A NOTE ON q-BERNSTEIN POLYNOMIALS

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Abstract. Recently, Simsek-Acikgoz([17]) and Kim-Jang-Yi([9]) have studied the q-extension of Bernstein polynomials. In this paper we propose the q-extension of Bernstein polynomials of degree n, which are different q-Bernstein polynomials of Simsek-Acikgoz([17]) and Kim-Jang-Yi([9]). From these q-Bernstein polynomials, we derive some fermionic p-adic integral representations of several q-Bernstein type polynomials. Finally, we investigate some identities between q-Bernstein polynomials and q-Euler numbers.

§1. Introduction

Let $C[0,1]$ denote the set of continuous function on $[0,1]$. For $f \in C[0,1]$, Bernstein introduced the following well known linear operators (see [1, 3]):

$$\mathbb{B}_n(f|x) = \sum_{k=0}^{n} f(\frac{k}{n})\binom{n}{k}x^k(1-x)^{n-k} = \sum_{k=0}^{n} f(\frac{k}{n})B_{k,n}(x). \tag{1}$$

Here $\mathbb{B}_n(f|x)$ is called Bernstein operator of order n for f. For $k, n \in \mathbb{Z}_+(= \mathbb{N}\cup\{0\})$, the Bernstein polynomials of degree n is defined by

$$B_{k,n}(x) = \binom{n}{k}x^k(1-x)^{n-k}, \text{ (see [1, 2, 3]).} \tag{2}$$

A Bernoulli trial involves performing an experiment once and noting whether a particular event A occurs. The outcome of Bernoulli trial is said to be "success" if A occurs and a "failure" otherwise. Let k be the number of successes in n independent Bernoulli trials, the probabilities of k are given by the binomial probability law:

$$p_n(k) = \binom{n}{k}p^k(1-p)^{n-k}, \text{ for } k = 0, 1, \cdots, n,$$

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where $p_n(k)$ is the probability of k successes in n trials. For example, a communication system transmit binary information over channel that introduces random bit errors with probability $\xi = 10^{-3}$. The transmitter transmits each information bit three times, an a decoder takes a majority vote of the received bits to decide on what the transmitted bit was. The receiver can correct a single error, but it will make the wrong decision if the channel introduces two or more errors. If we view each transmission as a Bernoulli trial in which a "success" corresponds to the introduction of an error, then the probability of two or more errors in three Bernoulli trial is

$$p(k \geq 2) = \binom{3}{2}(0.001)^2(0.999) + \binom{3}{3}(0.001)^3 \approx 3(10^{-6}), \text{ see [18].}$$

By the definition of Bernstein polynomials(see Eq.(1) and Eq.(2)), we can see that Bernstein basis is the probability mass function of binomial distribution. In the reference [15] and [16], Phillips proposed a generalization of classical Bernstein polynomials based on q-integers. In the last decade some new generalizations of well known positive linear operators based on q-integers were introduced and studied by several authors(see [1-21]). Let $0 < q < 1$. Define the q-numbers of x by $[x]_q = \frac{1-q^x}{1-q}$ (see [1-21]). Recently, Simsek-Acikgoz([17]) and Kim-Jang-Yi([9]) have studied the q-extension of Bernstein polynomials, which are different Phillips q-Bernstein polynomials. Let p be a fixed odd prime number. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ denote the rings of p-adic integers, the fields of p-adic rational numbers, and the completion of algebraic closure of $\mathbb{Q}_p$, respectively. The p-adic absolute value in $\mathbb{C}_p$ is normalized in such way that $|p|_p = \frac{1}{p}$. As well known definition, Euler polynomials are defined by

$$(3) \qquad \frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \text{ (see [1-14]).}$$

In the special case, $x = 0$, $E_n(0) = E_n$ are called the n-th Euler numbers. By (3), we see that the recurrence formula of Euler numbers is given by

$$(4) \qquad E_0 = 1, \text{ and } (E+1)^n + E_n = 0 \text{ if } n > 0, \text{ (see [12]),}$$

with the usual convention of replacing E^n by E_n. When one talks of q-analogue, q is variously considered as an indterminate, a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, we normally assume $|q| < 1$. If $q \in \mathbb{C}_p$, we normally always assume that $|1-q|_p < 1$. As the q-extension of (4), author defined the q-Euler numbers as follows:

$$(5) \qquad E_{0,q} = 1, \text{ and } (qE_q+1)^n + E_{n,q} = 0 \text{ if } n > 0, \text{ (see [21]),}$$

with the usual convention of replacing E_q^n by $E_{n,q}$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable function on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the fermionic p-adic q-integral was defined by

$$(6) \qquad I_q(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{1+q^{p^N}} \sum_{x=0}^{p^N-1} f(x)(-q)^x, \text{ (see [12]).}$$

In the special case, $q=1$, $I_1(f)$ is called the fermionic p-adic integral on $\mathbb{Z}_p$ (see [12, 21]). By (6) and the definition of $I_1(f)$, we see that

$$I_1(f_1)+I_1(f)=2f(0), \text{ where } f_1(x)=f(x+1). \tag{7}$$

For $n\in\mathbb{N}$, let $f_n(x)=f(x+n)$. Then we can also see that

$$I_1(f_n)+(-1)^{n-1}I_1(f)=2\sum_{l=0}^{n-1}(-1)^{n-l-1}f(l), \text{ (see [21]).} \tag{8}$$

From (5), (7) and (8), we note that

$$\int_{\mathbb{Z}_p} e^{[x]_q t}d\mu_{-1}(x)=\sum_{n=0}^{\infty}E_{n,q}\frac{t^n}{n!}=\sum_{n=0}^{\infty}\left(\frac{2}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}\frac{(-1)^l}{1+q^l}\right)\frac{t^n}{n!}. \tag{9}$$

Thus we have

$$E_{n,q}=\frac{2}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}\frac{(-1)^l}{1+q^l}, \text{ (see [21]).}$$

In [21], the q-Euler polynomials are defined by

$$E_{n,q}(x)=\int_{\mathbb{Z}_p}[y+x]_q^n d\mu_{-1}(x)=\frac{1}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{q^{lx}}{1+q^l}. \tag{10}$$

By (9) and (100, we get

$$E_{n,q}(x)=\sum_{l=0}^{n}\binom{n}{l}q^{lx}E_{l,q}=(q^xE_q+1)^n, \tag{11}$$

with the usual convention of replacing E_q^n by $E_{n,q}$. In this paper we firstly consider the q-Bernstein polynomials of degree n in $\mathbb{R}$, which are different q-Bernstein polynomials of Simsek-Acikgoz([17]) and Kim-Jang-Yi([9]). From these q-Bernstein polynomials, we try to study for the fermionic p-adic integral representations of the several q-Bernstein type polynomials on $\mathbb{Z}_p$. Finally, we give some interesting identities between q-Bernstein polynomials and q-Euler numbers.

§2. q-Bernstein Polynomials

For $n,k\in\mathbb{Z}_+$, the generating function for $B_{k,n}(x)$ is introduced by Acikgoz and Araci as follows:

$$F^{(k)}(t,x)=\frac{te^{(1-x)t}x^k}{k!}=\sum_{n=0}^{\infty}B_{k,n}(x)\frac{t^n}{n!}, \text{ (see [1, 9, 10, 17]).} \tag{12}$$

For $k, n \in \mathbb{Z}_+$, $0 < q < 1$ and $x \in [0,1]$, consider the q-extension of (12) as follows:
(13)

$$F_q^{(k)}(t,x) = \frac{(t[x]_q)^k e^{[1-x]_{\frac{1}{q}} t}}{k!} = \frac{[x]_q^k}{k!} \sum_{n=0}^{\infty} \frac{[1-x]_{\frac{1}{q}}^n}{n!} t^{n+k} = \sum_{n=k}^{\infty} \left(\frac{n! [x]_q^k [1-x]_{\frac{1}{q}}^{n-k}}{(n-k)! k!} \right) \frac{t^n}{n!}$$
$$= \sum_{n=k}^{\infty} \binom{n}{k} [x]_q^k [1-x]_{\frac{1}{q}}^{n-k} \frac{t^n}{n!} = \sum_{n=k}^{\infty} B_{k,n}(x,q) \frac{t^n}{n!}.$$

Because $B_{k,0}(x,q) = B_{k,1}(x,q) = \cdots = B_{k,k-1}(x,q) = 0$, we obtain the following generating function for $B_{k,n}(x,q)$:

$$F_q^{(k)}(t,x) = \frac{(t[x]_q)^k e^{[1-x]_{\frac{1}{q}} t}}{k!} = \sum_{n=0}^{\infty} B_{k,n}(x,q) \frac{t^n}{n!}, \text{ where } k \in \mathbb{Z}_+ \text{ and } x \in [0,1].$$

Thus, for $k, n \in \mathbb{Z}_+$, we note that

$$B_{k,n}(x,q) = \binom{n}{k} [x]_q^k [1-x]_{\frac{1}{q}}^{n-k}, \text{ if } n \geq k\ , \qquad (14)$$
$$= 0, \text{ if } k < n.$$

By (14), we easily get $\lim_{q \to 1} B_{k,n}(x,q) = B_{k,n}(x)$. For $0 \leq k \leq n$, we have

$$[1-x]_{\frac{1}{q}} B_{k,n-1}(x,q) + [x]_q B_{k-1,n-1}(x,q)$$
$$= [1-x]_{\frac{1}{q}} \binom{n-1}{k} [x]_q^k [1-x]_{\frac{1}{q}}^{n-k-1} + [x]_q \binom{n-1}{k-1} [x]_q^{k-1} [1-x]_{\frac{1}{q}}^{n-k}$$
$$= \binom{n-1}{k} [x]_q^k [1-x]_{\frac{1}{q}}^{n-k} + \binom{n-1}{k-1} [x]_q^k [1-x]_{\frac{1}{q}}^{n-k} = \binom{n}{k} [x]_q^k [1-x]_{\frac{1}{q}}^{n-k},$$

and the derivative of the q-Bernstein polynomials of degree n are also polynomials of degree $n-1$.

$$\frac{d}{dx} B_{k-1,n}(x,q)$$
$$= k \binom{n}{k} [x]_q^{k-1} [1-x]_{\frac{1}{q}}^{n-k} \left(\frac{\log q}{q-1} \right) q^x + \binom{n}{k} [x]_q^k (n-k) [1-x]_{\frac{1}{q}}^{n-k-1} \left(\frac{\log q}{1-q} \right) q^x$$
$$= \frac{\log q}{q-1} q^x \left(n \binom{n-1}{k-1} [x]_q^{k-1} [1-x]_{\frac{1}{q}}^{n-k} - n \binom{n-1}{k} [x]_q^k [1-x]_{\frac{1}{q}}^{n-1-k} \right)$$
$$= n \left(B_{k-1,n-1}(x,q) - B_{k,n-1}(x,q) \right) \frac{\log q}{q-1} q^x.$$

Therefore, we obtain the following theorem.

Theorem 1. *For $k, n \in \mathbb{Z}_+$ and $x \in [0,1]$, we have*

$$[1-x]_{\frac{1}{q}} B_{k,n-1}(x,q) + [x]_q B_{k-1,n-1}(x,q) = B_{k,n}(x,q),$$

and

$$\frac{d}{dx} B_{k,n}(x,q) = n\left(B_{k-1,n-1}(x,q) - B_{k,n-1}(x,q)\right) \frac{\log q}{q-1} q^x.$$

Let f be a continuous function on $[0,1]$. Then the q-Bernstein operator of order n for f is defined by

$$\mathbb{B}_{n,q}(f|x) = \sum_{k=0}^{n} f(\frac{k}{n}) B_{k,n}(x,q), \text{ where } 0 \le x \le 1 \text{ and } n \in \mathbb{Z}_+. \tag{15}$$

By (14) and (15), we see that

$$\mathbb{B}_{n,q}(1|x) = \sum_{k=0}^{n} B_{k,n}(x,q) = \sum_{k=0}^{n} \binom{n}{k} [x]_q^k [1-x]_{\frac{1}{q}}^{n-k} = \left([x]_q + [1-x]_{\frac{1}{q}}\right)^n = 1.$$

Also, we get from (15) that for $f(x) = x$,

$$\mathbb{B}_{n,q}(x|x) = \sum_{k=0}^{n} \frac{k}{n} \binom{n}{k} [x]_q^k [1-x]_{\frac{1}{q}}^{n-k} = \sum_{k=0}^{n-1} \binom{n-1}{k} [x]_q^{k+1} [1-x]_{\frac{1}{q}}^{n-k-1} = [x]_q.$$

The q-Bernstein polynomials are symmetric polynomials in the following sense:

$$B_{n-k,n}(1-x, \frac{1}{q}) = \binom{n}{n-k} [1-x]_{\frac{1}{q}}^{n-k} [x]_q^k = B_{k,n}(x,q).$$

Thus, we obtain the following theorem.

Theorem 2. *For $n, k \in \mathbb{Z}_+$ and $(x \in [0,1]$, we have*

$$B_{n-k,n}(1-x, \frac{1}{q}) = B_{k,n}(x,q).$$

Moreover, $\mathbb{B}_{n,q}(1|x) = 1$ and $\mathbb{B}_{n,q}(x|x) = [x]_q$.

From (15), we note that

$$\begin{aligned} \mathbb{B}_{n,q}(f|x) &= \sum_{k=0}^{n} f(\frac{k}{n}) B_{k,n}(x,q) = \sum_{k=0}^{n} f(\frac{k}{n}) \binom{n}{k} [x]_q^k [1-x]_{\frac{1}{q}}^{n-k} \\ &= \sum_{k=0}^{n} f(\frac{k}{n}) \binom{n}{k} [x]_q^k \sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^j [x]_q^j. \end{aligned} \tag{16}$$

By the definition of binomial coefficient, we easily get

$$\binom{n}{k}\binom{n-k}{j} = \binom{n}{k+j}\binom{k+j}{k}.$$

Let $k+j=m$. Then we have

$$\binom{n}{k}\binom{n-k}{j} = \binom{n}{m}\binom{m}{k}. \tag{17}$$

From (16) and (17), we have

$$\mathbb{B}_{n,q}(f|x) = \sum_{m=0}^{n}\binom{n}{m}[x]_q^m \sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k} f(\frac{k}{n}). \tag{18}$$

Therefore, we obtain the following theorem.

Theorem 3. *For $f \in C[0,1]$ and $n \in \mathbb{Z}_+$, we have*

$$\mathbb{B}_{n,q}(f|x) = \sum_{m=0}^{n}\binom{n}{m}[x]_q^m \sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k} f(\frac{k}{n}).$$

It is well known that the second kind stirling numbers are defined by

$$\frac{(e^t-1)^k}{k!} = \frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}e^{lt} = \sum_{n=0}^{\infty} s(n,k)\frac{t^n}{n!}, \text{ for } k \in \mathbb{N}, \text{ (see [12, 21])}. \tag{19}$$

Let Δ be the shift difference operator with $\Delta f(x) = f(x+1) - f(x)$. By iterative process, we easily get

$$\Delta^n f(0) = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k} f(k). \tag{20}$$

From (19) and (20), we can easily derive the following equation (21).

$$\frac{1}{k!}\Delta^k 0^n = s(n,k). \tag{21}$$

By (18) and (20) we obtain the following theorem.

Theorem 4. *For $f \in C[0,1]$ and $n \in \mathbb{Z}_+$, we have*

$$\mathbb{B}_{n,q}(f|x) = \sum_{k=0}^{n}\binom{n}{k}[x]_q^k \Delta^k f(\frac{0}{n}).$$

In the special case, $f(x) = x^m (m \in \mathbb{Z}_+)$, we have the following corollary.

Corollary 5. *For $x \in [0,1]$ and $m, n \in \mathbb{Z}_+$, we have*

$$n^m \mathbb{B}_{n,q}(x^m|x) = \sum_{k=0}^{n} \binom{n}{k} [x]_q^k \Delta^k 0^m,$$

and

$$n^m \mathbb{B}_{n,q}(x^m|x) = \sum_{k=0}^{n} \binom{n}{k} [x]_q^k k! s(m,k).$$

For $x, t \in \mathbb{C}$ and $n \in \mathbb{Z}_+$ with $n \geq k$, consider

$$\frac{n!}{2\pi i} \int_C \frac{([x]_q t)^k}{k!} e^{([1-x]_{\frac{1}{q}} t)} \frac{dt}{t^{n+1}}, \tag{22}$$

where C is a circle around the origin and integration is in the positive direction. We see from the definition of the q-Bernstein polynomials and the basic theory of complex analysis including Laurent series that

$$\int_C \frac{([x]_q t)^k}{k!} e^{([1-x]_{\frac{1}{q}} t)} \frac{dt}{t^{n+1}} = \sum_{m=0}^{\infty} \int_C \frac{B_{k,m}(x,q) t^m}{m!} \frac{dt}{t^{n+1}} = 2\pi i \left(\frac{B_{k,n}(x,q)}{n!} \right). \tag{23}$$

We get from (22) and (23) that

$$\frac{n!}{2\pi i} \int_C \frac{([x]_q t)^k}{k!} e^{([1-x]_{\frac{1}{q}} t)} \frac{dt}{t^{n+1}} = B_{k,n}(x,q), \tag{24}$$

and

$$\begin{aligned} \int_C \frac{([x]_q t)^k}{k!} e^{([1-x]_{\frac{1}{q}} t)} \frac{dt}{t^{n+1}} &= \frac{[x]_q^k}{k!} \sum_{m=0}^{\infty} \left(\frac{[1-x]_{\frac{1}{q}}^m}{m!} \int_C t^{m-n-1+k} dt \right) \\ &= 2\pi i \left(\frac{[x]_q^k [1-x]_{\frac{1}{q}}^{n-k}}{k!(n-k)!} \right) = \frac{2\pi i}{n!} \binom{n}{k} [x]_q^k [1-x]_{\frac{1}{q}}^{n-k}. \end{aligned} \tag{25}$$

By (22) and (25), we see that

$$\frac{n!}{2\pi i} \int_C \frac{([x]_q t)^k}{k!} e^{([1-x]_{\frac{1}{q}} t)} \frac{dt}{t^{n+1}} = \binom{n}{k} [x]_q^k [1-x]_{\frac{1}{q}}^{n-k}. \tag{26}$$

From (24) and (26), we note that

$$B_{k,n}(x,q) = \binom{n}{k} [x]_q^k [1-x]_{\frac{1}{q}}^{n-k}.$$

By the definition of q-Bernstein polynomials, we easily get

$$\begin{aligned}
&\left(\frac{n-k}{n}\right) B_{k,n}(x,q) + \left(\frac{k+1}{n}\right) B_{k+1,n}(x,q) \\
&= \left(\frac{(n-1)!}{k!(n-k-1)!}\right) [x]_q^k [1-x]_{\frac{1}{q}}^{n-k} + \left(\frac{(n-1)!}{k!(n-k-1)!}\right) [x]_q^{k+1} [1-x]_{\frac{1}{q}}^{n-k-1} \\
&= \left([1-x]_{\frac{1}{q}} + [x]_q\right) B_{k,n-1}(x,q) = B_{k,n-1}(x,q).
\end{aligned}$$

Therefore, we can write q-Bernstein polynomials as a linear combination of polynomials of higher order.

Theorem 6. *For $k, n \in \mathbb{Z}_+$ and $x \in [0,1]$, we have*

$$\left(\frac{n+1-k}{n+1}\right) B_{k,n+1}(x,q) + \left(\frac{k+1}{n+1}\right) B_{k+1,n+1}(x,q) = B_{k,n}(x,q).$$

We easily get from (14) that for $n, k \in \mathbb{N}$,

$$\begin{aligned}
&\left(\frac{n-k+1}{k}\right) \left(\frac{[x]_q}{[1-x]_{\frac{1}{q}}}\right) B_{k-1,n}(x,q) \\
&= \left(\frac{n-k+1}{k}\right) \left(\frac{[x]_q}{[1-x]_{\frac{1}{q}}}\right) \binom{n}{k-1} [x]_q^{k-1} [1-x]_{\frac{1}{q}}^{n-k+1} \\
&= \left(\frac{n!}{k!(n-k)!}\right) [x]_q^k [1-x]_{\frac{1}{q}}^{n-k} = B_{k,n}(x,q).
\end{aligned}$$

Therefore, we obtain the following corollary.

Corollary 7. *For $k, n \in \mathbb{N}$ and $x \in [0,1]$, we have*

$$\left(\frac{n-k+1}{k}\right) \left(\frac{[x]_q}{[1-x]_{\frac{1}{q}}}\right) B_{k-1,n}(x,q) = B_{k,n}(x,q).$$

By (14) and binomial theorem, we easily see that

$$B_{k,n}(x,q) = \binom{n}{k} [x]_q^k \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l [x]_q^l = \sum_{l=k}^{n} \binom{l}{k} \binom{n}{k} (-1)^{l-k} [x]_q^l.$$

Therefore, we obtain the following theorem.

Theorem 8. *For $k, n \in \mathbb{Z}_+$ and $x \in [0,1]$, we have*

$$B_{k,n}(x,q) = \sum_{l=k}^{n} \binom{l}{k}\binom{n}{k}(-1)^{l-k}[x]_q^l.$$

It is possible to write $[x]_q^k$ as a linear combination of the q-Bernstein polynomials by using the degree evaluation formulae and mathematical induction. We easily see from the property of the q-Bernstein polynomials that

$$\sum_{k=1}^{n}\left(\frac{k}{n}\right)B_{k,n}(x,q) = \sum_{k=0}^{n-1}\binom{n-1}{k}[x]_q^{k+1}[1-x]_{\frac{1}{q}}^{n-k-1} = [x]_q,$$

and that

$$\sum_{k=2}^{n}\frac{\binom{k}{2}}{\binom{n}{2}}B_{k,n}(x,q) = \sum_{k=0}^{n-2}\binom{n-2}{k}[x]_q^{k+2}[1-x]_{\frac{1}{q}}^{n-2-k} = [x]_q^2.$$

Continuing this process, we get

$$\sum_{k=j}^{n}\frac{\binom{k}{j}}{\binom{n}{j}}B_{k,n}(x,q) = [x]_q^j, \text{ for } j \in \mathbb{Z}_+.$$

Therefore, we obtain the following theorem.

Theorem 9. *For $n, j \in \mathbb{Z}_+$ and $x \in [0,1]$, we have*

$$\sum_{k=j}^{n}\frac{\binom{k}{j}}{\binom{n}{j}}B_{k,n}(x,q) = [x]_q^j.$$

In [7], the q-stirling numbers of the second kind are defined by

$$s_q(n,k) = \frac{q^{-\binom{k}{2}}}{[k]_q!}\sum_{j=0}^{k}(-1)^j q^{\binom{j}{2}}\binom{k}{j}_q [k-j]_q^n, \tag{27}$$

where $\binom{k}{j}_q = \frac{[k]_q!}{[j]_q![k-j]_q!}$ and $[k]_q! = \prod_{i=1}^{k}[i]_q$. For $n \in \mathbb{Z}_+$, it is known that

$$[x]_q^n = \sum_{k=0}^{n} q^{\binom{k}{2}}\binom{x}{k}_q [k]_q! s_q(n,k), \text{ (see [7, 21])}. \tag{28}$$

By (27), (28) and Theorem 7, we obtain the following corollary.

Corollary 10. *For $n, j \in \mathbb{Z}_+$ and $x \in [0,1]$, we have*

$$\sum_{k=j}^{n} \frac{\binom{k}{j}}{\binom{n}{j}} B_{k,n}(x,q) = \sum_{k=0}^{j} q^{\binom{k}{2}} \binom{x}{k}_q [k]_q! s_q(j,k).$$

§3. On fermionic p-adic integral representations of q-Bernstein polynomials

In this section we assume that $q \in \mathbb{C}_p$ with $|1-q|_p < 1$. From (10) we note that
(29)

$$E_{n,\frac{1}{q}}(1-x) = \int_{\mathbb{Z}_p} [1-x+x_1]_{\frac{1}{q}}^n d\mu_{-1}(x_1) = (-1)^n q^n \int_{\mathbb{Z}_p} [x+x_1]_q^n d\mu_{-1}(x_1), \text{ (see [21])}.$$

From (29) we have

$$\int_{\mathbb{Z}_p} [1-x]_{\frac{1}{q}}^n d\mu_{-1}(x) = q^n(-1)^n \int_{\mathbb{Z}_p} [x-1]_q^n d\mu_{-1}(x)$$
$$= \int_{\mathbb{Z}_p} (1-[x]_q)^n \, d\mu_{-1}(x) = (-1)^n q^n E_{n,\frac{1}{q}}(-1) = E_{n,q}(2).$$

By (5) and (10), we easily get

$$E_{n,q}(2) = 2 + E_{n,q}, \text{ if } n > 0.$$

Thus, we obtain the following theorem.

Theorem 11. *For $n \in \mathbb{N}$, we have*

$$\int_{\mathbb{Z}_p} [1-x]_{\frac{1}{q}}^n d\mu_{-1}(x) = \int_{\mathbb{Z}_p} (1-[x]_q)^n \, d\mu_{-1}(x) = 2 + \int_{\mathbb{Z}_p} [x]_q^n d\mu_{-1}(x).$$

By using Theorem 11, we derive our main results in this section. Taking the fermionic p-adic integral on $\mathbb{Z}_p$ for one q-Bernstein polynomials in (14), we get

$$\int_{\mathbb{Z}_p} B_{k,n}(x,q) d\mu_{-1}(x) = \binom{n}{k} \int_{\mathbb{Z}_p} [x]_q^k [1-x]_{\frac{1}{q}}^{n-k} d\mu_{-1}(x)$$

(30)
$$= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \int_{\mathbb{Z}_p} [x]_q^{k+l} d\mu_{-1}(x)$$
$$= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l E_{k+l,q}.$$

From (14) and Theorem 2, we note that

$$
\begin{aligned}
\int_{\mathbb{Z}_p} B_{k,n}(x,q)d\mu_{-1}(x) &= \int_{\mathbb{Z}_p} B_{n-k,n}(1-x,\frac{1}{q})d\mu_{-1}(x) \\
&= \binom{n}{k}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k+j}\int_{\mathbb{Z}_p}[1-x]_{\frac{1}{q}}^{n-j}d\mu_{-1}(x).
\end{aligned} \tag{31}
$$

For $n>k$, by (31) and Theorem 11, we get

$$
\begin{aligned}
\int_{\mathbb{Z}_p} B_{k,n}(x,q)d\mu_{-1}(x) &= \binom{n}{k}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k+j}\left(2+\int_{\mathbb{Z}_p}[x]_q^{n-j}d\mu_{-1}(x)\right) \\
&= 2+E_{n,q},\ \text{ if } k=0 \\
&= \binom{n}{k}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k+j}E_{n-j,q},\ \text{ if } k>0.
\end{aligned} \tag{32}
$$

From $m,n,k\in\mathbb{Z}_+$ with $m+n>2k$, the fermionic p-adic integral for multiplication of two q-Bernstein polynomials on $\mathbb{Z}_p$ can be given by the following relation:

$$
\begin{aligned}
&\int_{\mathbb{Z}_p} B_{k,n}(x,q)B_{k,m}(x,q)d\mu_{-1}(x) = \binom{n}{k}\binom{m}{k}\int_{\mathbb{Z}_p}[x]_q^{2k}[1-x]_{\frac{1}{q}}^{n+m-2k}d\mu_{-1}(x) \\
&= \binom{n}{k}\binom{m}{k}\sum_{j=0}^{2k}\binom{2k}{j}(-1)^{j+2k}\int_{\mathbb{Z}_p}[1-x]_{\frac{1}{q}}^{n+m-j}d\mu_{-1}(x) \\
&= \binom{n}{k}\binom{m}{k}\sum_{j=0}^{2k}(-1)^{j+2k}\left(2+\int_{\mathbb{Z}_p}[x]_q^{n+m-j}d\mu_{-1}(x)\right).
\end{aligned} \tag{33}
$$

From (33), we have

$$
\begin{aligned}
\int_{\mathbb{Z}_p} B_{k,n}(x,q)B_{k,m}(x,q)d\mu_{-1}(x) &= 2+E_{n+m,q},\ \text{ if } k=0 \\
&= \binom{n}{k}\binom{m}{k}\sum_{j=0}^{2k}\binom{2k}{j}(-1)^{j+2k}E_{n+m-j,q},\ \text{ if } k>0.
\end{aligned}
$$

For $m,k\in\mathbb{Z}_+$, it is difficult to show that

(34)

$$
\int_{\mathbb{Z}_p} B_{k,n}(x,q)B_{k,m}(x,q)d\mu_{-1}(x) = \binom{n}{k}\binom{m}{k}\sum_{j=0}^{n+m-2k}\binom{n+m-2k}{j}(-1)^{j}E_{j+2k,q}.
$$

Continuing this process we obtain the following theorem.

Theorem 12. *(I). For $n_1, \cdots, n_s, k \in \mathbb{Z}_+$ $(s \in \mathbb{N})$ with $n_1 + \cdots + n_s > sk$, we have*

$$\int_{\mathbb{Z}_p} \left(\prod_{i=1}^{s} B_{k,n_i}(x,q) \right) d\mu_{-1}(x) = 2 + E_{n_1+\cdots+n_s,q}, \ \text{if } k = 0,$$

and

$$\int_{\mathbb{Z}_p} \left(\prod_{i=1}^{s} B_{k,n_i}(x,q) \right) d\mu_{-1}(x) = \prod_{i=1}^{s} \binom{n_i}{k} \sum_{j=0}^{sk} \binom{sk}{j} (-1)^{sk-j} E_{n_1+\cdots+n_s-j,q}, \ \text{if } k > 0.$$

(II). Let $k, n_1, \cdots n_s \in \mathbb{Z}_+$ $(s \in \mathbb{N})$. Then we have

$$\begin{aligned} &\int_{\mathbb{Z}_p} \left(\prod_{i=1}^{s} B_{k,n_i}(x,q) \right) d\mu_{-1}(x) \\ &= \left(\prod_{i=1}^{s} \binom{n_i}{k} \right) \sum_{j=0}^{\sum_{i=1}^{s} n_i - sk} \binom{\sum_{i=1}^{s} n_i - sk}{j} (-1)^j E_{j+sk,q}. \end{aligned}$$

By Theorem 12, we obtain the following corollary.

Corollary 13. *For $n_1, \cdots, n_s, k \in \mathbb{Z}_+$ $(s \in \mathbb{N})$ with $n_1 + \cdots + n_s > sk$, we have*

$$\sum_{j=0}^{\sum_{i=1}^{s} n_i - sk} \binom{\sum_{i=1}^{s} n_i - sk}{j} (-1)^j E_{j+sk,q} = 2 + E_{n_1+\cdots+n_s,q}, \ \text{if } k = 0,$$

and

$$\begin{aligned} &\sum_{j=0}^{\sum_{i=1}^{s} n_i - sk} \binom{\sum_{i=1}^{s} n_i - sk}{j} (-1)^j E_{j+sk,q} \\ &= \sum_{j=0}^{sk} \binom{sk}{j} (-1)^{sk-j} E_{n_1+\cdots+n_s-j,q}, \ \text{if } k > 0. \end{aligned}$$

Let $m_1, \cdots, m_s, n_1, \cdots, n_s, k \in \mathbb{Z}_+$ $(s \in \mathbb{N})$ with $m_1 n_1 + \cdots + m_s n_s > (m_1 + \cdots + m_s)k$. By the definition of $B_{k,n_s}^{m_s}(x,q)$, we can also easily see that

$$\begin{aligned} &\int_{\mathbb{Z}_p} \left(\prod_{i=1}^{s} B_{k,n_i}^{m_i}(x,q) \right) d\mu_{-1}(x) \\ &= \prod_{i=1}^{s} \binom{n_i}{k}^{m_i} \sum_{j=0}^{k \sum_{i=1}^{s} m_i} \binom{k \sum_{i=1}^{s} m_i}{j} (-1)^{k \sum_{i=1}^{s} m_i - j} \int_{\mathbb{Z}_p} [1-x]_{\frac{1}{q}}^{\sum_{i=1}^{s} n_i m_i - j} d\mu_{-1}(x) \\ &= \prod_{i=1}^{s} \binom{n_i}{k}^{m_i} \sum_{j=0}^{k \sum_{i=1}^{s} m_i} \binom{k \sum_{i=1}^{s} m_i}{j} (-1)^{k \sum_{i=1}^{s} m_i - j} \left(2 + E_{\sum_{i=1}^{s} m_i n_i - j, q} \right). \end{aligned}$$

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