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Some Families of Generating Functions for the Bessel Polynomials

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The main object of this paper is to show how readily some general results on bilinear, bilateral, or mixed multilateral generating functions for the Bessel polynomials would provide unifications (and generalizations) of numerous generating functions which were proven recently by using group-theoretic techniques. © 1997 Academic Press

1. INTRODUCTION

Almost half a century ago, Krall and Frink [8] initiated a systematic investigation of what are now well-known in the mathematical literature as the Bessel polynomials. In their terminology, the *generalized Bessel polynomials* $y_n(x; \alpha, \beta)$ are defined by

$$y_n(x;\alpha,\beta) := \sum_{k=0}^{n} \binom{n}{k} \binom{\alpha+n+k-2}{k} k! \left(\frac{x}{\beta}\right)^k = {}_2F_0\left(-n, \alpha+n-1; -; -\frac{x}{\beta}\right) \tag{1.1}$$

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and the *simple Bessel polynomials* $y_n(x)$ are defined by

$$y_n(x) := \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} k!\left(\frac{x}{2}\right)^k = y_n(x;2,2)$$
$$= {}_2F_0\left(-n, n+1; -; -\frac{x}{2}\right). \tag{1.2}$$

These polynomials are orthogonal and satisfy the following second-order linear differential equation:

$$x^2\frac{d^2y}{dx^2} + (\alpha x + \beta)\frac{dy}{dx} - n(n+\alpha-1)y = 0. \tag{1.3}$$

More importantly, they arise naturally in a number of seemingly diverse contexts; e.g., in connection with the solution of the classical wave equation in spherical polar coordinates [8], in network synthesis and design [5], in the representation of the energy spectral functions for a family of isotropic turbulence fields [12], and so on. For further information and details about these polynomials and their applications, the interested reader may be referred to an excellent monograph on this subject by Grosswald [6].

The present investigation is motivated essentially by several recent works, using group-theoretic techniques of Louis Weisner (1899–1988) in conjunction with the differential equation (1.3), on *bilateral* generating functions for the Bessel polynomials. We begin by recalling here some of these results in the following (corrected or modified) forms:

THEOREM A (cf. Chongdar [4]). *If there exists a generating function of the form*:

$$F_\alpha(x,t) = \sum_{n=0}^{\infty} a_n y_n(x;\alpha,\beta)t^n, \tag{1.4}$$

then

$$\sum_{n=0}^{\infty} \xi_n^{(\alpha)}(x,z)t^n = (1-xt)^{1-\alpha}e^{\beta t}F_\alpha\left(\frac{x}{1-xt}, \frac{zt}{1-xt}\right), \tag{1.5}$$

where

$$\xi_n^{(\alpha)}(x,z) := \sum_{k=0}^{n} a_k y_n(x;\alpha-n+k,\beta)\frac{\beta^{n-k}}{(n-k)!}z^k. \tag{1.6}$$

THEOREM B (cf. Majumdar [9]). *If there exists a generating function of the form*:

$$G_\alpha(x,t) = \sum_{n=0}^{\infty} a_n y_n(x; \alpha - m, \beta) t^n, \tag{1.7}$$

then

$$\sum_{n=0}^{\infty} \eta_n^{(\alpha)}(x,z) t^n = (1-xt)^{1-\alpha+m} e^{\beta t} G_\alpha\left(\frac{x}{1-xt}, \frac{zt}{1-xt}\right), \tag{1.8}$$

where

$$\eta_n^{(\alpha)}(x,z) := \sum_{k=0}^{n} a_k y_n(x; \alpha - m - n + k, \beta) \frac{\beta^{n-k}}{(n-k)!} z^k. \tag{1.9}$$

THEOREM C (cf. Mukherjee and Chongdar [10]). *If there exists a generating function of the form*:

$$H_\alpha(x,t) = \sum_{n=0}^{\infty} a_n y_n(x; \alpha + n, \beta) t^n, \tag{1.10}$$

then

$$\sum_{n=0}^{\infty} \zeta_n^{(\alpha)}(x,z) t^n = (1-xt)^{1-\alpha} e^{\beta t} H_\alpha\left(\frac{x}{1-xt}, \frac{zt}{(1-xt)^2}\right), \tag{1.11}$$

where

$$\zeta_n^{(\alpha)}(x,z) := \sum_{k=0}^{n} a_k y_n(x; \alpha - n + 2k, \beta) \frac{\beta^{n-k}}{(n-k)!} z^k. \tag{1.12}$$

We remark in passing that, unlike the situation with regard to the parameter $\alpha + n$ on the right-hand side of the generating function (1.10), the parameter $\alpha - m$ occurring in the generating function (1.7) is *independent* of the index of summation. Furthermore, the summand in (1.11) *cannot* be expressed as the product used by Mukherjee and Chongdar [10, p. 478].

2. EQUIVALENCE OF THEOREM A AND THEOREM B

Obviously, the parameter m in the generating function (1.7) is redundant. In fact, we readily find from (1.4) and (1.7) that

$$F_{\alpha-m}(x,t) = G_\alpha(x,t) \tag{2.1}$$

or, equivalently,

$$F_\alpha(x,t) = G_{\alpha+m}(x,t) \tag{2.2}$$

for any (real or complex) parameter m. Thus, since

$$\xi_n^{(\alpha-m)}(x,z) = \eta_n^{(\alpha)}(x,z), \tag{2.3}$$

Theorem B would follow immediately from Theorem A upon replacing α trivially by $\alpha - m$.

3. A DIRECT PROOF OF THEOREM C AND ITS GENERALIZATION

We have observed, in the preceding section, that Theorem B is substantially the same as Theorem A. In our attempt to give a *direct* proof of Theorem C, we are led naturally to the following unification (and generalization) of numerous families of generating functions for the Bessel polynomials, including (for example) Theorem A (and hence also Theorem B) *and* Theorem C of the preceding section.

THEOREM 1. *Corresponding to a sequence* $\{\Omega_n(z_1,\dots,z_s)\}_{n=0}^{\infty}$ *of* s *complex variables*

$$z_1,\dots,z_s \qquad (s \in \mathbb{N} := \{1,2,3,..\}),$$

let

$$\Lambda_m^{(\sigma)}[x;z_1,\dots,z_s;t] := \sum_{n=0}^{\infty} a_n y_{m+n}(x;\alpha+\sigma n,\beta)\,\Omega_n(z_1,\dots,z_s)\frac{t^n}{n!}$$

$$(a_n \neq 0; m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}), \tag{3.1}$$

where σ *is a (real or complex) parameter. Suppose also that* $\Phi_n^{(\alpha)}(x;z_1,\dots,z_s;\omega)$ *is a polynomial of degree* n *in* ω *(with coefficients* *depending upon* $\alpha,\ \beta,\ \sigma,\ n,\ x$ *as well as on* $z_1, \dots, z_s$) *defined by*

$$\Phi_n^{(\alpha)}(x; z_1, \dots, z_s; \omega) := \sum_{k=0}^{n} \binom{n}{k} a_k y_{m+n}(x; \alpha - n + (\sigma + 1)k, \beta) \cdot \Omega_k(z_1, \dots, z_s)\, \omega^k. \tag{3.2}$$

Then

$$\sum_{n=0}^{\infty} \Phi_n^{(\alpha)}(x; z_1, \dots, z_s; \omega) \frac{t^n}{n!} = \left(1 - \frac{xt}{\beta}\right)^{1-\alpha-m} e^t \cdot \Lambda_m^{(\sigma)}\left[\frac{\beta x}{\beta - xt}; z_1, \dots, z_s; \omega t \left(1 - \frac{xt}{\beta}\right)^{-\sigma-1}\right] \qquad (|t| < |\beta/x|), \tag{3.3}$$

provided that each member of (3.3) *exists*.

Proof. If, for convenience, we denote the first member of the assertion (3.3) by $\mathcal{S}$, and make use of the definition (3.2), we shall obtain

$$\mathcal{S} = \sum_{n=0}^{\infty} t^n \sum_{k=0}^{n} a_k y_{m+n}(x; \alpha - n + (\sigma + 1)k, \beta) \cdot \Omega_k(z_1, \dots, z_s) \frac{\omega^k}{(n-k)!k!}$$

$$= \sum_{k=0}^{\infty} a_k \Omega_k(z_1, \dots, z_s) \frac{(\omega t)^k}{k!} \cdot \sum_{n=0}^{\infty} y_{m+k+n}(x; \alpha + \sigma k - n, \beta) \frac{t^n}{n!}, \tag{3.4}$$

where we have inverted the order of the double summation involved.

The inner sum in (3.4) can be evaluated by appealing to the familiar generating function (cf., e.g., Srivastava and Manocha [14, p. 419, Eq. 8.4(8)]):

$$\sum_{n=0}^{\infty} y_{m+n}(x; \alpha - n, \beta) \frac{t^n}{n!} = \left(1 - \frac{xt}{\beta}\right)^{1-\alpha-m} e^t y_m\left(\frac{\beta x}{\beta - xt}; \alpha, \beta\right) \qquad (m \in \mathbb{N}_0; |t| < |\beta/x|) \tag{3.5}$$

with, of course, m and α replaced by $m+k$ and $\alpha+\sigma k$, respectively ($k \in \mathbb{N}_o$). We thus find from (3.4) that

$$S = \left(1 - \frac{xt}{\beta}\right)^{1-\alpha-m} e^t \sum_{k=0}^{\infty} \frac{a_k}{k!} \Omega_k(z_1, \ldots, z_s)$$

$$\cdot y_{m+k}\left(\frac{\beta x}{\beta - xt}; \alpha + \sigma k, \beta\right)\left\{\omega t\left(1 - \frac{xt}{\beta}\right)^{-\sigma-1}\right\}^k \quad (|t| < |\beta/x|), \tag{3.6}$$

which, in view of the definition (3.1), is precisely the second member of the assertion (3.3).

This evidently completes the proof of Theorem 1 under the assumption that the double series involved in the first two steps of our proof are absolutely convergent. Thus, in general, Theorem 1 holds true (at least as a relation between *formal* power series) for those values of the various parameters and variables involved for which each member of the assertion (3.3) exists.

Alternatively, the assertion (3.3) can be deduced from Theorem 3 of Chen and Srivastava [3] by setting

$$\mu = 0, \qquad \rho = \sigma + 1, \qquad \text{and} \qquad p = q = 1,$$

and making some obvious notational changes.

4. APPLICATIONS OF THEOREM 1

First of all, we observe that Theorem C emerges as an immediate consequence of the assertion (3.3) of Theorem 1 when we set

$$m = 0, \qquad \sigma = 1, \qquad \Omega_n(z_1, \ldots, z_s) \equiv 1 \; (n \in \mathbb{N}_0), \qquad \text{and} \qquad \omega = \frac{z}{\beta},$$

and replace t and a_n by βt and $n!a_n$ ($n \in \mathbb{N}_0$), respectively. As a matter of fact, the assertion (3.3) can also be specialized suitably in order to derive Theorem A (and hence also Theorem B).

With a view to deriving Theorem A as a special case of the assertion (3.3), we set

$$m = \sigma = 0, \qquad \Omega_n(z_1, \ldots, z_s) \equiv 1 \; (n \in \mathbb{N}_0), \qquad \text{and} \qquad \omega = \frac{z}{\beta},$$

and (as before) replace t and a_n be βt and $n!a_n$ ($n \in \mathbb{N}_0$), respectively.

Thus Theorem 1 (proven in the preceding section) contains each of the results (Theorem A, Theorem B, and Theorem C) as its special cases.

Next we turn to some further applications of Theorem 1 when the multivariable function $\Omega_n(z_1, \ldots, z_s)$ $(n \in \mathbb{N}_0; s \in \mathbb{N})$ is expressed in terms of simpler functions of one and more variables. Indeed, the other main result of Chongdar [4, p. 151, Theorem 2] corresponds to a special case of Theorem 1 of Section 3 when we set

$$m = \sigma = 0, \qquad s = 1, \qquad z_1 = y, \qquad t \to \beta t, \qquad \text{and}$$
$$a_n \to n! a_n \ (n \in \mathbb{N}_0).$$

If, in Theorem 1 of Section 3, we set

$$s = 1, \qquad z_1 = z, \qquad \text{and} \qquad \Omega_n(z) = L_N^{(\gamma+n)}(z) \ (n, N \in \mathbb{N}_0; \gamma \in \mathbb{C}),$$

where $L_N^{(\gamma)}(z)$ denotes the classical Laguerre polynomials defined by (cf., e.g., Szegö [15, Chap. 5])

$$L_N^{(\gamma)}(z) := \sum_{k=0}^{N} \binom{\gamma+N}{N-k} \frac{(-z)^k}{k!}$$
$$= \binom{\gamma+N}{N} {}_1F_1(-N; \gamma+1; z), \tag{4.1}$$

we shall readily obtain a class of bilateral generating functions for the Bessel or Laguerre polynomials, given by

THEOREM 2. *If*

$$\Xi_m^{(\sigma)}(x, z, t) := \sum_{n=0}^{\infty} a_n y_{m+n}(x; \alpha + \sigma n, \beta) L_N^{(\gamma+n)}(z) \frac{t^n}{n!}$$
$$(a_n \neq 0; m \in \mathbb{N}_0; \gamma, \sigma \in \mathbb{C}) \tag{4.2}$$

and

$$\Psi_n^{(\alpha)}(x, z, \omega) := \sum_{k=0}^{n} \binom{n}{k} a_k y_{m+n}(x; \alpha - n + (\sigma+1)k, \beta)$$
$$\cdot L_N^{(\gamma+k)}(z)\, \omega^k, \tag{4.3}$$

then

$$\sum_{n=0}^{\infty} \Psi_n^{(\alpha)}(x, z, \omega) \frac{t^n}{n!} = \left(1 - \frac{xt}{\beta}\right)^{1-\alpha-m} e^t \cdot \Xi_m^{(\sigma)}\left[\frac{\beta x}{\beta - xt}, z, \omega t\left(1 - \frac{xt}{\beta}\right)^{-\sigma-1}\right] \qquad (|t| < |\beta/x|), \quad (4.4)$$

provided that each member of (4.4) *exists*.

A rather involved version of a *special* case of Theorem 2 when

$$m = \gamma = 0 \qquad \text{and} \qquad \sigma = -1$$

happens to be the *main* result of a recent paper by Hazra (Chakraborty) and Basu [7], which they proved by using the aforementioned group-theoretic techniques. It may be of interest to give here a simple (direct) proof this special case of Theorem 2 *without* using group-theoretic techniques. We first recall the *main* result of Hazra (Chakraborty) and Basu [7] in the following (corrected and slightly modified) form (cf. [7, p. 369]):

THEOREM D. *If*

$$\mathsf{G}(x, z, w) := \sum_{n=0}^{\infty} a_n y_n(x; \alpha - n, \beta) L_N^{(n)}(z) w^n, \qquad (4.5)$$

then

$$(1 - wx)^{1-\alpha} e^{(\beta-1)w} \mathsf{G}\left(\frac{x}{1 - wx}, z + w, wv\right) = \sum_{n,p,q=0}^{\infty} \frac{(-1)^q}{p!q!} a_n v^n \beta^p w^{p+q+n} y_{n+p}(x; \alpha - n - p, \beta) L_N^{(n+q)}(z). \qquad (4.6)$$

The assertion (4.6) can be proven *directly* (that is, without using the group-theoretic techniques applied by the earlier authors [7]) by first rewriting the second member of (4.6) as

$$\Delta := \sum_{n,p,q=0}^{\infty} \frac{(-1)^q}{p!q!} a_n v^n \beta^p w^{p+q+n} y_{n+p}(x; \alpha - n - p, \beta) L_N^{(n+q)}(z)$$

$$= \sum_{n,q=0}^{\infty} \frac{(-w)^q}{q!} a_n L_N^{(n+q)}(z)(wv)^n$$

$$\cdot \sum_{p=0}^{\infty} y_{n+p}(x; \alpha - n - p, \beta) \frac{(\beta w)^p}{p!}. \tag{4.7}$$

The innermost sum in (4.7) can easily be evaluated by means of the known result (3.5), and we thus obtain

$$\Delta = (1 - wx)^{1-\alpha} e^{\beta w} \sum_{n,q=0}^{\infty} \frac{(-w)^q}{q!} a_n L_N^{(n+q)}(z)(wv)^n$$

$$\cdot y_n\left(\frac{x}{1 - wx}; \alpha - n, \beta\right)$$

$$= (1 - wx)^{1-\alpha} e^{\beta w} \sum_{n=0}^{\infty} a_n y_n\left(\frac{x}{1 - wx}; \alpha - n, \beta\right)(wv)^n$$

$$\cdot \sum_{q=0}^{\infty} L_N^{(n+q)}(z) \frac{(-w)^q}{q!}. \tag{4.8}$$

In order to sum the inner series in (4.8), we recall another well-known result (cf., e.g., Buchholz [1, p. 142, Eq. (18)]:

$$\sum_{k=0}^{\infty} L_n^{(\alpha+k)}(x) \frac{t^k}{k!} = e^t L_n^{(\alpha)}(x - t) \qquad (\alpha \in \mathbb{C}), \tag{4.9}$$

which, in view of the elementary identity:

$$\frac{\partial^k}{\partial t^k}\left\{e^t L_n^{(\alpha)}(x - t)\right\} = e^t L_n^{(\alpha+k)}(x - t) \qquad (k \in \mathbb{N}_0; \alpha \in \mathbb{C}), \tag{4.10}$$

is an immediate consequence of the Taylor expansion of

$$e^t L_n^{(\alpha)}(x - t)$$

in powers of t. Applying (4.9) in (4.8), we find that

$$\Delta = (1 - wx)^{1-\alpha} e^{(\beta-1)w} \sum_{n=0}^{\infty} a_n y_n\left(\frac{x}{1-wx}; \alpha - n, \beta\right) L_N^{(n)}(z+w)(wv)^n, \tag{4.11}$$

which, upon interpretation by means of the definition (4.5), is precisely the first member of the assertion (4.6) of Theorem D.

A closer examination of Theorem D will immediately expose the fact that the argument v, occurring in (4.6), can (for the sake of simplicity) be replaced by v/w. Furthermore, since (4.9) holds true for an essentially arbitrary parameter α, our direct proof of Theorem D can be applied *mutatis mutandis* in order to drive the following generalization of Theorem D:

THEOREM 3. *If*

$$H_{\kappa,m}^{(\gamma,\delta)}[x,z,w] := \sum_{n=0}^{\infty} a_n y_{m+n}(x; \alpha - \kappa n, \beta) L_N^{(\gamma+\delta n)}(z) w^n$$

$$(a_n \neq 0; m \in \mathbb{N}_0; \kappa, \gamma, \delta \in \mathbb{C}), \tag{4.12}$$

then

$$(1-wx)^{1-\alpha-m} e^{(\beta-1)w} H_{\kappa,m}^{(\gamma,\delta)}\left[\frac{x}{1-wx}, z+w, v(1-wx)^{\kappa-1}\right]$$
$$= \sum_{n,p,q=0}^{\infty} \frac{(-1)^q}{p!q!} a_n v^n \beta^p w^{p+q} y_{m+n+p}(x; \alpha - \kappa n - p, \beta)$$
$$\cdot L_N^{(\gamma+\delta n+q)}(z) \qquad (|wx| < 1), \tag{4.13}$$

provided that each member of (4.13) *exists*.

Clearly, since

$$H_{1,0}^{(0,1)}[x,z,w] = G(x,z,w), \tag{4.14}$$

which follows readily from the definitions (4.5) and (4.12), Theorem 3 would reduce immediately to Theorem D when we set

$$m = \gamma = 0 \qquad \text{and} \qquad \kappa = \delta = 1,$$

and replace v trivially by wv.

Finally, in view of the elementary identity (4.9), the definition (4.3) can be rewritten in the form:

$$\Psi_n^{(\alpha)}(x, z, \omega) = e^u \sum_{k=0}^{n} \binom{n}{k} a_k \omega^k y_{m+n}(x; \alpha - n + (\sigma + 1)k, \beta) \cdot \sum_{q=0}^{\infty} L_N^{(\gamma+k+q)}(z - u) \frac{(-u)^q}{q!}. \tag{4.15}$$

Thus, by appealing also to the familiar result (3.5), it is easy to verify that Theorem 3 (with, of course, $\delta = 0$) is equivalent to Theorem 2. For various general families of bilinear, bilateral, and mixed multilateral generating functions for the simple as well as generalized Bessel polynomials, the interested reader may be referred to the results of Srivastava [11, pp. 228–229, Corollaries 1 and 2], which were subsequently reproduced in the latest treatise on the subject of generating functions by Srivastava and Manocha [14, p. 421, Corollaries 1 and 2], Chen *et al*. [2, pp. 359–361, Theorems 2 and 3; p. 364, Theorem 6], and Chen and Srivastava [3, pp. 153–155, Theorems 1, 2, and 3] (especially [3, Theorem 3] which we referred to already in Section 3 in connection with our proof of Theorem 1), and indeed also to a recent work of Srivastava [13, p. 129, Theorem].

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