltd, Chichester, 1983)
- [6] A. P. Prudnikov, Yu. A. Brychkov and O. I. Marichev, *Integrals and Series, vol. 2: Special Functions*, (Gordon and Breach, New York, 1998)
- [7] J. M. Sixdeniers and K. A. Penson, *On the completeness of coherent states generated by binomial distribution*, J. Phys. **A33**, 2907 (2000)
- [8] J. M. Sixdeniers and K. A. Penson, *On the completeness of photon-added coherent states*, J. Phys. **A34**, 2859 (2001)
- [9] J. M. Sixdeniers, K. A. Penson and J. R. Klauder, *Tricomi coherent states*, Int. J. Mod. Phys. B **15**, 4231 (2001)
- [10] J. M. Sixdeniers, K. A. Penson and A. I. Solomon, *Mittag-Leffler coherent states*, J. Phys. **A32**, 7543 (1999)
- [11] N. J. A. Sloane, On-Line Encyclopedia of Integer Sequences, published electronically at: <http://www.research.att.com/~njas/sequences/>
- [12] I. N. Sneddon, *The Use of Integral Transforms*, (McGraw-Hill, New York, 1974)
- [13] R. P. Stanley, *Enumerative Combinatorics*, Vol. 2, (Cambridge University Press, 1999)
- [14] E. P. Wigner, *Characteristic vectors of bordered matrices with infinite dimensions*, Ann.Math.**62**, 548 (1955)

(Concerned with sequences [A000108](#), [A000110](#), [A001761](#), [A060593](#), [A064299](#).)

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