Journal of Integer Sequences, Vol. 15 (2012),
Article 12.1.7

Series with Central Binomial Coefficients, Catalan Numbers, and Harmonic Numbers

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Abstract

We present several generating functions for sequences involving the central binomial coefficients and the harmonic numbers. In particular, we obtain the generating functions for the sequences $\binom{2n}{n}H_n$, $\binom{2n}{n}\frac{1}{n}H_n$, $\binom{2n}{n}\frac{1}{n+1}H_n$, and $\binom{2n}{n}n^m$. The technique is based on a special Euler-type series transformation formula.

1 Introduction and main results

The central binomial coefficients are defined by

$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2}$$

$(n = 0, 1, \ldots,)$ and are closely related to the Catalan numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

Many facts about these coefficients and Catalan numbers can be found in the recent book of Koshy [9]. Henry Gould has collected numerous identities involving central binomial coefficients in [5] and a large list of references on Catalan numbers in [6]. Riordan's book [13] is also a good reference.

Our focus here will be on power series involving these numbers. Several interesting powers series with central binomial coefficients were obtained and discussed by Lehmer [10] (see also some corrections by Mathar [11]). Other examples were given by Weinzierl [14] and Zucker [15]. Hansen's Table [8] contains such series too, for instance, entries (5.9.23), (5.18.9), (5.24.15), (5.24.30), (5.25.7), (5.27.9), (5.27.12), (5.27.17).

The generating functions of the numbers $\binom{2n}{n}$ and C_n are well-known, [8, (5.24.15)], [10],

$$\sum_{n=0}^{\infty} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}}, \tag{1}$$

$$\sum_{n=0}^{\infty} C_n\, x^n = \frac{2}{1+\sqrt{1-4x}}. \tag{2}$$

For both series we need $|4x| < 1$. The series (2) follows easily from (1) by integration. In this note we present a method of generating power series involving central binomial coefficients by using appropriate binomial transforms. Our results include several interesting power series where the coefficients are products of central binomial coefficients and harmonic numbers H_n, and also products of Catalan numbers and harmonic numbers. As usual,

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n},$$

for $n \geq 1$ and $H_0 = 0$.

Theorem 1. *For every $|x| < \frac{1}{4}$,*

$$\sum_{n=0}^{\infty} \binom{2n}{n} H_n (-1)^{n-1} x^n = \frac{2}{\sqrt{1+4x}} \log \frac{2\sqrt{1+4x}}{1+\sqrt{1+4x}} \tag{3}$$

or,

$$\sum_{n=0}^{\infty} \binom{2n}{n} H_n x^n = \frac{2}{\sqrt{1-4x}} \log \frac{1+\sqrt{1-4x}}{2\sqrt{1-4x}} \tag{4}$$

where the first series, (3), converges also for $x = \frac{1}{4}$.

With $x = \frac{1}{4}$ in (3) we find

$$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{(-1)^{n-1}}{4^n} H_n = \sqrt{2} \log \frac{2\sqrt{2}}{1+\sqrt{2}}.$$

With $x = \frac{1}{8}$ in (4) we have

$$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{H_n}{8^n} = 2\sqrt{2} \log \frac{1+\sqrt{2}}{2}.$$

Integrating the power series (4) (using the substitution $1 - 4x = y^2$ for the RHS) we obtain the following corollary involving the Catalan numbers.

Corollary 2. *For every* $|x| \le \frac{1}{4}$,

$$\sum_{n=0}^{\infty} C_n H_n x^{n+1} = \sqrt{1-4x}\log(2\sqrt{1-4x}) - (1+\sqrt{1-4x})\log(1+\sqrt{1-4x}) + \log 2 \qquad (5)$$

In particular, with $x = \frac{1}{4}$,

$$\sum_{n=0}^{\infty} \frac{C_n H_n}{4^n} = 4\log 2,$$

and when $x = \frac{-1}{4}$,

$$\sum_{n=0}^{\infty} (-1)^{n-1} \frac{C_n H_n}{4^{n+1}} = (1 + \frac{3}{2}\sqrt{2})\log 2 - (1+\sqrt{2})\log(1+\sqrt{2}).$$

Some numerical series involving central binomial coefficients and harmonic numbers have been computed by a different method in the papers [1, 3, 4]. Two connections of our results with certain series in [1] are specified in Section 3.

Next we turn to series with coefficients of the form $\binom{2n}{n} n^m$. Applying the operator $\left(x\frac{d}{dx}\right)$ to (1) Lehmer [10] computed

$$\sum_{n=0}^{\infty} \binom{2n}{n} n x^n = \frac{2x}{(1-4x)\sqrt{1-4x}}, \qquad (6)$$

and repeating the procedure,

$$\sum_{n=0}^{\infty} \binom{2n}{n} n^2 x^n = \frac{2x(2x+1)}{(1-4x)^2\sqrt{1-4x}}. \qquad (7)$$

Continuing like this is unpleasant, but fortunately a general formula can be obtained by a different method.

Theorem 3. *For every positive integer* m *and every* $|x| < \frac{1}{4}$ *we have*

$$\sum_{n=0}^{\infty} \binom{2n}{n} n^m x^n = \frac{1}{\sqrt{1-4x}} \sum_{k=0}^{m} S(m,k) \binom{2k}{k} k! \left(\frac{x}{1-4x}\right)^k, \qquad (8)$$

where $S(m,k)$ *are the Stirling numbers of the second kind.*

Remarkably, the central binomial coefficients appear on both sides of this equation!

Information about $S(m,n)$ can be found in the classical book [7]. When $m = 0$, $S(0,0) = 1$ and (8) turns into (1). When $m = 1$, $S(1,0) = 0, S(1,1) = 1$ and the RHS in (8) becomes

$$\frac{1}{\sqrt{1-4x}} \binom{2}{1} \left(\frac{x}{1-4x}\right) = \frac{2x}{(1-4x)\sqrt{1-4x}},$$

which is the RHS in (6).

When $m = 2$, $S(2,0) = 0,\ S(2,1) = S(2,2) = 1$ and the RHS in (8) becomes

$$\frac{1}{\sqrt{1-4x}}\binom{2}{1}\left(\frac{x}{1-4x}\right) + \ 2\binom{4}{2}\left(\frac{x}{1-4x}\right)^2 = \frac{4x^2+2x}{(1-4x)^2\sqrt{1-4x}}.$$

Thus (8) turns into (7). When $m = 3$, $S(3,0) = 0,\ \ S(3,1) = S(3,3) = 1,\ S(3,2) = 3$, and simple computation gives

$$\sum_{n=0}^{\infty}\binom{2n}{n}n^3x^n = \frac{2x(4x^2+10x+1)}{(1-4x)^3\sqrt{1-4x}}.$$

From (8) we have the immediate corollary:

Corollary 4. *Let $P_q(z) = a_q z^q + a_{q-1}z^{q-1} + \ldots + a_0$ be a polynomial. Then*

$$\sum_{n=0}^{\infty}\binom{2n}{n}P_q(n)x^n = \frac{1}{\sqrt{1-4x}}\sum_{m=0}^{q}a_m\sum_{k=0}^{m}S(m,k)\binom{2k}{k}k!\left(\frac{x}{1-4x}\right)^k$$

$$= \frac{1}{\sqrt{1-4x}}\sum_{k=0}^{q}\binom{2k}{k}k!\left(\frac{x}{1-4x}\right)^k\sum_{m=k}^{q}a_m S(m,k)$$

The proofs of these theorems are given in Section 2. In Section 3 we present some more corollaries and some new series, including the generating function for the sequence $\binom{2n}{n}\frac{1}{n}H_n$.

2 Proofs of the theorems

Let

$$f(z) = \sum_{k=0}^{\infty}a_k z^k \tag{9}$$

be a function analytical in a neighborhood of the origin. The proof of Theorem 1 is based on the following Euler-type series transformation formula .

Proposition 5. *For any complex number α and $|z|$ small enough to provide convergence we have*

$$\sum_{n=0}^{\infty}\binom{\alpha}{n}(-1)^n a_n z^n = (z+1)^{\alpha}\sum_{n=0}^{\infty}\left(\frac{z}{z+1}\right)^n\binom{\alpha}{n}(-1)^n\left\{\sum_{k=0}^{n}\binom{n}{k}a_k\right\}. \tag{10}$$

(cf. [12, p. 294, (1.20)]). The proof is given in the Appendix. This formula and several other series transformation formulas of the same type can be found in [2].

Proof. Setting $\alpha = \frac{-1}{2}$ in (10) and using the simple fact that $\binom{-1/2}{n}(-1)^n = \frac{1}{4^n}\binom{2n}{n}$ we obtain

$$\sum_{n=0}^{\infty}\binom{2n}{n}a_n\frac{z^n}{4^n} = \frac{1}{\sqrt{1+z}}\sum_{n=0}^{\infty}\left(\frac{z}{z+1}\right)^n\frac{1}{4^n}\binom{2n}{n}\left\{\sum_{k=0}^{n}\binom{n}{k}a_k\right\}.$$

With $z = 4x$ this becomes

$$\sum_{n=0}^{\infty} \binom{2n}{n} a_n x^n = \frac{1}{\sqrt{1+4x}} \sum_{n=0}^{\infty} \left(\frac{x}{1+4x} \right)^n \binom{2n}{n} \left\{ \sum_{k=0}^{n} \binom{n}{k} a_k \right\}. \tag{11}$$

Now we set $a_k = (-1)^{k-1} H_k$ and use the well-known binomial formula

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{k-1} H_k = \frac{1}{n}$$

$(n = 1, 2, \ldots)$ to obtain from (11)

$$\sum_{n=0}^{\infty} \binom{2n}{n} (-1)^{n-1} H_n x^n = \frac{1}{\sqrt{1+4x}} \sum_{n=1}^{\infty} \left(\frac{x}{1+4x} \right)^n \binom{2n}{n} \frac{1}{n}. \tag{12}$$

Next we turn to the representation (see, for instance [9, p. 87] or [10, (6)]).

$$\sum_{n=1}^{\infty} \binom{2n}{n} \frac{1}{n} z^n = 2 \log \left(\frac{1 - \sqrt{1-4z}}{2z} \right) = 2 \log \left(\frac{2}{1 + \sqrt{1-4z}} \right). \tag{13}$$

Using this in the RHS of (12) with $z = \frac{x}{1+4x}$ we obtain (6). Thus Theorem 1 is proved. $\square$

We can state now the following observation.

Method for obtaining new generating functions from old ones. *Suppose we know the function*

$$g(z) = \sum_{n=0}^{\infty} \binom{2n}{n} b_n z^n \tag{14}$$

in explicit compact form and suppose also that we can compute the sequence $a_k,\ k = 0, 1, \ldots,$ *explicitly from*

$$a_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} b_k, n = 0, 1, \ldots, \tag{15}$$

(binomial transform). Then we have from (11) *the new generating function*

$$f(x) = \sum_{n=0}^{\infty} \binom{2n}{n} a_n t^n = \frac{1}{\sqrt{1+4t}}\, g\left(\frac{t}{1+4\,t} \right). \tag{16}$$

Note that (15) can be inverted in the following manner (see [2, 13]),

$$b_n = \sum_{k=0}^{n} \binom{n}{k} a_k. \tag{17}$$

Using this method we shall prove also Theorem 3. The theorem follows from the well-known binomial transform

$$n! S(m, n) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} k^m \tag{18}$$

for any non-negative integer m. This is, in fact, the classical representation of the Stirling numbers of the second kind (see [7]). The inverse of (18) is

$$\sum_{k=0}^{n} \binom{n}{k} k! S(m,k) = n^m.$$

We set now $a_k = k!S(m,k),\ b_k = k^m,\ k = 0, 1, \ldots,$ and apply the above method. Note that the sequence a_k is finite, as $S(m,k) = 0$ when $k > m$. From (16),

$$\sum_{n=0}^{m} \binom{2n}{n} n! S(m,n) t^n = \frac{1}{\sqrt{1+4t}} \sum_{n=0}^{\infty} \left(\frac{t}{1+4t}\right)^n \binom{2n}{n} n^m \tag{19}$$

which after the substitution

$$\frac{t}{1+4t} = x$$

becomes (8).

3 Several more series

In this section we present some more power series related to those above. We start with the generating function for the Catalan numbers.

$$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{x^n}{n+1} = \frac{2}{1+\sqrt{1-4x}}. \tag{20}$$

Integrating both sides yields the representation (substitution $1 - 4x = y^2$ for the RHS)

$$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{x^{n+1}}{(n+1)^2} = x + \frac{1}{4}\left(1+\sqrt{1-4x}\right)^2 - \frac{3}{2}\left(1+\sqrt{1-4x}\right) + \log\left(1+\sqrt{1-4x}\right) + 2 - \log 2.$$

or, starting from $n = 1$,

$$\sum_{n=1}^{\infty} \binom{2n}{n} \frac{x^{n+1}}{(n+1)^2} = \frac{1}{4}\left(1+\sqrt{1-4x}\right)^2 - \frac{3}{2}\left(1+\sqrt{1-4x}\right) + \log\left(1+\sqrt{1-4x}\right) + 2 - \log 2. \tag{21}$$

Also, we can write (20) in the form (starting the summation from $n = 1$)

$$\sum_{n=1}^{\infty} \binom{2n}{n} \frac{x^n}{n+1} = \frac{2}{1+\sqrt{1-4x}} - 1 = \frac{4x}{\left(1+\sqrt{1-4x}\right)^2}$$

and subtracting this from (13) we obtain (cf. [11])

$$\sum_{n=1}^{\infty} \binom{2n}{n} \frac{x^n}{n(n+1)} = 1 - \frac{2}{1+\sqrt{1-4x}} + 2 \log \frac{2}{1+\sqrt{1-4x}}. \tag{22}$$

Next, since

$$H_{n+1} = H_n + \frac{1}{n+1}, \quad \frac{H_{n+1}}{n+1} = \frac{H_n}{n+1} + \frac{1}{(n+1)^2},$$

we can write

$$\sum_{n=1}^{\infty} \binom{2n}{n} \frac{H_{n+1}}{n+1} x^{n+1} = \sum_{n=1}^{\infty} \binom{2n}{n} \frac{H_n}{n+1} x^{n+1} + \sum_{n=1}^{\infty} \binom{2n}{n} \frac{1}{(n+1)^2} x^{n+1}$$

and from (5) and (21) we obtain after adding and simplifying

$$\sum_{n=1}^{\infty} \binom{2n}{n} \frac{H_{n+1}}{n+1} x^{n+1} = 1 - x + \sqrt{1-4x} \left(\log \frac{2\sqrt{1-4x}}{1+\sqrt{1-4x}} - 1 \right), \tag{23}$$

or, by adding x to both sides and starting the summation from $n = 0$,

$$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{H_{n+1}}{n+1} x^{n+1} = 1 + \sqrt{1-4x} \left(\log \frac{2\sqrt{1-4x}}{1+\sqrt{1-4x}} - 1 \right).$$

With $x = \frac{1}{4}$ we have from (23),

$$\sum_{n=1}^{\infty} \binom{2n}{n} \frac{H_{n+1}}{4^n (n+1)} = 3$$

which confirms [1, (3.23)]. With $x = \frac{-1}{4}$ in (23),

$$\sum_{n=1}^{\infty} \binom{2n}{n} \frac{(-1)^{n-1} H_{n+1}}{4^n (n+1)} = 5 + 4\sqrt{2} \left(\log \frac{2\sqrt{2}}{1+\sqrt{2}} - 1 \right) \doteq 0.238892684$$

Some of the above generating functions are quite simple. They are possibly just lucky exceptions. The next series is somewhat similar to (21) and (22), but the generating function is more involved.

Proposition 6. *We have*

$$\sum_{n=1}^{\infty} \binom{2n}{n} \frac{x^n}{n^2} = 2\,\mathrm{Li}_2 \left(\frac{1-\sqrt{1-4x}}{2} \right) - \log^2 \left(1 + \sqrt{1-4x} \right) - 2 \log 2 \log \frac{1-\sqrt{1-4x}}{x} + 3(\log 2)^2. \tag{24}$$

where the first function on the RHS is the dilogarithm

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}.$$

Proof. Starting from (13),

$$\sum_{n=1}^{\infty} \binom{2n}{n} \frac{x^n}{n} = 2 \log \left(\frac{2}{1+\sqrt{1-4x}} \right) = 2\log 2 - 2\log\left(1+\sqrt{1-4x}\right),$$

we divide both sides by x and then integrate to find

$$\sum_{n=1}^{\infty} \binom{2n}{n} \frac{x^n}{n^2} = 2\log 2 \log x - 2\int \frac{\log(1+\sqrt{1-4x})}{x}\, dx.$$

With the substitution $y = \sqrt{1-4x}$ and by using partial fractions

$$-2\int \frac{\log(1+\sqrt{1-4x})}{x}\, dx = -\log^2\left(1+\sqrt{1-4x}\right) + 2\int \frac{\log(1+y)}{1-y}\, dy.$$

Further, setting $1 - y = u$ (so that now $u = 1 - \sqrt{1-4x}$),

$$\begin{aligned} 2\int \frac{\log(1+y)}{1-y}\, dy &= -2\int \frac{\log(2-u)}{u}\, du = -2\int \frac{\log 2 + \log(1-u/2)}{u}\, du \\ &= -2\log 2 \log\left(1-\sqrt{1-4x}\right) + 2\sum_{k=1}^{\infty} \frac{(u/2)^k}{k^2} \\ &= -2\log 2 \log\left(1-\sqrt{1-4x}\right) + 2\,\mathrm{Li}_2\left(\frac{1-\sqrt{1-4x}}{2}\right). \end{aligned}$$

Bringing all pieces together and computing the constant of integration (for $x = 0$), we obtain (24). □

Setting $x = \frac{1}{4}$ in (24) yields

$$\sum_{n=1}^{\infty} \binom{2n}{n} \frac{1}{4^n n^2} = 2\,\mathrm{Li}_2\left(\frac{1}{2}\right) - (\log 2)^2 = \frac{\pi^2}{6} - 2(\log 2)^2.$$

Now we shall derive from Theorem 1 the generating function for the numbers $\binom{2n}{n}\frac{H_n}{n}$ which are "close" to $C_n H_n$ and to the coefficients in (23). The generating function, however, is not so simple as the one in (5).

Corollary 7. *For every $|x| \le 1/4$ and with $y = \sqrt{1-4x}$ we have*

$$\begin{aligned} \sum_{n=1}^{\infty} \binom{2n}{n} \frac{H_n}{n} x^n &= 2\log y \log\frac{1+y}{1-y} + 2\log 2\log(1+y) - \log^2(1+y) \\ &+2\,\mathrm{Li}_2(-y) - 2\,\mathrm{Li}_2(y) - 2\,\mathrm{Li}_2\left(\frac{1-y}{2}\right) - (\log 2)^2 + \frac{\pi^2}{2}. \end{aligned} \tag{25}$$

Proof. For the proof we first write (4) in the form

$$\sum_{n=1}^{\infty} \binom{2n}{n} H_n x^n = \frac{2\log\left(1+\sqrt{1-4x}\right)}{\sqrt{1-4x}} - \frac{2\log 2}{\sqrt{1-4x}} - \frac{2\log\sqrt{1-4x}}{\sqrt{1-4x}},$$

then divide both sides by x and integrate. The integrals are solved with the same techniques as those in the previous proof. Details are left to the reader. To evaluate the constant of integration we set $x = 0$ and use the fact that

$$2\operatorname{Li}_2(1) = \frac{\pi^2}{3}, \quad 2\operatorname{Li}_2(-1) = \frac{-\pi^2}{6}, \quad 2\operatorname{Li}_2\left(\frac{1}{2}\right) = \frac{\pi^2}{6} - (\log 2)^2.$$

□

Setting $x = 1/4$ in (25) yields

$$\sum_{n=1}^{\infty} \binom{2n}{n} \frac{H_n}{4^n n} = \frac{\pi^2}{3},$$

in accordance with [1, (3.8)].

We finish with a generalization of (20).

Proposition 8. *For every $m = 0, 1, 2\ldots,$*

$$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{x^n}{n+m+1} = \frac{1}{2^{2m+1}x^{m+1}} \sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^k}{2k+1} \left[1 - (1-4x)^k \sqrt{1-4x}\,\right]. \tag{26}$$

In particular, for $m = 0$ this reduces to (20). For $m = 1$ and after simplification (26) becomes

$$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{x^n}{n+2} = \frac{2\left(2+\sqrt{1-4x}\right)}{3\left(1+\sqrt{1-4x}\right)^2}.$$

Proof. Let $f(x)$ be the LHS in (26). Then

$$\frac{d}{dx}\left(x^{m+1} f(x)\right) = \sum_{n=0}^{\infty} \binom{2n}{n} x^{n+m} = \frac{x^m}{\sqrt{1-4x}}$$

and therefore,

$$x^{m+1} f(x) = \int_0^x \frac{t^m}{\sqrt{1-4t}}\, dt = \frac{-1}{2^{2m+1}} \int_1^{\sqrt{1-4x}} (1-y^2)^m \, dy$$

with the substitution $y = \sqrt{1-4t}$. The next step is to expand the binomial inside the integral and integrate termwise. Details are left to the reader.

□

4 Appendix

Here we prove the formula

$$\sum_{n=0}^{\infty} \binom{\alpha}{n} (-1)^n a_n z^n = (z+1)^{\alpha} \sum_{n=0}^{\infty} \left(\frac{z}{z+1} \right)^n \binom{\alpha}{n} (-1)^n \left\{ \sum_{k=0}^{n} \binom{n}{k} a_k \right\}. \tag{27}$$

Proof. Let L be a circle centered at origin and inside a disk where the function (9) is holomorphic. For the coefficients a_k from (9) we have

$$a_k = \frac{1}{2\pi i} \oint_L \frac{1}{\lambda^k} \frac{f(\lambda)}{\lambda} \, d\lambda. \tag{28}$$

Multiplying both sides by $\binom{n}{k}$ and summing for k we find

$$\sum_{k=0}^{n} \binom{n}{k} a_k = \frac{1}{2\pi i} \oint_L \left\{ \sum_{k=0}^{n} \binom{n}{k} \frac{1}{\lambda^k} \right\} \frac{f(\lambda)}{\lambda} \, d\lambda = \frac{1}{2\pi i} \oint_L \left(1 + \frac{1}{\lambda} \right)^n \frac{f(\lambda)}{\lambda} \, d\lambda. \tag{29}$$

Let z be a complex number inside the circle L with $|z|$ small enough to assure convergence in the following expansions.
Multiplying in (29) by $\left(\frac{-z}{z+1}\right)^n \binom{\alpha}{n}$ and summing for n we arrive at the representation

$$\sum_{n=0}^{\infty} \left(\frac{-z}{z+1} \right)^n \binom{\alpha}{n} \left\{ \sum_{k=0}^{n} \binom{n}{k} a_k \right\} = \frac{1}{2\pi i} \oint_L \left\{ \sum_{n=0}^{\infty} \left(\frac{-z(1+\lambda)}{\lambda(z+1)} \right)^n \binom{\alpha}{n} \right\} \frac{f(\lambda)}{\lambda} \, d\lambda \tag{30}$$

$$= \frac{1}{(1+z)^{\alpha} 2\pi i} \oint_L \left\{ \left(1 - \frac{z}{\lambda} \right)^{\alpha} \right\} \frac{f(\lambda)}{\lambda} \, d\lambda.$$

Expanding now the binomial inside the integral, integrating termwise, and using (28) again we obtain

$$\frac{1}{2\pi i} \oint_L \left\{ \left(1 - \frac{z}{\lambda} \right)^{\alpha} \right\} \frac{f(\lambda)}{\lambda} \, d\lambda = \sum_{n=0}^{\infty} \binom{\alpha}{n} (-z)^n a_n. \tag{31}$$

Thus (27) follows from (30) and (31). □

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2010 *Mathematics Subject Classification*: Primary 11B83; Secondary 05A10.
Keywords: central binomial coefficient, Catalan number, harmonic number, generating functions, Euler series transformation, binomial transform.

Received June 3 2011; revised version received December 16 2011. Published in *Journal of Integer Sequences*, December 27 2011.

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