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# Certain classes of finite-series relationships and generating functions involving the generalized Bessel polynomials

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**Abstract**

Recently, by making use of the familiar group-theoretic (Lie algebraic) method, a certain mixed trilateral finite-series relationship was proven for the generalized Bessel polynomials. The main object of this paper is to derive several substantially more general families of bilinear, bilateral, and mixed multilateral finite-series relationships and generating functions for these and other related classes of hypergeometric polynomials. A duly corrected and modified analogue of the aforementioned trilateral finite-series relationship is shown to follow by suitably specializing one of the general results presented here. Several closely related (presumably new) finite-series relationships and generating functions, some of which also involve (for example) the Stirling numbers of the second kind, are onsidered rather briefly.



*Keywords:* Finite-series relationships; Generating functions; Bessel polynomials; Generalized hypergeometric function; Group-theoretic (Lie algebraic) method; Hypergeometric polynomials; Laguerre polynomials; Stirling numbers of the second kind; Kronecker delta

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## 1. Introduction, definitions, and preliminaries

The *simple* Bessel polynomials $y_n(x)$ defined by

$$y_n(x) := \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} k! \left(\frac{x}{2}\right)^k \tag{1.1}$$

and the *generalized* Bessel polynomials $y_n(x;\alpha,\beta)$ or $Y_n^{\alpha}(x)$ defined by

$$y_n(x;\alpha,\beta) := \sum_{k=0}^{n} \binom{n}{k}\binom{\alpha+n+k-2}{k} k! \left(\frac{x}{\beta}\right)^k =: Y_n^{\alpha-2}\left(\frac{2x}{\beta}\right) \tag{1.2}$$

are known to arise naturally in a number of seemingly diverse contexts (see, for details [5,6,8,14]). Clearly, the definitions (1.1) and (1.2) immediately imply the relationship:

$$y_n(x) = y_n(x;2,2) = Y_n^0(x) \quad (n \in \mathbb{N}_0 := \{0,1,2,\ldots\}). \tag{1.3}$$

Furthermore, in terms of a generalized hypergeometric function ${}_pF_q(z)$ with $p$ numerator and $q$ denominator parameters, it is easily seen from the definition (1.2) that

$$y_n(x;\alpha,\beta) = {}_2F_0\left(-n;\alpha+n+1;\text{-}\,;-\frac{x}{\beta}\right) = Y_n^{\alpha-2}\left(\frac{2x}{\beta}\right). \tag{1.4}$$

Indeed, by reversal of the order of terms of the hypergeometric polynomials in (1.4), it is also observed that

$$y_n(x;\alpha,\beta) = n!\left(-\frac{x}{\beta}\right)^n L_n^{(1-\alpha-2n)}\left(\frac{\beta}{x}\right) \tag{1.5}$$

or, equivalently,

$$L_n^{(\alpha)}(x) = \frac{(-x)^n}{n!}\, y_n\left(\frac{\beta}{x};1-\alpha-2n,\beta\right), \tag{1.6}$$

where $L_n^{(\alpha)}(x)$ denotes the classical Laguerre polynomials defined by (cf., e.g. [17, p. 101])

$$L_n^{(\alpha)}(x) := \sum_{k=0}^{n} \binom{n+\alpha}{n-k}\frac{(-x)^k}{k!} = \binom{n+\alpha}{n} {}_1F_1(-n;\alpha+1;x). \tag{1.7}$$

Recently, by making use of the familiar group-theoretic (Lie algebraic) method, which is described fairly adequately by (for example) Miller [10], McBride [9], and Srivastava and Manocha [16], Mukherjee [11] proved a certain mixed trilateral finite-series relationship for the generalized Bessel polynomials $y_n(x;\alpha,\beta)$ defined by (1.2). We choose first to recall here the *main* result of Mukherjee [11] in the following (corrected *as well as* modified) form:

**Theorem 1** (cf. Mukherjee [11, p. 87]). *If there exists a generating relation of the form*:

$$G(x,u,w) = \sum_{k=0}^{n} a_k \ y_{n-k}(x;\alpha+k,\beta) \ g_k(u)w^k, \tag{1.8}$$

*then*

$$(1-t)^n \ G\Big(\frac{x}{1-t}, u, \frac{t}{1-t}\Big) = \sum_{k=0}^{n} a_k \ \sigma_{n-k}^{(\alpha+k)}(x,t) \ g_k(u)t^k, \tag{1.9}$$

*where*

$$\sigma_n^{(\alpha)}(x,t) := \sum_{l=0}^{n} \binom{n}{l} y_{n-l}(x;\alpha+l,\beta)(-t)^l. \tag{1.10}$$

Theorem 1 is actually stated and proved by Mukherjee [11] not only with the parameter $\alpha$ unnecessarily constrained to be a nonnegative integer, but also with $n$ replaced trivially by $n-\kappa$ for an obviously redundant parameter $\kappa$ $(\alpha, n, \kappa, n-\kappa \in \mathbb{N}_0)$. As a matter of fact, Theorem 1 would follow rather simply as an immediate consequence of the known result (cf., e.g., [15, p. 106, Eq. (1.26)]):

$$\sum_{k=0}^{n} \binom{n}{k} y_{n-k}(x;\alpha+k,\beta)t^k = (1+t)^n \ y_n\bigg(\frac{x}{1+t};\alpha,\beta\bigg), \tag{1.11}$$

which, in view of the relationship (1.5) or (1.6), is essentially the same as the relatively more familiar *classical* result (cf., e.g., [4, p. 348, Eq. (27)], [1, p. 142, Eq. (18)], and [7, p. 319, Entry (48.19.2)]):

$$\sum_{k=0}^{n} L_{n-k}^{(\alpha+k)}(x)\frac{t^k}{k!} = L_n^{(\alpha)}(x-t), \tag{1.12}$$

which, in turn, is the Taylor expansion of $L_n^{(\alpha)}(x-t)$ in powers of $t$, since

$$\frac{\partial^k}{\partial t^k}\left\{L_n^{(\alpha)}(x-t)\right\} = \begin{cases} L_{n-k}^{(\alpha+k)}(x-t) & (k=0,1,\ldots,n), \\ 0 & (k=n+1,n+2,n+3,\ldots). \end{cases} \tag{1.13}$$

By applying the known result (1.11), we find from the definition (1.10) that

$$\sigma_n^{(\alpha)}(x,t) = (1-t)^n \ y_n\Big(\frac{x}{1-t};\alpha,\beta\Big) \quad (n \in \mathbb{N}_0) \tag{1.14}$$

so that, in view of the hypothesis (1.8), we have

$$\sum_{k=0}^{n} a_k \ \sigma_{n-k}^{(\alpha+k)}(x,t) \ g_k(u)t^k = (1-t)^n \sum_{k=0}^{n} a_k \ y_{n-k}\Big(\frac{x}{1-t};\alpha+k,\beta\Big)g_k(u)\Big(\frac{t}{1-t}\Big)^k$$

$$= (1-t)^n G\Big(\frac{x}{1-t},u,\frac{t}{1-t}\Big), \tag{1.15}$$

which is precisely the left-hand side of the assertion (1.9) of Theorem 1.

In this paper, we first derive a much deeper application of the known result (1.11). We then present a similar application of another known result (cf., e.g., [15, p. 104, Eq. (1.19)]):

$$\sum_{k=0}^{n} \binom{n}{k} (\alpha+n-1)_k y_{n-k}(x;\alpha+2k,\beta)\left(\frac{t}{\beta}\right)^k = y_n(x+t;\alpha,\beta), \tag{1.16}$$

which, in view of the relationship (1.5) or (1.6), is essentially the same as a fairly well-known (*rather classical*) result (cf. [18, p. 85, Eq. (9)] and [9, p. 35, Eq. (1)]):

$$\sum_{k=0}^{n} \binom{\alpha+n}{k} L_{n-k}^{(\alpha)}(x)t^k = (1+t)^n L_n^{(\alpha)}\left(\frac{x}{1+t}\right), \tag{1.17}$$

which, in turn, is an *obvious* special case of the familiar generating function (cf. [16, p. 132, Eq. 2.5(5) *et seq.*]):

$$\sum_{k=0}^{\infty} \frac{(\lambda)_k}{(\alpha+1)_k} L_k^{(\alpha)}(x)t^k = (1-t)^{-\lambda} \ {}_1F_1\Big(\lambda;\alpha+1;-\frac{xt}{1-t}\Big) \quad (|t|<1) \tag{1.18}$$

when $\lambda = -n$ $(n \in \mathbb{N}_0)$, $(\lambda)_\kappa := \Gamma(\lambda+\kappa)/\Gamma(\lambda)$ being the Pochhammer symbol (or the *shifted* factorial, since $(1)_k = k!$ for $k \in \mathbb{N}_0$).

## 2. Generating functions based upon the formulas (1.11) and (1.16)

By applying the formula (1.11), we first prove

**Theorem 2.** *Corresponding to an identically nonvanishing function* $\Omega_\mu(\xi_1,\ldots,\xi_s)$ *of* $s$ (*real or complex*) *variables* $\xi_1,\ldots,\xi_s$ $(s \in \mathbb{N} := \mathbb{N}_0 \setminus \{0\})$ *and of* (*complex*) *order* $\mu$, *let*

$$\Lambda_{n,p,q}^{(1)}[x;\xi_1,\ldots,\xi_s;z] := \sum_{k=0}^{[n/q]} a_k \ y_{n-qk}(x;\alpha+(\rho+1)qk,\beta)\Omega_{\mu+pk}(\xi_1,\ldots,\xi_s)z^k$$

$$(a_k \neq 0;\ n,k \in \mathbb{N}_0;\ p,q \in \mathbb{N}), \tag{2.1}$$

*where $\rho$ is a suitable complex parameter. Suppose also that*

$$\Theta_{k,n,p}^{\alpha,q,\rho}(x;\xi_1,\ldots,\xi_s;\eta):=\sum_{l=0}^{[k/q]}\binom{n-ql}{k-ql}a_l y_{n-k}(x;\alpha+\rho ql+k,\beta)\cdot\Omega_{\mu+pl}(\xi_1,\ldots,\xi_s)\eta^l. \tag{2.2}$$

*Then*

$$\sum_{k=0}^{n}\Theta_{k,n,p}^{\alpha,q,\rho}(x;\xi_1,\ldots,\xi_s;\eta)t^k=(1+t)^n\Lambda_{n,p,q}^{(1)}\left[\frac{x}{1+t};\xi_1,\ldots,\xi_s;\frac{\eta t^q}{(1+t)^q}\right], \tag{2.3}$$

*provided that each member of* (2.3) *exists.*

**Proof.** For convenience, let $\Delta(x,t)$ denote the first member of the assertion (2.3). Then, upon substituting for the polynomials

$$\Theta_{k,n,p}^{\alpha,q,\rho}(x;\xi_1,\ldots,\xi_s;\eta)$$

from the definition (2.2) into the left-hand side of (2.3), we obtain

$$\begin{aligned}\Delta(x,t)&=\sum_{k=0}^{n}t^k\sum_{l=0}^{[k/q]}\binom{n-ql}{k-ql}a_l\ y_{n-k}(x;\alpha+\rho ql+k,\beta)\Omega_{\mu+pl}(\xi_1,\ldots,\xi_s)\eta^l\\&=\sum_{l=0}^{[\eta/q]}a_l\ \Omega_{\mu+pl}(\xi_1,\ldots,\xi_s)(\eta t^q)^l\sum_{k=0}^{n-ql}\binom{n-ql}{k}\\&\quad\cdot y_{n-ql-k}(x;\alpha+(\rho+1)ql+k,\beta)t^k,\end{aligned}$$

which, in view of (1.11) with

$$n\mapsto n-ql \quad\text{and}\quad \alpha\mapsto\alpha+(\rho+1)ql \quad (l\in\mathbb{N}_0),$$

yields

$$\begin{aligned}\Delta(x,t)&=(1+t)^n\sum_{l=0}^{[n/q]}a_l\ y_{n-ql}\left(\frac{x}{1+t};\alpha+(\rho+1)ql,\beta\right)\\&\quad\cdot\Omega_{\mu+pl}(\xi_1,\ldots,\xi_l)\left(\frac{\eta t^q}{(1+t)^q}\right)^l,\end{aligned}$$

and the assertion (2.3) follows immediately by means of the definition (2.1). □

In a similar manner, by appealing to the formula (1.16), we are led fairly easily to

**Theorem 3.** *Corresponding to an identically nonvanishing function $\Omega_\mu(\xi_1,\ldots,\xi_s)$ of $s$ (real or complex) variables $\xi_1,\ldots,\xi_s$ ($s\in\mathbb{N}$) and of (complex) order $\mu$, let*

$$\Lambda^{(2)}_{n,p,q}[x;\xi_1,\ldots,\xi_s;z] := \sum_{k=0}^{[n/q]} a_k\ y_{n-qk}(x;\alpha+\rho k,\beta)\Omega_{\mu+pk}(\xi_1,\ldots,\xi_s)z^k$$
$$(a_k \neq 0;\ n,k\in\mathbb{N}_0;\ p,q\in\mathbb{N}), \tag{2.4}$$

*where $\rho$ is a suitable complex parameter. Suppose also that*

$$\Phi^{\alpha,q,\rho}_{k,n,p}(x;\xi_1,\ldots,\xi_s;\eta) := \sum_{l=0}^{[k/q]} \binom{n-ql}{k-ql}(\alpha+(\rho-1)ql+n-1)_{k-ql} \cdot y_{n-k}(x;\alpha+(\rho-2)ql+2k,\beta)\Omega_{\mu+pl}(\xi_1,\ldots,\xi_s)\eta^l. \tag{2.5}$$

*Then*

$$\sum_{k=0}^{n} \Phi^{\alpha,q,\rho}_{k,n,p}(x;\xi_1,\ldots,\xi_s;\eta)t^k = \Lambda^{(2)}_{n,p,q}[x+\beta t;\xi_1,\ldots,\xi_s;\eta t^q], \tag{2.6}$$

*provided that each member of* (2.6) *exists.*

## 3. Finite-series relationships involving the Stirling numbers of the second kind

We follow the work of Riordan [12, p. 90 *et seq.*] and denote by $S(n,k)$ the Stirling numbers of the second kind, defined by

$$S(n,k) := \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n, \tag{3.1}$$

so that

$$S(n,0)=\delta_{n,0}\ (n\in\mathbb{N}_0),\quad S(n,1)=S(n,n)=1,\quad \text{and}\quad S(n,k)=0\ (k>n), \tag{3.2}$$

where $\delta_{m,n}$ denotes the Kronecker delta.

Starting from the generating functions (1.11) and (1.16), and making use of the definition (3.1), it is not difficult to derive the following (presumably new) finite-series relationships (associated with the Stirling numbers $S(n,k)$ of the second kind) for the generalized Bessel polynomials:

$$\sum_{k=0}^{m}\binom{m}{k}k^n\ y_{m-k}(x;\alpha+k,\beta)z^k = (1+z)^m\sum_{k=0}^{\min(m,n)}\binom{m}{k}k!\ S(n,k)y_{m-k}\left(\frac{x}{1+z};\alpha+k,\beta\right)\left(\frac{z}{1+z}\right)^k$$
$$(m,n\in\mathbb{N}_0); \tag{3.3}$$

$$\sum_{k=0}^{m}\binom{m}{k}k^{n}(\alpha+m-1)_{k}\ y_{m-k}(x;\alpha+2k,\beta)z^{k}$$
$$=\sum_{k=0}^{\min(m,n)}\binom{m}{k}k!\ S(n,k)y_{m-k}(x+\beta z;\alpha+2k,\beta)z^{k}\quad(m,n\in\mathbb{N}_0).\tag{3.4}$$

Equivalently, for the classical Laguerre polynomials, the finite-series relationships (1.12) and (1.17) would readily yield

$$\sum_{k=0}^{m}\frac{k^{n}}{k!}L_{m-k}^{(\alpha+k)}(x)z^{k}=\sum_{k=0}^{\min(m,n)}S(n,k)L_{m-k}^{(\alpha+k)}(x-z)z^{k}\quad(m,n\in\mathbb{N}_0)\tag{3.5}$$

and

$$\sum_{k=0}^{m}\binom{\alpha+m}{k}k^{n}L_{m-k}^{(\alpha)}(x)z^{k}=(1+z)^{m}\sum_{k=0}^{\min(m,n)}\binom{\alpha+m}{k}k!\ S(n,k)\cdot L_{m-k}^{(\alpha)}\left(\frac{x}{1+z}\right)\left(\frac{z}{1+z}\right)^{k}\quad(m,n\in\mathbb{N}_0),\tag{3.6}$$

respectively.

In their special cases when $n=0$, the finite-series relationships (3.3) to (3.6), associated with the Stirling numbers $S(n,k)$ defined by (3.1), would reduce immediately to the known results (1.11), (1.16), (1.12), and (1.17), respectively.

## 4. Further remarks and observations

First of all, a duly corrected *and* modified analogue of the *main* result of Mukherjee [11, p. 87] can be deduced from Theorem 2 by first setting

$$p=q=s=1$$

and then making some obvious notational changes.

Each of our results (Theorems 2 and 3) can easily be restated in terms of the classical Laguerre polynomials by using the relationship (1.5) or (alternatively) by applying the known results (1.12) and (1.17) *directly*. For the sake of completeness, however, we merely state these variants of Theorems 2 and 3 as Theorems 4 and 5.

**Theorem 4.** *Corresponding to an identically nonvanishing function* $\Omega_\mu(\xi_1,\ldots,\xi_s)$ *of* $s$ (*real or complex*) *variables* $\xi_1,\ldots,\xi_s$ $(s\in\mathbb{N})$ *and of* (*complex*) *order* $\mu$, *let*

$$\Lambda^{(3)}_{n,p,q}[x;\xi_1,\ldots,\xi_s;z] := \sum_{k=0}^{[n/q]} a_k\, L^{(\alpha+(\rho+1)qk)}_{n-qk}(x)\Omega_{\mu+pk}(\xi_1,\ldots,\xi_s)z^k$$
$$(a_k \neq 0;\ n,k\in\mathbb{N}_0;\ p,q\in\mathbb{N}), \tag{4.1}$$

*where $\rho$ is a suitable complex parameter. Suppose also that*

$$\Psi^{\alpha,q,\rho}_{k,n,p}(x;\xi_1,\ldots,\xi_s;\eta) := \sum_{l=0}^{[k/q]} \frac{a_l}{(k-ql)!} L^{(\alpha+\rho ql+k)}_{n-k}(x)\Omega_{\mu+pl}(\xi_1,\ldots,\xi_s)\eta^l. \tag{4.2}$$

*Then*

$$\sum_{k=0}^{n} \Psi^{\alpha,q,\rho}_{k,n,p}(x;\xi_1,\ldots,\xi_s;\eta)t^k = \Lambda^{(3)}_{n,p,q}[x-t;\xi_1,\ldots,\xi_s;\eta t^q], \tag{4.3}$$

*provided that each member of* (4.3) *exists.*

**Theorem 5.** *Corresponding to an identically nonvanishing function $\Omega_\mu(\xi_1,\ldots,\xi_s)$ of $s$ (real or complex) variables $\xi_1,\ldots,\xi_s$ ($s\in\mathbb{N}$) and of (complex) order $\mu$, let*

$$\Lambda^{(4)}_{n,p,q}[x;\xi_1,\ldots,\xi_s;z] := \sum_{k=0}^{[n/q]} a_k\, L^{(\alpha+\rho qk)}_{n-qk}(x)\Omega_{\mu+pk}(\xi_1,\ldots,\xi_s)z^k$$
$$(a_k \neq 0;\ n,k\in\mathbb{N}_0;\ p,q\in\mathbb{N}), \tag{4.4}$$

*where $\rho$ is a suitable complex parameter. Suppose also that*

$$\Xi^{\alpha,q,\rho}_{k,n,p}(x;\xi_1,\ldots,\xi_s;\eta) := \sum_{l=0}^{[k/q]} \binom{\alpha+(\rho-1)ql+n}{k-ql} a_l \cdot L^{(\alpha+\rho ql)}_{n-k}(x)\Omega_{\mu+pl}(\xi_1,\ldots,\xi_s)\eta^l. \tag{4.5}$$

*Then*

$$\sum_{k=0}^{n} \Xi^{\alpha,q,\rho}_{k,n,p}(x;\xi_1,\ldots,\xi_s;\eta)t^k = (1+t)^n \Lambda^{(4)}_{n,p,q}\left[\frac{x}{1+t};\xi_1,\ldots,\xi_s;\frac{\eta t^q}{(1+t)^q}\right], \tag{4.6}$$

*provided that each member of* (4.6) *exists.*

Other *linear* generating functions for the Bessel polynomials, which have already been exploited extensively in deriving the corresponding substantially general families of bilinear, bilateral, and mixed multilateral generating functions, include (for example) the following results:

$$\sum_{k=0}^{\infty} y_{n+k}(x)\frac{t^k}{k!} = (1-2xt)^{-(1/2)(n+1)} \exp\left(x^{-1}\left\{1-\sqrt{1-2xt}\right\}\right) \cdot y_n\left(\frac{x}{\sqrt{1-2xt}}\right) \quad \left(n\in\mathbb{N}_0;\ |xt|<\frac{1}{2}\right), \tag{4.7}$$

which yields Corollary 2 of Srivastava [13, Part I, p. 229] (see also [16, p. 421, Corollary 2]), given by

**Theorem 6.** *Corresponding to an identically nonvanishing function* $\Omega_\mu(\xi_1,\dots,\xi_s)$ *of* $s$ (*real or complex*) *variables* $\xi_1,\dots,\xi_s$ $(s\in\mathbb{N})$ *and of* (*complex*) *order* $\mu$, *let*

$$\Lambda^{(5)}_{m,p,q}[x;\xi_1,\dots,\xi_s;z] := \sum_{n=0}^{\infty} a_n\ y_{m+qn}(x)\Omega_{\mu+pn}(\xi_1,\dots,\xi_s)\frac{z^n}{(qn)!} \quad (a_n\neq 0;\ m\in\mathbb{N}_0;\ p,q\in\mathbb{N}). \tag{4.8}$$

*Suppose also that*

$$M^{p,\mu}_{n,q}(\xi_1,\dots,\xi_s;\eta) := \sum_{k=0}^{[n/q]}\binom{n}{qk}a_k\ \Omega_{\mu+pk}(\xi_1,\dots,\xi_s)\eta^k. \tag{4.9}$$

*Then*

$$\sum_{n=0}^{\infty} y_{m+n}(x)M^{p,\mu}_{n,q}(\xi_1,\dots,\xi_s;\eta)\frac{t^n}{n!} = (1-2xt)^{-(1/2)(m+1)}\exp\left(x^{-1}\left\{1-\sqrt{1-2xt}\right\}\right) \cdot \Lambda^{(5)}_{m,p,q}\left[\frac{x}{\sqrt{1-2xt}};\xi_1,\dots,\xi_s;\eta\left(\frac{t}{\sqrt{1-2xt}}\right)^q\right] \quad \left(|t|<\frac{1}{2}|x|^{-1}\right), \tag{4.10}$$

*provided that each member of* (4.10) *exists.*

$$\sum_{k=0}^{\infty} y_{n+k}(x;\alpha-k,\beta)\frac{t^k}{k!} = \left(1-\frac{xt}{\beta}\right)^{1-\alpha-n} \mathrm{e}^t y_n\left(\frac{\beta x}{\beta-xt};\alpha,\beta\right) \quad (n\in\mathbb{N}_0;\ |t|<|\beta/x|), \tag{4.11}$$

which yields Theorem 3 of Chen and Srivastava [3, p. 154], that is,

**Theorem 7.** *Corresponding to an identically nonvanishing function* $\Omega_\mu(\xi_1,\dots,\xi_s)$ *of* $s$ (*real or complex*) *variables* $\xi_1,\dots,\xi_s$ $(s\in\mathbb{N})$ *and of* (*complex*) *order* $\mu$, *let*

$$\Lambda^{(6)}_{m,p,q}[x;\xi_1,\ldots,\xi_s;z] := \sum_{n=0}^{\infty} a_n\; y_{m+qn}(x;\alpha+(\rho-1)qn,\beta) \cdot \Omega_{\mu+pn}(\xi_1,\ldots,\xi_s)\frac{z^n}{(qn)!}$$
$$(a_n \neq 0;\; m\in\mathbb{N}_0;\; p,q\in\mathbb{N}), \tag{4.12}$$

*where* $\rho$ *is a suitable complex parameter. Suppose also that*

$$N^{\alpha,\mu,\rho}_{n,p,q}(x;\xi_1,\ldots,\xi_s;\eta) := \sum_{k=0}^{[n/q]} \binom{n}{qk} a_k\; y_{m+n}(x;\alpha-n+\rho qk,\beta) \cdot \Omega_{\mu+pk}(\xi_1,\ldots,\xi_s)\eta^k. \tag{4.13}$$

*Then*

$$\sum_{n=0}^{\infty} N^{\alpha,\mu,\rho}_{n,p,q}(x;\xi_1,\ldots,\xi_s;\eta)\frac{t^n}{n!} = \left(1-\frac{xt}{\beta}\right)^{1-\alpha-m} \mathrm{e}^t \Lambda^{(6)}_{m,p,q}\left[\frac{\beta x}{\beta-xt};\xi_1,\ldots,\xi_s;\eta t^q\left(1-\frac{xt}{\beta}\right)^{-\rho q}\right]$$
$$(|t|<|\beta/x|), \tag{4.14}$$

*provided that each member of* (4.14) *exists.*

$$\sum_{k=0}^{\infty}\binom{\alpha+n+k-2}{k} y_n(x;\alpha+k,\beta)t^k = (1-t)^{1-\alpha-n}\; y_n\left(\frac{x}{1-t};\alpha,\beta\right) \quad (n\in\mathbb{N}_0;\; |t|<1), \tag{4.15}$$

which yields Theorem 5 of Chen et al. [2, p. 363], given by

**Theorem 8.** *Corresponding to an identically nonvanishing function* $\Omega_\mu(\xi_1,\ldots,\xi_s)$ *of* $s$ (*real or complex*) *variables* $\xi_1,\ldots,\xi_s$ $(s\in\mathbb{N})$ *and of* (*complex*) *order* $\mu$, *let*

$$\Lambda^{(7)}_{p,q}[x;\xi_1,\ldots,\xi_s;z] := \sum_{k=0}^{\infty} a_k\; y_n(x;\alpha+(\rho+1)qk,\beta)\Omega_{\mu+pk}(\xi_1,\ldots,\xi_s)z^k$$
$$(a_k\neq 0;\; n\in\mathbb{N}_0;\; p,q\in\mathbb{N}), \tag{4.16}$$

*where* $\rho$ *is a suitable complex parameter. Suppose also that*

$$P^{\alpha,\mu,\rho}_{k,p,q}(x;\xi_1,\ldots,\xi_s;\eta) := \sum_{l=0}^{[k/q]} \binom{\alpha+n+k+\rho ql-2}{k-ql} a_l y_n(x;\alpha+k+\rho ql,\beta) \cdot \Omega_{\mu+pl}(\xi_1,\ldots,\xi_s)\eta^l. \tag{4.17}$$

*Then*

$$\sum_{k=0}^{\infty} P_{k,p,q}^{\alpha,\mu,\rho}(x;\xi_1,\ldots,\xi_s;\eta)t^k = (1-t)^{1-\alpha-n}\Lambda_{p,q}^{(7)}\left[\frac{x}{1-t};\xi_1,\ldots,\xi_s;\frac{\eta t^q}{(1-t)^{(\rho+1)q}}\right] \quad (|t|<1), \tag{4.18}$$

*provided that each member of* (4.18) *exists.*

$$\sum_{k=0}^{\infty} y_n(x;\alpha-k,\beta)\frac{t^k}{k!} = \left(1-\frac{xt}{\beta}\right)^n \mathrm{e}^t\, y_n\left(\frac{\beta x}{\beta-xt};\alpha,\beta\right) \quad (n\in\mathbb{N}_0), \tag{4.19}$$

which yields Theorem 6 of Chen et al. [2, p. 364], that is,

**Theorem 9.** *Corresponding to an identically nonvanishing function* $\Omega_\mu(\xi_1,\ldots,\xi_s)$ *of* $s$ (*real or complex*) *variables* $\xi_1,\ldots,\xi_s$ $(s\in\mathbb{N})$ *and of* (*complex*) *order* $\mu$, *let*

$$\Lambda_{p,q}^{(8)}[x;\xi_1,\ldots,\xi_s;z] := \sum_{k=0}^{\infty} a_k\, y_n(x;\alpha+(\rho-1)qk,\beta)\Omega_{\mu+pk}(\xi_1,\ldots,\xi_s)\frac{z^k}{(qk)!} \quad (\alpha_k\neq 0;\ n\in\mathbb{N}_0;\ p,q\in\mathbb{N}), \tag{4.20}$$

*where* $\rho$ *is a suitable complex parameter. Suppose also that*

$$Q_{k,p,q}^{\alpha,\mu,\rho}(x;\xi_1,\ldots,\xi_s;\eta) := \sum_{l=0}^{[k/q]}\binom{k}{ql} a_l\, y_n(x;\alpha-k+\rho ql,\beta) \cdot \Omega_{\mu+pl}(\xi_1,\ldots,\xi_s)\eta^l. \tag{4.21}$$

*Then*

$$\sum_{k=0}^{\infty} Q_{k,p,q}^{\alpha,\mu,\rho}(x;\xi_1,\ldots,\xi_s;\eta)\frac{t^k}{k!} = \left(1-\frac{xt}{\beta}\right)^n e^t\Lambda_{p,q}^{(8)}\left[\frac{\beta x}{\beta-xt};\xi_1,\ldots,\xi_s;\eta t^q\right], \tag{4.22}$$

*provided that each member of* (4.22) *exists.*

$$\sum_{k=0}^{\infty} y_{n+k}(x;\alpha-2k,\beta)\frac{(-\beta t)^k}{k!} = (1-xt)^{\alpha-2}\exp\left(-\frac{\beta t}{1-xt}\right) \cdot y_n(x(1-xt);\alpha,\beta) \quad (n\in\mathbb{N}_0;\ |t|<|x|^{-1}), \tag{4.23}$$

which yields Srivastava's theorem [15, p. 129]:

**Theorem 10.** *Corresponding to an identically nonvanishing function* $\Omega_\mu(\xi_1,\ldots,\xi_s)$ *of* $s$ (*real or complex*) *variables* $\xi_1,\ldots,\xi_s$ $(s\in\mathbb{N})$ *and of* (*complex*) *order* $\mu$, *let*

$$\Lambda^{(9)}_{m,p,q}[x;\xi_1,\ldots,\xi_s;z] := \sum_{n=0}^{\infty} a_n \; y_{m+qn}(x;\alpha+(\rho-2)qn,\beta) \cdot \Omega_{\mu+pn}(\xi_1,\ldots,\xi_s)\frac{z^n}{(qn)!}$$

$$(a_n \neq 0;\; m\in\mathbb{N}_0;\; p,q\in\mathbb{N}), \tag{4.24}$$

*where* $\rho$ *is a suitable complex parameter. Suppose also that*

$$R^{\alpha,\mu,\rho}_{n,p,q}(x;\xi_1,\ldots,\xi_s;\eta)\frac{t^n}{n!} := \sum_{k=0}^{[n/q]} \binom{n}{qk} a_k \; y_{m+n}(x;\alpha-2n+\rho qk,\beta) \cdot \Omega_{\mu+pk}(\xi_1,\ldots,\xi_s)\eta^k. \tag{4.25}$$

*Then*

$$\sum_{n=0}^{\infty} R^{\alpha,\mu,\rho}_{n,p,q}(x;\xi_1,\ldots,\xi_s;\eta)\frac{t^n}{n!} = \left(1+\frac{xt}{\beta}\right)^{\alpha-2}\exp\left(\frac{\beta t}{\beta+xt}\right) \cdot \Lambda^{(9)}_{m,p,q}\left[x\left(1+\frac{xt}{\beta}\right);\xi_1,\ldots,\xi_s;\eta t^q\left(1+\frac{xt}{\beta}\right)^{(\rho-2)q}\right] \quad (|t|<|\beta/x|), \tag{4.26}$$

*provided that each member of* (4.26) *exists.*

Just as in *all* of the aforementioned theorems on bilinear, bilateral, and mixed multilateral generating functions, for *each* suitable choice of the coefficients $a_n$ $(n\in\mathbb{N}_0)$, if we express the multivariable function

$$\Omega_\mu(\xi_1,\ldots,\xi_s) \quad (s\in\mathbb{N}\setminus\{1\}) \tag{4.27}$$

as an appropriate product of several relatively simpler (known or new) functions, *each* of the results (which we presented in Section 2) can be shown to yield various families of *mixed* multilateral generating functions for the generalized Bessel polynomials. We choose to leave the detailed demonstration of this observation as an exercise for the interested reader.

In the preceding section, we made use of the definition (3.1) in conjunction with the generating functions (1.11) and (1.16) in order to derive the finite-series relationships (3.3) and (3.4) associated with the Stirling numbers $S(n,k)$. As a matter of fact, we can similarly apply each of the generating functions (4.7), (4.11), (4.15), (4.19), and (4.23). Thus we obtain the following generating functions (associated with the Stirling numbers $S(n,k)$ of the second kind) for the simple as well as generalized Bessel polynomials:

$$\sum_{k=0}^{\infty} \frac{k^n}{k!} y_k\left(\frac{x}{\sqrt{1+2xz}}\right)\left(\frac{z}{\sqrt{1+2xz}}\right)^k$$

$$= \sqrt{1+2xz}\,\exp\left(-x^{-1}\left\{1-\sqrt{1+2xz}\right\}\right)\sum_{k=0}^{n} S(n,k)y_k(x)z^k$$

$$\left(n \in \mathbb{N}_0;\ |z| < \frac{1}{2}|x|^{-1}\right), \tag{4.28}$$

$$\sum_{k=0}^{\infty} \frac{k^n}{k!} y_k(x;\alpha-k,\beta)z^k = \left(1-\frac{xz}{\beta}\right)^{1-\alpha} \mathrm{e}^z \sum_{k=0}^{n} S(n,k)y_k\left(\frac{\beta x}{\beta - xz};\alpha-k,\beta\right)z^k$$

$$(n \in \mathbb{N}_0;\ |z| < |\beta/x|), \tag{4.29}$$

$$\sum_{k=0}^{\infty} \binom{\alpha+N+k-2}{k} k^n y_N(x;\alpha+k,\beta)z^k$$

$$= (1-z)^{1-\alpha-N}\sum_{k=0}^{n}\binom{\alpha}{k}k!\,y_N\left(\frac{x}{1-z};\alpha+k,\beta\right)\left(\frac{z}{1-z}\right)^k$$

$$(n \in \mathbb{N}_0;\ N \in \mathbb{N}_0;\ |z| < 1), \tag{4.30}$$

$$\sum_{k=0}^{\infty} \frac{k^n}{k!} y_N(x;\alpha-k,\beta)z^k = \left(1-\frac{xz}{\beta}\right)^{n} \mathrm{e}^z \sum_{k=0}^{n} S(n,k)y_N\left(\frac{\beta x}{\beta - xz};\alpha-k,\beta\right)z^k$$

$$(n \in \mathbb{N}_0;\ N \in \mathbb{N}_0), \tag{4.31}$$

and

$$\sum_{k=0}^{\infty} \frac{k^n}{k!} y_k(x;\alpha-2k,\beta)z^k$$

$$= \left(1+\frac{xz}{\beta}\right)^{\alpha-2}\exp\left(\frac{\beta z}{\beta+xz}\right)$$

$$\cdot \sum_{k=0}^{n} S(n,k)y_k\left(x\left(1+\frac{xz}{\beta}\right);\alpha-2k,\beta\right)\left[z\left(1+\frac{xz}{\beta}\right)^{-2}\right]^k$$

$$(n \in \mathbb{N}_0;\ |z| < |\beta/x|), \tag{4.32}$$

respectively.

Evidently, by appealing to the relationship (1.5), one can easily rewrite each of the above generating functions in terms of the classical Laguerre polynomials $L_n^{(\alpha)}(x)$ defined by (1.7). In the cases of the generating function (4.28) and Theorem 6 *as well*, one can make make use of the relationship:

$$y_n(x) = n!\left(-\frac{x}{2}\right)^n L_n^{(-2n-1)}\left(\frac{2}{x}\right), \tag{4.33}$$

which, in view of (1.3), follows from (1.5) *with* $\alpha = \beta = 2$.

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