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Catalan Numbers, the Hankel Transform, and Fibonacci Numbers

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Abstract

We prove that the Hankel transformation of a sequence whose elements are the sums of two adjacent Catalan numbers is a subsequence of the Fibonacci numbers. This is done by finding the explicit form for the coefficients in the three-term recurrence relation that the corresponding orthogonal polynomials satisfy.

1. Introduction

Let $A = \{a_0, a_1, a_2, ...\}$ be a sequence of real numbers. The Hankel matrix generated by A is the infinite matrix $H = [h_{i,j}]$, where $h_{i,j} = a_{i+j-2}$, i.e.,

$$H = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \dots \\ a_1 & a_2 & a_3 & a_4 & \dots \\ a_2 & a_3 & a_4 & a_5 & \dots \\ a_3 & a_4 & a_5 & a_6 & \dots \\ a_4 & a_5 & a_6 & a_7 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

The *Hankel matrix* H_n *of order* n is the upper-left $n \times n$ submatrix of H and the *Hankel determinant of order* n of A, denoted by h_n, is the determinant of the corresponding Hankel matrix.

For a given sequence $A = \{a_0, a_1, a_2, ...\}$, the *Hankel transform* of A is the corresponding sequence of Hankel determinants $\{h_0, h_1, h_2, \dots\}$ (see Layman [5]).

The elements of the sequence in which we are interested (A005807 of the On-Line Encyclopedia of Integer Sequences (EIS) [10], also INRIA [3]) are the sums of two adjacent Catalan numbers:

$$\begin{aligned} a_n &= c(n) + c(n+1) = \frac{1}{n+1}\binom{2n}{n} + \frac{1}{n+2}\binom{2n+2}{n+1} \\ &= \frac{(2n)!(5n+4)}{n!(n+2)!} \qquad (n = 0, 1, 2, \dots). \end{aligned}$$

This sequence starts as follows:

$$2, \quad 3, \quad 7, \quad 19, \quad 56, \quad 174\dots$$

In a comment stored with sequence A001906 Layman conjectured that the Hankel transformation of $\{a_n\}_{n\geq 0}$ equals the sequence A001906, i.e., the bisection of Fibonacci sequence. In this paper we shall prove a slight generalization of Layman's conjecture.

The generating function $G(x)$ for the sequence $\{a_n\}_{n\geq 0}$ is given by

$$G(x) = \sum_{n=0}^{\infty} a_n x^n = \frac{1}{x}\left(\frac{(1-\sqrt{1-4x})(1+x)}{2x} - 1\right) \tag{1}$$

It is known (for example, see Krattenthaler [4]) that the Hankel determinant h_n of order n of the sequence $\{a_n\}_{n\geq 0}$ equals

$$h_n = a_0^n \beta_1^{n-1} \beta_2^{n-2} \cdots \beta_{n-2}^2 \beta_{n-1}, \tag{2}$$

where $\{\beta_n\}_{n\geq 1}$ is the sequence given by:

$$G(x) = \sum_{n=0}^{\infty} a_n x^n = \cfrac{a_0}{1 + \alpha_0 x - \cfrac{\beta_1 x^2}{1 + \alpha_1 x - \cfrac{\beta_2 x^2}{1 + \alpha_2 x - \cdots}}} \tag{3}$$

The sequences $\{\alpha_n\}_{n\geq 0}$ and $\{\beta_n\}_{n\geq 1}$ are the coefficients in the recurrence relation

$$P_{n+1}(x) = (x-\alpha_n)P_n(x) - \beta_n P_{n-1}(x)$$

where $\{P_n(x)\}_{n\geq 0}$ is the monic polynomial sequence orthogonal with respect to the functional L determined by

$$L[x^n] = a_n \quad (n = 0, 1, 2, \ldots). \tag{4}$$

In the next section this functional is constructed and a theorem concerning the polynomials $\{P_n(x)\}_{n\geq 0}$ and the sequences $\{\alpha_n\}_{n\geq 0}$ and $\{\beta_n\}_{n\geq 1}$ is proved.

2. Main Theorem

We would like to express $L[f]$ in the form:

$$L[f(x)] = \int_R f(x) d\psi(x),$$

where $\psi(x)$ is a distribution, or, even more, to find the weight function $w(x)$ such that $w(x) = \psi'(x)$.

Denote by $F(z)$ the function

$$F(z) = \sum_{k=0}^{\infty} a_k z^{-k-1},$$

From the generating function (1), we have:

$$F(z) = z^{-1}\, G\big(z^{-1}\big) = \frac{1}{2}\left\{ z - 1 - (z+1)\sqrt{1-\frac{4}{z}} \right\}. \tag{5}$$

From the theory of distribution functions (see Chihara [1]), we have Stieltjes inversion function

$$\psi(t) - \psi(s) = -\frac{1}{\pi}\int_s^t \Im F(x+iy)dx. \tag{6}$$

Since $F(\bar{z}) = \overline{F(z)}$, it can be written in the form

$$\psi(t) - \psi(0) = -\frac{1}{2\pi i}\lim_{y\to 0^+}\int_0^t \Big[F(x+iy) - F(x-iy) \Big] dx. \tag{7}$$

Knowing that

$$\int_0^t F(x+a)dx = \frac{1}{4}\left\{ a^2\,\sqrt{1-\frac{4}{a}} - 2t + 2at + t^2 - (a+t)^2\sqrt{1-\frac{4}{a+t}} \right\}$$
$$-2\log\Big(-2+a+a\,\sqrt{1-\frac{4}{a}}\Big) + 2\log\Big(-2+a+t+(a+t)\,\sqrt{1-\frac{4}{a+t}}\Big),$$

we find the distribution function

$$\psi(t)=\begin{cases}\frac{1}{4\pi}\left\{t\ \sqrt{t(4-t)}-8\Big(\pi-\arctan\frac{\sqrt{(4-t)t}}{2-t}\Big)\right\}, & 0\le t<2;\\ \frac{1}{4\pi}\left\{t\ \sqrt{t(4-t)}-8\arctan\frac{\sqrt{(4-t)t}}{t-2}\right\}; & 2\le t\le 4.\end{cases}$$

After differentiation of $\psi(t)$ and simplification of the resulting expression, we finally have:

$$w(x)=\frac{1}{2}(x+1)\sqrt{\frac{4}{x}-1},\quad x\in(0,4). \tag{8}$$

In this way, we obtained the positive-definite L that satisfies (4) and proved that the corresponding orthogonal polynomial sequence exists. We have

Theorem 1. *The monic polynomial sequence $\{P_n(x)\}$ orthogonal with respect to the linear functional*

$$L(f):=\frac{1}{2\pi}\int_0^4 f(x)(x+1)\sqrt{\frac{4}{x}-1}dx, \tag{9}$$

satisfies the three-term recurrence relation

$$P_{n+1}(x)=(x-\alpha_n)P_n(x)-\beta_nP_{n-1}(x), \tag{10}$$

with

$$\alpha_n=2-\frac{1}{F_{2n+1}F_{2n+3}},\quad \beta_n=1+\frac{1}{F_{2n+1}^2},\qquad k\ge 0 \tag{11}$$

where F_i is the i-th Fibonacci number.

Example 1. The first members of this sequence are:

$$\begin{aligned}P_0(x)&=1;\\ P_1(x)&=x-\frac{3}{2};\\ P_2(x)&=x^2-\frac{17}{5}x+\frac{8}{5};\\ P_3(x)&=x^3-\frac{70}{13}x^2+\frac{95}{13}x-\frac{21}{13};\\ P_4(x)&=x^4-\frac{251}{34}x^3+\frac{290}{17}x^2-\frac{435}{34}x+\frac{55}{34}.\end{aligned}$$

Notice that $P_n(0)=(-1)^nF_{2n+2}/F_{2n+1}$.

Proof of Theorem 1. Denoting by $W_n(x)=P_n^{(1/2,-1/2)}(x)$ $(n\ge 0)$ a special Jacobi polynomial, which is also known as *the Chebyshev polynomial of the fourth kind*.

The sequence of these polynomials is orthogonal with respect to $p^{(1/2,-1/2)}(x)=(1-x)^{1/2}(1+x)^{-1/2}$ on the interval $(-1,1)$. These polynomials can be expressed

(Szegö [9]) by

$$W_n(\cos\theta) = \frac{\sin(n+\frac{1}{2})\theta}{2^n \sin\frac{1}{2}\theta}.$$

and satisfy the three-term recurrence relation (Chihara [1]):

$$\begin{gathered} W_{n+1}(x) = (x-\alpha_n^*)\, W_n(x) - \beta_n^* W_{n-1}(x) \quad (n=0,1,\ldots), \\ W_{-1}(x)=0, \quad W_0(x)=1, \end{gathered}$$

where

$$\alpha_0^* = -\frac{1}{2}, \quad \alpha_n^* = 0, \qquad \beta_0^* = \pi, \quad \beta_n^* = \frac{1}{4} \qquad (n \geq 1).$$

If we use the weight function $\hat{p}(t) = (t-c)p^{(1/2,-1/2)}(t)$, then the corresponding coefficients $\hat{\alpha}_n$ and $\hat{\beta}_n$ can be evaluated as follows (see, for example, Gautschi [2])

$$\hat{\alpha}_n = c - \frac{W_{n+1}(c)}{W_n(c)} - \beta_{n+1}^* \frac{W_n(c)}{W_{n+1}(c)}, \tag{12}$$

$$\hat{\beta}_n = \beta_n^* \frac{W_{n-1}(c) W_{n+1}(c)}{W_n^2(c)}, \quad n \in \mathbb{N}. \tag{13}$$

Here, we use $c = -3/2$ and $\hat{p}(x) = (x+3/2)(1-x)^{1/2}(1+x)^{-1/2}$.

If we write $\lambda_n = W_n(-3/2)$ then, using the three-term recurrence relation for $W_n(x)$, we have

$$4\lambda_{n+1} + 6\lambda_n + \lambda_{n-1} = 0,$$

with initial values $\lambda_0 = 1, \quad \lambda_1 = -1$.

So, we find

$$\lambda_n = W_n(-3/2) = \frac{(-1)^n}{2\sqrt{5}\ 4^n} \left\{ (\sqrt{5}+1)(3+\sqrt{5})^n + (\sqrt{5}-1)(3-\sqrt{5})^n \right\}.$$

Denoting by

$$\phi = \frac{1+\sqrt{5}}{2}, \qquad \overline{\phi} = \frac{1-\sqrt{5}}{2} \tag{14}$$

the golden section numbers, we can write:

$$\lambda_n = W_n(-3/2) = \frac{(-1)^n}{\sqrt{5}\ 2^n}(\phi^{2n+1} - \overline{\phi}^{2n+1}) = \frac{(-1)^n}{2^n} F_{2n+1}. \tag{15}$$

In order to simplify further algebraic manipulations we shall use

$$F_{2n-1}F_{2n+3} = F_{2n+1}^2 + 1 \tag{16}$$

This formula is a special case of the identity (Vajda [12]):

$$G(n+i)H(n+k) - G(n)H(n+i-k) = (-1)^n (G(i)H(k) - G(0)H(i+k)) \tag{17}$$

where G and H are sequences that satisfy the same recurrence relation as the Fibonacci numbers with possibly different initial conditions. However, we take both G and H to be the Fibonacci numbers and $n \to 2n+1$, $i=2, k=-2$.

Now

$$\begin{aligned}\hat{\beta}_n &= \frac{1}{4}\frac{\lambda_{n-1}\lambda_{n+1}}{\lambda_n^2} = \frac{1}{4}\frac{F_{2n-1}F_{2n+3}}{F_{2n+1}^2} \\ &= \frac{1}{4}\left\{1+\frac{1}{F_{2n+1}^2}\right\} \end{aligned} \tag{18}$$

and

$$\begin{aligned}\hat{\alpha}_n &= -\frac{3}{2} - \frac{\lambda_{n+1}}{\lambda_n} - \frac{1}{4}\frac{\lambda_n}{\lambda_{n+1}} \\ &= \frac{-3F_{2n+1}F_{2n+3} + F_{2n+3}^2 + F_{2n+1}^2}{2F_{2n+1}F_{2n+3}} \\ &= \frac{F_{2n+2}^2 - F_{2n+1}F_{2n+3}}{2F_{2n+1}F_{2n+3}} \\ &= -\frac{1}{2F_{2n+1}F_{2n+3}}.\end{aligned}$$

If a new weight function $p(x)$ is introduced by

$$p(x) = \hat{p}(ax+b)$$

then we have

$$\alpha_n = \frac{\hat{\alpha}_n - b}{a}, \qquad \beta_n = \frac{\hat{\beta}_n}{a^2} \qquad (n \geq 0).$$

Now, by using $x \mapsto x/2 - 1$, i.e., $a = 1/2$ and $b = -1$, we have the wanted weight function

$$w(x) = \hat{p}(\frac{x}{2} - 1) = \frac{1}{2}(x+1)\sqrt{\frac{4-x}{x}}.$$

Thus

$$\alpha_n = 2 - \frac{5}{(\phi^{2n+1} - \overline{\phi}^{2n+1})(\phi^{2n+3} - \overline{\phi}^{2n+3})} = 2 - \frac{1}{F_{2n+1}F_{2n+3}} \tag{19}$$

and

$$\beta_n = 1 + \frac{5}{(\phi^{2n+1} - \overline{\phi}^{2n+1})^2} = 1 + \frac{1}{F_{2n+1}^2} \tag{20}$$

finishing the proof of (1) .□

3. Layman's conjecture

By making use of (2) we have that:

$$h_n = a_0^n\left(1+\frac{1}{F_3^2}\right)^{n-1}\left(1+\frac{1}{F_5^2}\right)^{n-2}\cdots\left(1+\frac{1}{F_{2n-1}^2}\right) \tag{21}$$

Using (16) we can write (21) as:

$$h_n = a_0^n \left(\frac{F_1 F_5}{F_3^2}\right)^{n-1} \left(\frac{F_3 F_7}{F_5^2}\right)^{n-2} \left(\frac{F_5 F_9}{F_7^2}\right)^{n-3} \cdots \frac{F_{2n-3} F_{2n+1}}{F_{2n-1}^2} \tag{22}$$

Since $a_0 = 2 = F_3$ the corresponding factors cancel, therefore:

$$h_n = F_{2n+1} \qquad\qquad (n \geq 0),$$

thus proving that Hankel transform of A005807 equals A001519 -sequence of Fibonacci numbers with odd indices.

As we have mentioned in the introduction, Layman observed that the Hankel transform of A005807 equals A001906 -sequence of Fibonacci numbers with even indices. This sequence is obtained if we start the Hankel matrix from $a_1 = 3$, i.e., determinants will have a_1 on the position $(1, 1)$.

The proof of this fact is almost identical with the proof presented here, and so we do not include it. Notice that now we construct $L[x^n] = a_{n+1}$ and that $a_1 = 3 = F_4$; in (17) we take $n \to 2n$. We also use the easily provable fact $P_n(0) = (-1)^n F_{2n+2}/F_{2n+1}$ (see Example 1).

Finally we mention that, following Layman [5], it is known that the Hankel transform is invariant with the respect to the Binomial and Invert transform, so all the sequences obtained from A005807 using these two transformations have the Hankel transform shown here.

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(Mentions sequences A005807, A001906, A001519.)

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