CATALAN CONTINUED FRACTIONS AND INCREASING SUBSEQUENCES IN PERMUTATIONS

PETTER BRÄNDÉN, ANDERS CLAESSON, AND EINAR STEINGRÍMSSON

Abstract. We call a Stieltjes continued fraction with monic monomial numerators a Catalan continued fraction. Let $e_k(\pi)$ be the number of increasing subsequences of length $k+1$ in the permutation π. We prove that any Catalan continued fraction is the multivariate generating function of a family of statistics on the 132-avoiding permutations, each consisting of a (possibly infinite) linear combination of the e_ks. Moreover, there is an invertible linear transformation that translates between linear combinations of e_ks and the corresponding continued fractions.

Some applications are given, one of which relates fountains of coins to 132-avoiding permutations according to number of inversions. Another relates ballot numbers to such permutations according to number of right-to-left maxima.

1. Introduction and main results

We denote by $\mathcal{S}_n$ the set of permutation on $\{1, 2, \ldots, n\}$. Given $\pi = a_1 a_2 \cdots a_n$ in $\mathcal{S}_n$ and $\tau = b_1 b_2 \cdots b_k$ in $\mathcal{S}_k$, we say that π has j occurrences of the *pattern* τ if there are exactly j different sequences $1 \le i_1 < i_2 < \cdots < i_k \le n$ such that the numbers $a_{i_1} a_{i_2} \cdots a_{i_k}$ are in the same relative order as $b_1 b_2 \cdots b_k$. We use the symbol τ also for the permutation statistics defined by $\tau(\pi) = j$ if π has j occurrences of the pattern τ. If $\tau(\pi) = 0$ we say that π is *τ-avoiding*.

Everywhere in this paper a permutation on $S \subset \mathbb{N}$, with $|S| = n$, will be identified with the permutation in $\mathcal{S}_n$ whose letters are in the same relative order as the letters of the given permutation on S. As an example, the permutation 17358 on $\{1, 3, 5, 7, 8\}$ is identified with 14235 in $\mathcal{S}_5$.

Let $\mathcal{S}_n(132)$ be the set of 132-avoiding permutations of length n, and let $\mathcal{S}(132) = \bigcup_{n\ge 0} \mathcal{S}_n(132)$. Suppose $\pi = \pi_1 n \pi_2 \in \mathcal{S}_n(132)$. Then each letter in π_1 must be greater than any letter in π_2, where both π_1 and π_2 must necessarily be 132-avoiding. Conversely, every permutation of this form is clearly 132-avoiding. This observation immediately yields a functional relation for the generating function, $C(x)$, for the number of 132-avoiding permutations according to length, namely

$$C(x) = 1 + xC(x)^2. \tag{1}$$

Readers unfamiliar with the symbolic method implicitly used in this derivation may consult, for example, [3]. Solving for $C(x)$ in (1) we obtain

$$C(x) = \frac{1 - \sqrt{1 - 4x}}{2x},$$

which is the familiar generating function of the Catalan numbers, $C_n = \frac{1}{n+1}\binom{2n}{n}$. Thus we have derived the well known fact [5, p. 239] that the cardinality of $\mathcal{S}_n(132)$

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is the nth Catalan number. Rewriting (1) in the form

$$C(x) = \frac{1}{1 - xC(x)}$$

and iterating this identity we arrive at the formal continued fraction expansion

$$C(x) = \cfrac{1}{1 - \cfrac{x}{1 - \cfrac{x}{\ddots}}},$$

which is the simplest instance of the continued fractions studied in this paper.

A *Stieltjes continued fraction* is a continued fraction of the form

$$C = \cfrac{1}{1 - \cfrac{m_1}{1 - \cfrac{m_2}{\ddots}}},$$

where each m_i is a monomial in some set of variables. We define a *Catalan continued fraction* to be a Stieltjes continued fraction with monic monomial numerators.

For $k \geq 1$, we denote by e_{k-1} the pattern/statistic $1\,2 \cdots k$. Thus $e_0(\pi)$ is the length $|\pi|$ of π, and $e_1(\pi)$ counts the number of non-inversions in π. We also define $e_{-1}(\pi) = 1$ for all permutations π (that is, we declare all permutations to have exactly one increasing subsequence of length 0).

The main purpose of this paper is to show that any Catalan continued fraction is the multivariate generating function of a family of statistics, consisting of linear combinations of the e_ks. Moreover, there is an invertible linear transformation that translates between linear combinations of e_ks and the corresponding continued fractions.

A theorem of Robertson, Wilf and Zeilberger [12] gives a simple continued fraction that records the joint distribution of the patterns 12 and 123 on permutations avoiding the pattern 132.

Generalizations of this theorem have already been given, by Krattenthaler [6], by Mansour and Vainshtein [8] and by Jani and Rieper [4]. However, in none of these papers is there explicit mention of the *joint* distribution of the statistics under consideration. We now state this theorem; it is a generalization of [12, Theorem 1]. Moreover, this theorem is implicit in [8, Proposition 2.3] and it also follows, with minor changes, from the corresponding proofs in [4, Corollary 7] and [6, Theorem 1].

Theorem 1. *The following continued fraction expansion holds:*

$$\sum_{\pi \in \mathcal{S}(132)} \prod_{k \geq 0} x_k^{e_k(\pi)} = \cfrac{1}{1 - \cfrac{x_0^{\binom{0}{0}}}{1 - \cfrac{x_0^{\binom{1}{0}} x_1^{\binom{1}{1}}}{1 - \cfrac{x_0^{\binom{2}{0}} x_1^{\binom{2}{1}} x_2^{\binom{2}{2}}}{1 - \cfrac{x_0^{\binom{3}{0}} x_1^{\binom{3}{1}} x_2^{\binom{3}{2}} x_3^{\binom{3}{3}}}{\ddots}}}}}$$

in which the $(n+1)$st numerator is $\prod_{k=0}^{n} x_k^{\binom{n}{k}}$.

Proof. Let $\pi = \pi_1 n \pi_2 \in \mathcal{S}_n(132)$. Since every increasing subsequence of length $k+1$ is contained either in π_1, or in π_2, or may consist of a subsequence of length k in π_1 ending with the n in $\pi_1 n \pi_2$, we have

$$e_k(\pi) = e_k(\pi_1) + e_{k-1}(\pi_1) + e_k(\pi_2), \quad k \geq 0.$$

Let $\mathbf{x} = (x_0, x_1, \ldots)$, where the x_is are indeterminates, and let

$$w(\pi; \mathbf{x}) = \prod_{k \geq 0} x_k^{e_k(\pi)}.$$

Then $w(\pi; \mathbf{x}) = x_0 w(\pi_1; \mathbf{x}^*) w(\pi_2; \mathbf{x})$, where $\mathbf{x}^* = (x_0 x_1, x_1 x_2, \ldots)$. Consequently, the generating function

$$C(\mathbf{x}) = \sum_{\pi \in \mathcal{S}(132)} w(\pi, \mathbf{x})$$

satisfies

$$C(\mathbf{x}) = 1 + x_0 C(\mathbf{x}^*) C(\mathbf{x}),$$

or, equivalently,

$$C(\mathbf{x}) = \frac{1}{1 - x_0 C(\mathbf{x}^*)},$$

and the theorem follows by induction. $\square$

To state and prove our main theorem we need some definitions: Let

$$\mathcal{A} = \{A : \mathbb{N} \times \mathbb{N} \to \mathbb{Z} \mid \forall n \, (A_{nk} = 0 \text{ for all but finitely many } k)\},$$

be the ring of all infinite matrices with a finite number of non zero entries in each row, with multiplication defined by $(AB)_{nk} = \sum_{i=0}^{\infty} A_{ni} B_{ik}$.

With each $A \in \mathcal{A}$ we now associate a family of statistics $\{\langle \mathbf{e}, A_k \rangle\}_{k \geq 0}$, defined on $\mathcal{S}(132)$, where $\mathbf{e} = (e_0, e_1, \ldots)$, A_k is the kth column of A, and

$$\langle \mathbf{e}, A_k \rangle = \sum_i A_{ik} e_i.$$

Let $\mathbf{q} = (q_0, q_1, \ldots)$, where the q_is are indeterminates. For each $A \in \mathcal{A}$ and $\pi \in \mathcal{S}(132)$ we define:

(1) the *weight* $\mu(\pi, A; \mathbf{q})$ of π with respect to A, by

$$\mu(\pi, A; \mathbf{q}) = \prod_{k \geq 0} q_k^{\langle \mathbf{e}, A_k \rangle (\pi)},$$

(2) the multivariate generating function, associated with A, of the statistics $\{\langle \mathbf{e}, A_k \rangle\}_{k \geq 0}$, by

$$F_A(\mathbf{q}) = \sum_{\pi \in \mathcal{S}(132)} \mu(\pi, A; \mathbf{q}),$$

(3) the Catalan continued fraction associated with A, by

$$C_A(\mathbf{q}) = \cfrac{1}{1 - \cfrac{\prod q_k^{A_{0k}}}{1 - \cfrac{\prod q_k^{A_{1k}}}{1 - \cfrac{\prod q_k^{A_{2k}}}{1 - \cfrac{\prod q_k^{A_{3k}}}{\ddots}}}}}.$$

Note that the product in part 1 above is finite by the definition of $\mathcal{A}$ together with the fact that $e_i(\pi) = 0$ whenever $i > |\pi|$.

In what follows we will use the convention that $\binom{n}{k} = 0$ whenever $n < k$ or $k < 0$.

Theorem 2. *Let $A \in \mathcal{A}$. Then*

$$F_A(\mathbf{q}) = C_{BA}(\mathbf{q}),$$

where $B = [\binom{i}{j}]$, and conversely

$$C_A(\mathbf{q}) = F_{B^{-1}A}(\mathbf{q}).$$

In particular, all Catalan continued fractions are generating functions of statistics on $\mathcal{S}(132)$ consisting of (possibly infinite) linear combinations of e_ks.

Proof. We have

$$\begin{aligned}
\mu(\pi, A; \mathbf{q}) &= \prod_{k\geq 0} q_k^{\langle \mathbf{e}, A_k\rangle(\pi)} \\
&= \prod_{k\geq 0}\prod_{j\geq 0} q_k^{A_{jk}e_j(\pi)} \\
&= \prod_{j\geq 0}\left(\prod_{k\geq 0} q_k^{A_{jk}}\right)^{e_j(\pi)}.
\end{aligned}$$

Let $x_j = \prod_{k\geq 0} q_k^{A_{jk}}$. Applying Theorem 1 we get a continued fraction in which the $(n+1)$st numerator is

$$\prod_{j\geq 0} x_j^{\binom{n}{j}} = \prod_{j\geq 0}\left(\prod_{k\geq 0} q_k^{A_{jk}}\right)^{\binom{n}{j}} = \prod_{k\geq 0} q_k^{\langle(\binom{n}{0},\binom{n}{1},\binom{n}{2},\ldots),A_k\rangle},$$

which is the $(n+1)$st numerator in $C_{BA}(\mathbf{q})$. Hence

$$F_A(\mathbf{q}) = C_{BA}(\mathbf{q}).$$

Observing that $B^{-1} = [(-1)^{i-j}\binom{i}{j}] \in \mathcal{A}$ we also get

$$C_A(\mathbf{q}) = F_{B^{-1}A}(\mathbf{q}).$$

□

Corollary 3. *If $f = \sum_{k\geq 0} \lambda_k e_k$ with $\lambda_k \in \mathbb{Z}$, then the generating function for the statistic f over $\mathcal{S}(132)$ admits the Catalan continued fraction expansion*

$$\sum_{\pi\in\mathcal{S}(132)} x^{f(\pi)} t^{|\pi|} = \cfrac{1}{1 - \cfrac{x^{f(e_0)}t}{1 - \cfrac{x^{f(e_1)-f(e_0)}t}{1 - \cfrac{x^{f(e_2)-f(e_1)}t}{\ddots}}}}.$$

where in the continued fraction e_{k-1} is the permutation $12\cdots k$.

Proof. The result follows from Theorem 2 and the observation

$$\begin{aligned}
f(e_n) - f(e_{n-1}) &= \sum_k \lambda_k \Big(e_k(e_n) - e_k(e_{n-1})\Big) \\
&= \sum_k \lambda_k \left(\binom{n+1}{k+1} - \binom{n}{k+1}\right) \\
&= \sum_k \lambda_k \binom{n}{k}.
\end{aligned}$$

□

2. Dyck paths

Before giving applications of Theorem 2 we review some theory on Dyck paths and their relation to 132-avoiding permutations.

A *Dyck path* of length $2n$ is a path in the integral plane from $(0,0)$ to $(2n,0)$, consisting of steps of type $u=(1,1)$ and $d=(1,-1)$ and never going below the x-axis. We call the steps of type u *up-steps* and those of type d we call *down-steps*. The *height* of a step in a Dyck path is the height above the x-axis of its left point.

A nonempty Dyck path w can be written uniquely as uw_1dw_2 where w_1 and w_2 are Dyck paths. This decomposition is called the *first return decomposition* of w, because the d in uw_1dw_2 corresponds to the first place, after $(0,0)$, where the path touches the x-axis.

In [6] a bijection Φ between $\mathcal{S}_n(132)$ and the set of Dyck paths of length $2n$ is studied. This bijection, as a function defined on $\mathcal{S}(132)$, can also be defined recursively by

$$\Phi(\varepsilon)=\varepsilon \quad\text{ and }\quad \Phi(\pi)=u\Phi(\pi_1)d\Phi(\pi_2),$$

where $\pi=\pi_1 n\pi_2\in\mathcal{S}_n(132)$ and ε is the empty permutation/Dyck path. For example, letting Φ operate on the permutation 453612 we successively obtain

$$453612 \;\rightarrow\; u453d12 \;\rightarrow\; uu4d3du1d \;\rightarrow\; uuudduddudd.$$

In what follows, when we talk about a correspondence between a Dyck path and a 132-avoiding permutation, we will always mean the correspondence defined by Φ.

Using Φ we can express $e_k(\pi)$ in terms of the Dyck path corresponding to π. Namely (see [6]),

$$e_k(\pi)=\sum_{d\ in\ \Phi(\pi)}\binom{h(d)-1}{k}, \tag{2}$$

where the sum is over all down-steps d in $\Phi(\pi)$ and $h(d)$ is the height of the left point of d. This can also be shown by induction over the length of π. Indeed, for a nonempty 132-avoiding permutation $\pi=\pi_1 n\pi_2$, we have

$$e_k(\pi)=e_k(\pi_1)+e_{k-1}(\pi_1)+e_k(\pi_2).$$

On the other hand, defining $f_k(w)=\sum_{d\ in\ w}\binom{h(d)-1}{k}$ for $w=uw_1dw_2$ we have

$$\begin{aligned} f_k(w) &= \sum_{d\ in\ w}\binom{h(d)-1}{k}\\ &= \sum_{d\ in\ w_1}\binom{h(d)}{k}+\sum_{d\ in\ w_2}\binom{h(d)-1}{k}\\ &= \sum_{d\ in\ w_1}\binom{h(d)-1}{k}+\sum_{d\ in\ w_1}\binom{h(d)-1}{k-1}+f_k(w_2)\\ &= f_k(w_1)+f_{k-1}(w_1)+f_k(w_2). \end{aligned}$$

Since $e_k(\varepsilon)=f_k(\varepsilon)$, it follows by induction over the length of π that $f_k(\Phi(\pi))=e_k(\pi)$, which is the same as (2).

3. Applications

We now give some applications of Theorem 2. Some of these relate known continued fractions to the statistics e_k, whereas others relate these statistics to various other combinatorial structures.

3.1. **A continued fraction of Ramanujan.** The continued fraction

$$R(q,t) = \cfrac{1}{1 - \cfrac{qt}{1 - \cfrac{q^3t}{1 - \cfrac{q^5t}{1 - \cfrac{q^7t}{\ddots}}}}}$$

was studied by Ramanujan (see [10, p. 126]). It was shown in [2] that the coefficient to t^nq^k in the expansion of $R(q,t)$ is the number of Dyck paths of length $2n$ and area k. Using the converse part of Theorem 2, we would like to find the linear combinations of the statistics e_ks that have as bivariate generating function the continued fraction $R(q,t)$. Comparing $R(q,t)$ with the $C_A(\mathbf{q})$ defined just before Theorem 2, we have

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots \\ 3 & 1 & 0 & 0 & \cdots \\ 5 & 1 & 0 & 0 & \cdots \\ 7 & 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Since

$$\sum_{k\geq 0}(2k+1)(-1)^{n-k}\binom{n}{k} = \delta_{n0} + 2\delta_{n1},$$

where δ_{ij} is the Kronecker delta, we get

$$B^{-1}A = \begin{pmatrix} 1 & 1 & 0 & \cdots \\ 2 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and hence, recalling that the coefficient of the linear combinations of the statistics e_k are the columns of this matrix, we have

$$R(q,t) = \sum_{\pi\in\mathcal{S}(132)} q^{e_0(\pi)+2e_1(\pi)}t^{|\pi|},$$

where we prefer to use two different notations $e_0(\pi)$ and $|\pi|$ for the length of π. Thus $R(q,t)$ records the statistic $e_0 + 2e_1$ on 132-avoiding permutations. In fact, the bijection Φ translates the statistic $e_0 + 2e_1$ into the sum of the heights of the steps in the corresponding Dyck path, which in turn is easily seen to equal area.

3.2. **Fountains of coins.** A *fountain of coins* is an arrangement of coins in rows such that the bottom row is full (that is, there are no "holes"), and such that each coin in a higher row rests on two coins in the row below (see Figure 1). Let $F(x,t) = \sum_{n,k} f(n,k)x^kt^n$, where $f(n,k)$ counts the number of fountains with n

coins in the bottom row and k coins in total. In [9] it is shown that

$$F(x,t) = \cfrac{1}{1 - \cfrac{xt}{1 - \cfrac{x^2t}{1 - \cfrac{x^3t}{1 - \cfrac{x^4t}{\ddots}}}}}$$

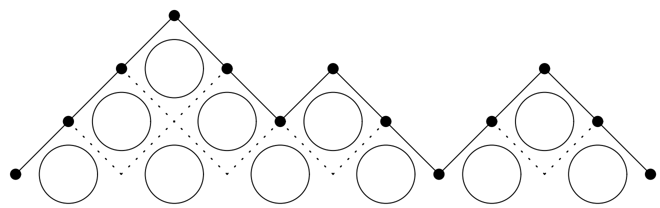
A straightforward application of Theorem 2 gives the following result.

Proposition 4. *The number $f(n,k)$ equals the number of permutations $\pi \in \mathcal{S}_n(132)$ with $(e_0 + e_1)\pi = k$. Equivalently, $f(n,k)$ equals the number of permutations in $\mathcal{S}_n(132)$ with $k - n$ non-inversions.*

If we reverse each permutation in $\mathcal{S}_n(132)$ we see that $f(n,k)$ also equals the number of 231-avoiding permutations in $\mathcal{S}_n$ with exactly $k - n$ inversions.

We also give a combinatorial proof of Proposition 4, by constructing a bijection between the set of Dyck paths of length $2n$ and the set of fountains with n coins in the bottom row. Let Ψ be the bijection that maps a Dyck path to the fountain obtained by placing coins at the centre of all lattice squares inside the path, in the way that Figure 1 suggests.

Figure 1. A fountain of coins and the corresponding Dyck path.



The ith *slant line* in a fountain is the sequence of coins starting with the ith coin from the left in the bottom row and continuing in the northeast direction. The height of a down-step thus corresponds to the number of coins in the slant line ending at the left point of the down-step d. Now, e_0 counts the number of coins in the bottom row and $\binom{h(d)-1}{1}$ is one less than the number of coins in the corresponding slant line (see the end of Section 2). Thus $e_0 + e_1$ counts the total number of coins in the fountain.

3.3. **Increasing subsequences.** The total number of increasing subsequences in a permutation is counted by $e_0 + e_1 + \cdots$. An application of Theorem 2 gives the following continued fraction for the distribution of $e_0 + e_1 + \cdots$:

$$\sum_{\pi \in \mathcal{S}(132)} x^{e_0\pi + e_1\pi + \cdots} t^{|\pi|} = \cfrac{1}{1 - \cfrac{xt}{1 - \cfrac{x^2t}{1 - \cfrac{x^4t}{1 - \cfrac{x^8t}{\ddots}}}}}$$

3.4. **Right-to-left maxima and ballot numbers.** We say that an increasing subsequence $\pi(i_1)\pi(i_2)\cdots\pi(i_k)$ of $\pi \in \mathcal{S}_n$ is *right maximal* if $\pi(i_k) < \pi(j)$ implies $j < i_k$ (so that the sequence can not be extended to the right).

Proposition 5. *Let $\pi \in \mathcal{S}_n(132)$ and let $m_k(\pi)$ be the number of right maximal increasing subsequences of π of length $k+1$. Then*

$$m_k(\pi) = e_k(\pi) - e_{k+1}(\pi) + e_{k+2}(\pi) - \cdots .$$

In particular, the number of right-to-left maxima in π equals

$$e_0(\pi) - e_1(\pi) + e_2(\pi) - e_3(\pi) + \cdots .$$

Proof. It suffices to prove that for all $\pi \in \mathcal{S}(132)$ and $k \geq 0$ we have $m_k(\pi) + m_{k+1}(\pi) = e_k(\pi)$. The statistic e_k counts all increasing sequences of length $k+1$ in π. If such a sequence is right maximal, it is counted by m_{k+1}. It therefore suffices to show that every increasing subsequence of length k that is not right maximal can be associated to a unique right maximal subsequence of length $k+1$, and conversely.

If an increasing subsequence of length k is not right maximal, it can be extended to a right maximal one of length $k+1$ and we show that this can only be done in one way. Suppose x is the last letter of the original sequence and that the sequence can be extended to a right maximal one by adjoining either y or z, where y comes before z in π. Then y must be greater than z, so x, y, z form a 132-sequence which is contrary to the assumption that π is 132-avoiding.

Conversely, deleting the last letter in a right maximal sequence of length $k+1$ clearly gives a non-right maximal sequence of length k. ☐

Define

$$M_k(x,t) = \sum_{\pi \in \mathcal{S}(132)} x^{m_k(\pi)} t^{|\pi|}.$$

To apply Corollary 3 we note that

$$m_k(e_n) - m_k(e_{n-1}) = \binom{n}{k} - \binom{n}{k+1} + \binom{n}{k+2} - \cdots = \binom{n-1}{k-1},$$

so the $(n+1)$st numerator in the Catalan continued fraction expansion of $M_k(x,t)$ is $tx^{\binom{n-1}{k-1}}$. Define

$$E_k(x,t) = \sum_{\pi \in \mathcal{S}(132)} x^{e_k(\pi)} t^{|\pi|}.$$

Since $\binom{n-1}{-1}$ is naturally defined to be δ_{n0}, Theorem 2 yields, for all $k \geq -1$, that $E_k(x,t)$ is the continued fraction with $(n+1)$st numerator $tx^{\binom{n}{k}}$. This leads to the following observation.

Proposition 6. *For all $k \geq 0$ we have*

$$M_k(x,t) = \frac{1}{1 - tE_{k-1}(x,t)}.$$

The *ballot number* $b(n,k)$ is the number of paths from $(0,0)$ to $(n+k, n-k)$ that do not go below the x-axis. It is well known that the ballot number $b(n,k)$ is equal to $\frac{n+1-k}{n+1}\binom{n+k}{n}$. Define $B(x,t) = \sum_{n,k} b(n,k) x^k t^n$. Then (see [11, p 152])

$$B(x,t) = \frac{C(xt)}{1 - tC(xt)},$$

where $C(x)$ is the generating function for the Catalan numbers.

Proposition 7. *The number of permutations in $\mathcal{S}_n(132)$ with k right-to-left maxima equals the ballot number*

$$b(n-1, n-k) = \frac{k}{2n-k}\binom{2n-k}{n},$$

and

$$b(n-1, k) = \frac{n-k}{n+k}\binom{n+k}{k}$$

counts the number of permutations of length n with k right maximal increasing subsequences of length two.

Proof. By Proposition 6,

$$M_0(x,t) = \frac{1}{1-xtC(t)}$$

records the distribution of right-to-left maxima. Since

$$B(x^{-1}, xt) = \frac{C(t)}{1-xtC(t)}$$

we have

$$M_0(x,t) = 1 + xtB(x^{-1}, xt) = 1 + \sum_{n,k} b(n-1, n-k)x^k t^n,$$

and the first assertion follows. For the second assertion, observe that by Proposition 6,

$$M_1(x,t) = \frac{1}{1-tC(xt)}.$$

Furthermore,

$$M_1(x,t) = M_0(x^{-1}, xt) = 1 + tB(x,t),$$

which concludes the proof. □

The first assertion of Proposition 7 can be proved bijectively using the map Φ in Section 2. In fact, the number of right-to-left maxima of π is equal to the *number of returns* in $\Phi(\pi)$, that is, the number of times the path $\Phi(\pi)$ intersects the x-axis. This number is known to have a distribution given by $b(n-1, n-k)$ (see [1]).

3.5. **Narayana numbers.** The generating function $N(x,t) = \sum_{n,k} N(n,k)x^k t^n$ for the Narayana numbers $N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k+1}$ satisfies the functional equation (see for example [13])

$$N(x,t) = 1 + xtN^2(x,t) - xtN(x,t) + tN(x,t).$$

Equivalently,

$$N(x,t) = \cfrac{1}{1 - \cfrac{t}{1 - xtN(x,t)}}.$$

This allows us to express $N(x,t)$ as a continued fraction:

$$N(x,t) = \cfrac{1}{1 - \cfrac{t}{1 - \cfrac{tx}{1 - \cfrac{t}{1 - \cfrac{tx}{\ddots}}}}}.$$

Proposition 8. *The statistic $s = e_1 - 2e_2 + 4e_3 - \cdots$ has the Narayana distribution over $\mathcal{S}(132)$, that is,*

$$\sum_{\pi \in \mathcal{S}(132)} x^{s(\pi)} t^{|\pi|} = \sum_{n,k} N(n,k) x^k t^n.$$

Proof. This follows immediately from Theorem 2 and the identity

$$\sum_{k \ odd} (-1)^{n-k} \binom{n}{k} = (-2)^{n-1}, \text{ for } n > 0.$$

□

Now

$$\sum_{k \geq 1} (-2)^{k-1} f_k(w) = \sum_{k \geq 1} \sum_{d \ in \ w} (-2)^{k-1} \binom{h(d)-1}{k} = \sum_{d \ in \ w} \frac{1 + (-1)^{h(d)}}{2}$$

so the interpretation of $e_1 - 2e_2 + 4e_3 - \cdots$ in terms of Dyck paths is the number of down-steps starting at even height, whose distribution is known [7] to be given by the Narayana numbers.

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Matematik, Chalmers tekniska högskola och Göteborgs universitet, S-412 96 Göteborg, Sweden

E-mail address: `branden@math.chalmers.se`, `claesson@math.chalmers.se`, `einar@math.chalmers.se`