TAIWANESE JOURNAL OF MATHEMATICS
Vol. 10, No. 6, pp. 1661-1670, December 2006
This paper is available online at http://www.math.nthu.edu.tw/tjm/

THE GENERALIZED BINET FORMULA, REPRESENTATION AND SUMS OF THE GENERALIZED ORDER-k PELL NUMBERS

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Abstract. In this paper we give a new generalization of the Pell numbers in matrix representation. Also we extend the matrix representation and we show that the sums of the generalized order-k Pell numbers could be derived directly using this representation. Further we present some identities, the generalized Binet formula and combinatorial representation of the generalized order-k Pell numbers.

1. Introduction

It is well-known that the Pell sequence $\{P_n\}$ is defined recursively by the equation, for $n \geq 1$

$$P_{n+1} = 2P_n + P_{n-1} \tag{1.1}$$

in which $P_0 = 0$, $P_1 = 1$.

In [3], Horadam showed that some properties involving Pell numbers. Also in [2], Ercolano gave the matrix method for generating the Pell sequence as follows:

$$M = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \tag{1.2}$$

and by taking succesive positive powers of the matrix M one can easily verify that

$$M^n = \begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix}.$$

Received August 5, 2004, revised August 31, 2005.
Communicated by Song-Sun Lin.
2000 *Mathematics Subject Classification*: 11B37, 15A36, 15A15.
Key words and phrases: Generalized order-k Pell numbers; Matrix method; Binet formula; Sum.

The Pell sequence is a special case of a sequence which is defined recursively as a linear combination of the preceding k terms:

$$a_{n+k} = c_0 a_n + c_1 a_{n+1} + \ldots + c_{k-1} a_{n+k-1}$$

where $c_0, c_1, \ldots, c_{k-1}$ are real contants. In [4], Kalman derived a number of closed-form formulas for the generalized sequence by companion matrix method as follows:

$$A_k = [a_{ij}]_{k\times k} = \begin{bmatrix} c_0 & c_1 & c_2 & \ldots & c_{k-2} & c_{k-1} \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{bmatrix}. \tag{1.3}$$

Then by an inductive argument he obtained that

$$A_k^n \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{bmatrix}.$$

Further in [7], we defined the generalized order-k Lucas sequence in matrix representation with employing the matrix methods of Kalman.

Also in [5], we gave the generalized Binet formula, combinatorial representation and some relations involving the generalized order-k Fibonacci and Lucas numbers.

Now we give a new generalization of the Pell numbers in matrix representation and extend the matrix representation so we give sums of the generalized Pell numbers could be derived directly using this representation.

2. The Main Results

Define k sequences of the generalized order-k Pell numbers as shown:

$$P_n^i = 2P_{n-1}^i + P_{n-2}^i + \ldots + P_{n-k}^i \tag{2.1}$$

for $n > 0$ and $1 \leq i \leq k$, with initial conditions

$$P_n^i = \begin{cases} 1 & \text{if } n = 1 - i, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{for } 1 - k \leq n \leq 0,$$

where P_n^i is the nth term of the ith sequence. When $k = 2$, the generalized order-k Pell sequence, $\{P_n^k\}$, is reduced to the usual Pell sequence, $\{P_n\}$.

When $i = k$ in (2.1), we call P_n^k the generalized k-Pell number.

By (2.1), we can write

$$\begin{bmatrix} P_{n+1}^i \\ P_n^i \\ P_{n-1}^i \\ \vdots \\ P_{n-k+2}^i \end{bmatrix} = \begin{bmatrix} 2 & 1 & \dots & 1 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} \begin{bmatrix} P_n^i \\ P_{n-1}^i \\ P_n^i \\ \vdots \\ P_{n-k+1}^i \end{bmatrix} \tag{2.2}$$

for the generalized order-k Pell sequences. Letting

$$R = [r_{ij}]_{k \times k} = \begin{bmatrix} 2 & 1 & \dots & 1 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}. \tag{2.3}$$

The matrix R is said to be generalized order-k Pell matrix.

To deal with the k sequences of the generalized order-k Pell sequences simultaneously, we define a $k \times k$ matrix E_n as follows:

$$E_n = [e_{ij}]_{k \times k} = \begin{bmatrix} P_n^1 & p_n^2 & \dots & p_n^k \\ P_{n-1}^1 & P_{n-1}^2 & \dots & P_{n-1}^k \\ \vdots & \vdots & & \vdots \\ P_{n-k+1}^1 & P_{n-k+1}^2 & \dots & P_{n-k+1}^k \end{bmatrix}. \tag{2.4}$$

Generalizing Eq. (2.2), we derive

$$E_{n+1} = R \cdot E_n. \tag{2.5}$$

Lemma 1. *Let E_n and R be as in (2.4) and (2.3), respectively. Then, for all integers $n \geq 0$*

$$E_{n+1} = R^{n+1}.$$

Proof. By (2.4), we have $E_{n+1} = R \cdot E_n$. Then, by an inductive argument, we may rewrite it as

$$E_{n+1} = R^n \cdot E_1. \tag{2.6}$$

Since by definition of the generalized order-k Pell number, $E_1 = R$; therefore

$$E_{n+1} = R^{n+1}.$$

So the proof is complete. ∎

Theorem 1. *Let E_n be as in (2.4). Then*

$$\det E_n = \begin{cases} 1 & \text{if } k \text{ is odd,} \\ (-1)^n & \text{if } k \text{ is even.} \end{cases}$$

Proof. From Lemma 1, we have $E_{n+1} = R^{n+1}$. Then

$$\det E_{n+1} = \det\left(R^{n+1}\right) = (\det R)^{n+1}$$

where $\det R = (-1)^{k+1}$. Thus

$$\det E_{n+1} = \begin{cases} 1 & \text{if } k \text{ is odd,} \\ (-1)^{n+1} & \text{if } k \text{ is even.} \end{cases}$$

So the proof is complete. ∎

Now we give some relations involving the generalized order-k Pell numbers.

Theorem 2. *Let P_n^i be the nth generalized order-k Pell number, for $1 \leq i \leq k$. Then, for all positive integers n and m*

$$P_{n+m}^i = \sum_{j=1}^{k} P_m^j P_{n-j+1}^i.$$

Proof. From Lemma 1, we know that $E_n = R^n$; we may rewrite it as

$$E_{n+1} = E_n E_1 = E_1 E_n. \tag{2.7}$$

In other words, E_1 is commutative under matrix multiplication. Hence, more generalizing Eq. (2.7), we can write

$$E_{n+m} = E_n E_m = E_m E_n. \tag{2.8}$$

Consequently, an element of E_{n+m} is the product of a row E_n and a column of E_m; that is

$$P_{n+m}^i = \sum_{j=1}^{k} P_m^j P_{n-j+1}^i.$$

Thus the proof is complete. ■

For example, if we take $k = 2$ in Theorem 2, we have

$$\begin{aligned} P^2_{n+m} &= \sum_{j=1}^{2} P^j_m P^2_{n-j+1} \\ &= P^1_m P^2_n + P^2_m P^2_{n-1} \end{aligned}$$

and, since $P^1_n = P^2_{n+1}$ for all $n \in \mathbb{Z}^+$ and $k = 2$, we obtain

$$P^2_{n+m} = P^2_{m+1} P^2_n + P^2_m P^2_{n-1}$$

where P^2_n is the usual Pell number. Indeed, we generalize the following relation involving the usual Pell numbers:

$$P_{n+m} = P_{m+1} P_n + P_m P_{n-1}.$$

Lemma 2. *Let P^i_n be the nth generalized order-k Pell number. Then*

$$\begin{aligned} &P^i_{n+1} = P^1_n + P^{i+1}_n, \quad \textit{for} \ \ 2 \le i \le k-1, \\ &P^1_{n+1} = 2P^1_n + P^2_n, \\ &P^k_{n+1} = P^1_n. \end{aligned} \tag{2.9}$$

Proof. From Eq. (2.7), we have $E_{n+1} = E_n E_1$. Since using a property of matrix multiplication, the proof is readily seen. ■

3. Sums of the Pell Numbers

Now we extend the matrix representation and show that the sums of the generalized Pell numbers.

To calculate the sums S_n, $n \ge 0$, of the generalized order-k Pell numbers, defined by

$$S_n = \sum_{i=0}^{n} P^1_i.$$

Let T be a $(k+1) \times (k+1)$ square matrix, such that

$$T = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & & & & \\ 0 & & R & & \\ \vdots & & & & \\ 0 & & & & \end{bmatrix} \tag{3.1}$$

where R is the $k \times k$ matrix as in (2.3).

Theorem 3. *Let S_n, $n \geq 0$, denote the sums of the generalized Pell numbers. Then S_n is $(2,1)$ entry of the matrix T^{n+1} in which T is the $(k+1) \times (k+1)$ matrix as in (3.1).*

Proof. Let C_n be a $(k+1) \times (k+1)$ square matrix, such that

$$C_n = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ S_{n-1} & & & & \\ S_{n-2} & & E_n & & \\ \vdots & & & & \\ S_{n-k} & & & & \end{bmatrix}$$

where E_n is the $k \times k$ matrix as in (2.4). Then, by Eq. (2.9) and

(3.2) $$S_{n+1} = P_{n+1}^1 + S_n,$$

we derive a recurrence equation

(3.3) $$C_{n+1} = C_n \cdot T.$$

Inductively, we also have

(3.4) $$C_{n+1} = C_1 \cdot T^n.$$

Since $S_{-i} = 0$, $1 \leq i \leq k$, we thus infer $C_1 = T$, and in general, $C_n = T^n$. Since $S_n = (C_{n+1})_{2,1}$ and $C_n = T^n$, the proof is readily seen. ∎

From Eqs. (3.3) and (3.4), we reach the following equation:

(3.5) $$C_{n+1} = C_n C_1 = C_1 C_n$$

which shows that C_1 is commutative as well under matrix multiplication. By an application of Eq. (3.5), the sums of the generalized order-k Pell numbers satisfy the follwing recurrence relation:

$$S_n = 1 + 2S_{n-1} + \sum_{i=2}^{k} S_{n-i}.$$

Substituting $S_n = P_n^1 + S_{n-1}$, an instance of Eq. (3.2), into Eq. (3.4), we express P_n^1 in terms of the sums of the generalized order-k Pell numbers:

(3.6) $$P_n^1 = 1 + \sum_{i=1}^{k} S_{n-i}.$$

When $k=2$, this equation is reduced to

$$P_n^1 = 1 + S_{n-1} + S_{n-2}.$$

So we derive the well-known result [3]:

$$\sum_{i=1}^{n} P_i = \frac{P_{n+1}+P_n-1}{2}$$

where P_n is the nth term of the usual Pell sequence.

4. Generalized Binet Formula

In [6], Levesque gave a Binet formula for the Fibonacci sequence. In this section, we derive a generalized Binet formula for the generalized order-k Pell sequence by using the determinant.

Lemma 3. *The equation* $x^{k+1}-3x^k+x^{k-1}+1=0$ *does not have multiple roots for* $k\geq 2$.

Proof. Let $f(x)=x^k-2x^{k-1}-x^{k-2}-\ldots-x-1$ and let $h(x)=(x-1)f(x)$. Then $h(x)=x^{k+1}-3x^k+x^{k-1}+1$. So 1 is a root but not a multiple root of $h(x)$, since $k\geq 2$ and $f(1)\neq 1$. Suppose that α is a multiple root of $h(x)$. Note that $\alpha\neq 0$ and $\alpha\neq 1$. Since α is a multiple root, $h(\alpha)=\alpha^{k+1}-3\alpha^k+\alpha^{k-1}+1=0$ and

$$\begin{aligned} h'(x) &= (k+1)\alpha^k - 3k\alpha^{k-1} + (k-1)\alpha^{k-2} \\ &= \alpha^{k-2}\left((k+1)\alpha^2 - 3k\alpha + k - 1\right) = 0. \end{aligned}$$

Thus $\alpha_{1,2}=\frac{3k\mp\sqrt{5k^2+4}}{2(k+1)}$ and hence, for α_1

$$\begin{aligned} 0 &= \alpha_1^{k-1}\left(-\alpha_1^2+3\alpha_1-1\right)-1 \\ &= \left(\frac{3k+\sqrt{5k^2+4}}{2(k+1)}\right)^{k-1}\left(\frac{5k-4+3\sqrt{5k^2+4}}{2(k+1)^2}\right)-1. \end{aligned} \tag{4.1}$$

We let $a_k=\left(\left(\frac{3k+\sqrt{5k^2+4}}{2(k+1)}\right)^{k-1}\left(\frac{5k-4+3\sqrt{5k^2+4}}{2(k+1)^2}\right)\right)$. Then we write Eq. (4.1) as follows:

$$0 = a_k - 1.$$

Since $a_k<a_{k+1}$ and $a_2=2,0887$ for $k\geq 2$, $a_k\neq 1$, a contradiction. Similarly, hence, for α_2

$$0 = \alpha_2^{k-1}\left(-\alpha_2^2+3\alpha_2-1\right)-1$$

$$(4.2) \qquad = \left(\frac{3k-\sqrt{5k^2+4}}{2(k+1)}\right)^{k-1}\left(\frac{5k-4-3\sqrt{5k^2+4}}{2(k+1)^2}\right)-1.$$

We let $b_k = \left(\left(\frac{3k-\sqrt{5k^2+4}}{2(k+1)}\right)^{k-1}\left(\frac{5k-4-3\sqrt{5k^2+4}}{2(k+1)^2}\right)\right)$. Then we write Eq. (4.2) as follows:

$$0 = b_k - 1.$$

Since $b_k > b_{k+1}$ and $b_2 = -8,88662\times 10^{-2}$ for $k \geq 2$, $b_k \neq 1$, a contradiction. Therefore, the equation $h(x)=0$ does not have multiple roots. ■

Consequently, from Lemma 3, it is seen that the equation $x^k - 2x^{k-1} - x^{k-2} - \ldots - x - 1 = 0$ does not have multiple roots for $k \geq 2$.

Let $f(\lambda)$ be the characteristic polynomial of the generalized order-k Pell matrix R. Then $f(\lambda) = \lambda^k - 2\lambda^{k-1} - \lambda^{k-2} - \ldots - \lambda - 1$, which is a well-known fact. Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the eigenvalues of R. Then, by Lemma 3, $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct. Let V be a $k \times k$ Vandermonde matrix as follows:

$$V = \begin{bmatrix} \lambda_1^{k-1} & \lambda_2^{k-1} & \ldots & \lambda_k^{k-1} \\ \lambda_1^{k-2} & \lambda_2^{k-2} & \ldots & \lambda_k^{k-2} \\ \vdots & \vdots & & \vdots \\ \lambda_1 & \lambda_2 & \ldots & \lambda_k \\ 1 & 1 & \ldots & 1 \end{bmatrix}.$$

Let w_k^i be a $k \times 1$ matrix as follows:

$$w_k^i = \begin{bmatrix} \lambda_1^{n+k-i} \\ \lambda_2^{n+k-i} \\ \vdots \\ \lambda_k^{n+k-i} \end{bmatrix}$$

and $V_j^{(i)}$ be a $k \times k$ matrix obtained from V by replacing the jth column of V by w_k^i. Then we obtain the generalized Binet formula for the generalized order-k Pell numbers with the following theorem.

Theorem 4. *Let P_n^i be the nth term of ith Pell sequence, for $1 \leq i \leq k$. Then*

$$P_{n-i+1}^j = \frac{\det\left(V_j^{(i)}\right)}{\det(V)}.$$

Proof. Since the eigenvaules of R are distinct, R is diagonizable. It is readily seen that $RV = VD$, where $D = diag\left(\lambda_1, \lambda_2, \ldots, \lambda_k\right)$. Since V is invertible
$V^{-1}RV = D$. Hence, R is similar to D. So we obtain $R^nV = VD^n$. Then we have the following linear system of equations:

$$\begin{aligned} e_{i1}\lambda_1^{k-1} + e_{i2}\lambda_1^{k-2} + \ldots + e_{ik} &= \lambda_1^{n+k-i} \\ e_{i1}\lambda_2^{k-1} + e_{i2}\lambda_2^{k-2} + \ldots + e_{ik} &= \lambda_2^{n+k-i} \\ \vdots \qquad\qquad & \quad \vdots \\ e_{i1}\lambda_k^{k-1} + e_{i2}\lambda_k^{k-2} + \ldots + e_{ik} &= \lambda_k^{n+k-i} \end{aligned}.$$

And, for each $j = 1, 2, \ldots, k$, we obtain

$$e_{ij} = \frac{\det\left(V_j^{(i)}\right)}{\det\left(V\right)}$$

where e_{ij} is the (i, j)th elements of the matrix E_n, $i.e.$, $e_{ij} = P_{n-i+1}^j$.
So the proof is complete. ■

Corollary 1. *Let P_n^k be the nth generalized k-Pell number. Then*

$$P_n^k = \frac{\det\left(V_k^{(1)}\right)}{\det\left(V\right)}.$$

Proof. Since e_{ij} is the (i, j)th elements of the matrix E_n, $i.e.$, $e_{ij} = P_{n-i+1}^j$. If we take $i = 1$ and $j = k$, then $e_{1,k} = P_n^k$. Then by using Theorem 4, the proof is immediately seen. ■

5. Combinatorial Representation

In this section we give a combinatorial representation of the generalized order-k Pell numbers. In [1], the authors obtained an explicit formula for the elements in the nth power of the companion matrix and gave some interesting applications. The matrix A_k be as in (1.3), then we find the following Theorem in [1].

Theorem 5. *The (i, j) entry $a_{ij}^{(n)}\left(c_1, c_2, \ldots, c_k\right)$ in the matrix $A_k^n\left(c_1, c_2, \ldots, c_k\right)$ is given by the following formula:*

$$a_{ij}^{(n)}\left(c_1, c_2, \ldots, c_k\right) = \sum_{(t_1, t_2, \ldots t_k)} \tfrac{t_j + t_{j+1} + \ldots + t_k}{t_1 + t_2 + \ldots + t_k} \times \tbinom{t_1 + t_2 + \ldots + t_k}{t_1, t_2, \ldots, t_k} c_1^{t_1} \ldots c_k^{t_k} \tag{5.1}$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \ldots + kt_k = n - i + j$*, and the coefficients in* (5.1) *is defined to be* 1 *if* $n = i - j$*.*

Then we have the following Corollary.

Corollary 2. *Let* P_n^i *be the generalized order-*k *Pell number, for* $1 \leq i \leq k$*. Then*

$$P_n^i = \sum_{(r_1, r_2, \ldots r_k)} \frac{r_k}{r_1 + r_2 + \ldots + r_k} \times \binom{r_1 + r_2 + \ldots + r_k}{r_1, r_2, \ldots, r_k} 2^{r_1}$$

where the summation is over nonnegative integers satisfying $r_1 + 2r_2 + \ldots + kr_k = n - i + k$.

Proof. In Theorem 5, if $j = k$ and $c_1 = 2$, then the proof is immediately seen from (2.4). ■

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