Journal of Integer Sequences, Vol. 16 (2013),
Article 13.6.8

A Generalization of the Catalan Numbers

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Abstract

In this paper, we generalize the Catalan number C_n to the (m,n)th Catalan number $C(m,n)$ using a combinatorial description, as follows: the number of paths in $\mathbb{R}^m$ from the origin to the point $\big(\underbrace{n,\dots,n}_{m-1},(m-1)n\big)$ with m kinds of moves such that the path never rises above the hyperplane $x_m = x_1 + \cdots + x_{m-1}$.

1 Introduction

Catalan numbers (A000108) are a very prominent sequence of numbers that arises in a wide varity of combinatorial situations [1, 2]. Stanley [10] gave a list of 66 different combinatorial descriptions of Catalan numbers and he added to the list some more [11]. Some of the specific instances are

- The number of movements in xy-plane from $(0,0)$ to (n,n) with two kinds of moves

  $$R : (x,y) \to (x+1,y), \quad U : (x,y) \to (x,y+1),$$

  such that the path never rises above the line $y = x$.

- Triangulations of a convex $(n+2)$-gon into n triangles by $n-1$ diagonals that do not intersect in their interiors.

- Binary parenthesizations of a string of $n+1$ letters.
- Binary trees with n vertices.

The solution to these problems is the nth Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n},$$

and the sequence $C_0, C_1, C_2, \dots, C_n, \dots$ is called the Catalan sequence.

There have been many attempts to generalize the Catalan numbers. Probably the most important generalization consists of the k-ary numbers or k-Catalan numbers, defined by

$$C_n^k = \frac{1}{kn+1}\binom{kn+1}{n} = \frac{1}{(k-1)n+1}\binom{kn}{n} = \frac{1}{n}\binom{kn}{n-1},$$

where $k, n \in \mathbb{N}$. Clearly, $C_n^2 = C_n$. The k-good paths (below the line $y = kx$) from $(0,-1)$ to $\big(n, (k-1)n-1\big)$, staircase tilings and k-ary trees are structures known to be enumerated by k-ary numbers [5, 6, 8, 10]. Moreover, Kim [7, Thm. 2] showed that C_n^k is the number of partitions of $n(k-1)+2$ polygon by $(k+1)$-gon where all vertices of all $(k+1)$-gon lie on the vertices of $n(k-1)+2$ polygon. Gould [3] developed a generalization that has both the Catalan numbers and the k-Catalan numbers as special cases, defined as

$$A_n(a,b) = \frac{a}{a+bn}\binom{a+bn}{n},$$

and showed the following convolution formula for these sequences:

$$\sum_{k=0}^{n} A_k(a,b)A_{n-k}(c,b) = A_n(a+c,b).$$

These numbers are also known as the Rothe numbers [9] and Rothe-Hagen coefficients of the first type [4]. Clearly, $A_n(1,2) = C_n$ and $A_n(1,k) = C_n^k$.

We know that one of the interpretations of the Catalan numbers is the movements in $\mathbb{R}^2$ with two kinds of moves such that the path never rises above the line $y = x$. In this paper, a new generalization of the Catalan numbers using this interpretation is introduced. Consider m kinds of moves in $\mathbb{R}^m$ such that they are one unit parallel to the positive axes. We show that the number of paths from the origin to the point $\big(\underbrace{n,\dots,n}_{m-1},(m-1)n\big)$ using these moves such that the path never rises above the hyperplane $x_m = x_1 + \cdots + x_{m-1}$ is

$$\frac{1}{n(m-1)+1}\binom{2n(m-1)}{\underbrace{n,\dots,n}_{m-1},\, n(m-1)}.$$

We call this number the (m,n)th Catalan number $C(m,n)$. Clearly, $C(2,n)$ is the ordinary nth Catalan number C_n.

2 Generalization

In this section, we prove the our main theorem. We show that the generalized Catalan numbers $C(m,n)$ are given by

$$\frac{1}{n(m-1)+1}\binom{2n(m-1)}{\underbrace{n,\ldots,n}_{m-1},n(m-1)}.$$

Theorem 1. *Let $\mathbb{R}^m$ be the m-dimensional vector space. Consider*

$$\begin{aligned} R_1 &: (x_1,x_2,\ldots,x_m) \longrightarrow (x_1+1,x_2,\ldots,x_m),\\ R_2 &: (x_1,x_2,\ldots,x_m) \longrightarrow (x_1,x_2+1,\ldots,x_m),\\ &\vdots\\ R_m &: (x_1,x_2,\ldots,x_m) \longrightarrow (x_1,x_2,\ldots,x_m+1), \end{aligned}$$

be m kinds of moves in $\mathbb{R}^m$ (i.e., R_i denotes the move one unit parallel to the x_i-axis in the positive direction). Then the number of paths from $\mathbf{0} = (\underbrace{0,\ldots,0}_{m})$ to the point $N := (\underbrace{n,\ldots,n}_{m-1},(m-1)n)$ using the moves $R_1,R_2,\ldots,R_m$ such that the path never rises above the hyperplane $x_m = x_1+\cdots+x_{m-1}$ is

$$\frac{1}{n(m-1)+1}\binom{2n(m-1)}{\underbrace{n,\ldots,n}_{m-1},n(m-1)}.$$

Proof. We call a path from $\mathbf{0}$ to N of n R_1's, n R_2's, …, n R_{m-1}'s, and $(m-1)n$ R_m's acceptable if the path never rises above the hyperplane $x_m = x_1+\cdots+x_{m-1}$ and unacceptable otherwise. Let A_n^m and U_n^m denote the number of acceptable and unacceptable paths, respectively. It is easy to see that each path from $\mathbf{0}$ to N corresponds to an arrangement of n R_1's, n R_2's, …, n R_{m-1}'s, and $(m-1)n$ R_m's. Then

$$A_n^m + U_n^m = \frac{\big(2n(m-1)\big)!}{n!^{m-1}\big(n(m-1)\big)!}.$$

Now, consider an unacceptable path and its arrangement $r_1,r_2,\ldots,r_{2n(m-1)}$, where $r_i \in \{R_1,R_2,\ldots,R_m\}$ indicates the ith step of the path. Since the path rises above the hyperplane, there is a first t such that the number of R_m's in $r_1,\ldots,r_t$ exceeds the sum of the numbers $R_1,R_2,\ldots,R_{m-1}$. Moreover, $r_t = R_m$. We only change $r_{t+1},\ldots,r_{2n(m-1)}$ the part of the path after the crossing in the arrangement as follows: Mark all the positions of the R_m's in that part of the path and fill those positions with the sequence (in order) consisting of all but the last of the non-R_m's. Then replace those non-R_m's that have been used in the replacement with R_m's. Here is an example: let $m=3$, $n=2$ and the path be given by $R_1R_3R_2R_3R_3R_1R_2R_3$. Then $t=5$, and the part of the path to be modified is $R_1R_2R_3$. There is just one position of the R_m's, so we replace R_3 with R_1, and then fill the R_1R_2 with R_3R_3 to obtain the modified sequence $R_1R_3R_2R_3R_3R_3R_3R_1$. The resulting arrangement $r_1', r_2', \ldots, r_{2n(m-1)}'$ is an arrangement of $(m-1)n+1$ R_m's, n R_1's, …, n R_{i-1}'s, n R_{i+1}'s, …, n R_{m-1}'s, and $n-1$ R_i's for a $1 \le i \le m-1$. It is not difficult to see that this process is reversible:

$$
\begin{array}{c}
r_1, r_2, \ldots, r_{2n(m-1)} \\
\Updownarrow \\
\underbrace{*, *, \ldots, *, R_m}_{\substack{rR_m\text{'s}, \\ r-1R_1,\ldots,R_{m-1}\text{'s}.}} \quad | \quad \underbrace{*, *, \ldots, *}_{\substack{(m-1)n-rR_m\text{'s}, \\ (m-1)n-r+1R_1,\ldots,R_{m-1}\text{'s}.}} \\
\Updownarrow \\
\underbrace{*, *, \ldots, *, R_m}_{\substack{rR_m\text{'s}, \\ r-1R_1,\ldots,R_{m-1}\text{'s}.}} \quad | \quad \underbrace{*, *, \ldots, *}_{\substack{(m-1)n-r+1R_m\text{'s}, \\ (m-1)n-rR_1,\ldots,R_{m-1}\text{'s}.}} \\
\Updownarrow \\
r_1', r_2', \ldots, r_{2n(m-1)}'
\end{array}
$$

Hence, there are as many unacceptable arrangements as there are arrangements of $(m-1)n+1$ R_m's, n R_1's, …, n R_{i-1}'s, n R_{i+1}'s, …, n R_{m-1}'s, and $n-1$ R_i's for a $1 \le i \le m-1$. Then

$$U_n^m = (m-1)\frac{\big(2n(m-1)\big)!}{n!^{m-2}(n-1)!\big(n(m-1)+1\big)!}.$$

So,

$$
\begin{aligned}
A_n^m =& \frac{\big(2n(m-1)\big)!}{n!^{m-1}\big(n(m-1)\big)!} - (m-1)\frac{\big(2n(m-1)\big)!}{n!^{m-2}(n-1)!\big(n(m-1)+1\big)!} \\
=& \frac{\big(2n(m-1)\big)!}{n!^{m-2}(n-1)!\big(n(m-1)\big)!}\Big(\frac{1}{n} - \frac{m-1}{n(m-1)+1}\Big) \\
=& \frac{\big(2n(m-1)\big)!}{n!^{m-1}\big(n(m-1)+1\big)!} \\
=& \frac{1}{n(m-1)+1}\binom{2n(m-1)}{\underbrace{n,\ldots,n}_{m-1}, n(m-1)}.
\end{aligned}
$$

□

We denote A_n^m in the above proof by $C(m,n)$. The first few generalized Catalan numbers are evaluated to be

$n\backslash m$	3	4	5
0	1	1	1
1	4	30	336
2	84	11880	3603600
3	2640	8168160	76881235200
4	100100	7207615800	2229760743210000
5	4232592	7336632122820	77015151194691790080
6	192203088	8193001579963200	2978057806800232994982144
7	9178678080	9763825599821779200	124625746332992720112321024000
8	455053212900	12209602888667136003480	5529032167369807343550830945418000

3 Acknowledgments

The author would like to thank the referee for his/her valuable comments and suggestions which have improved the clarity of the proof of the Theorem 1. This work is partially supported by the University of Kashan under grant number 233437/3.

References

[1] R. A. Brualdi, *Introductory Combinatorics*, 5th ed., Prentice-Hall, 2009.

[2] R. P. Grimaldi, *Discrete and Combinatorial Mathematics*, 5th ed., Addison-Wesley, 2004.

[3] H. W. Gould, *Combinatorial Identities*, Morgantown, West Virginia, 1972.

[4] H. W. Gould, *Fundamentals of Series*, eight tables based on seven unpublished manuscript notebooks (1945–1990), edited and compiled by J. Quaintance, May 2010. Available at http://www.math.wvu.edu/~gould/.

[5] S. Heubach, N. Y. Li, and T. Mansour, Staircase tilings and k-Catalan structures, *Discrete Math.* **308** (2008), 5954–5964.

[6] P. Hilton and J. Pedersen, Catalan numbers, their generalization, and their uses, *Math. Intelligencer* **13** (2) (1991), 64–75.

[7] D. Kim, On the (n, k)-th Catalan numbers, *Commun. Korean Math. Soc.* **23** (2008), 349–356.

[8] I. Pak, Reduced decompositions of permutations in terms of star transpositions, generalized Catalan numbers and k-ary trees, *Discrete Math.* **204** (1999), 329–335.

[9] S. L. Richardson, Jr., *Enumeration of the generalized Catalan numbers*, M.Sc. Thesis, Eberly College of Arts and Sciences, West Virginia University, Morgantown, West Virginia, 2005.

[10] R. P. Stanley, *Enumerative Combinatorics*, Vol. 2, Cambridge University Press, 1999.

[11] R. P. Stanley, *Catalan addendum*, preprint, May 25 2013.
Available at `http://www-math.mit.edu/~rstan/ec/catadd.pdf`.

2000 *Mathematics Subject Classification*: Primary 05A19; Secondary 05A10, 05A15.
Keywords: Catalan number, path.

(Concerned with sequence A000108.)

Received March 16 2013; revised version received July 3 2013. Published in *Journal of Integer Sequences*, July 30 2013.

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