q-Bernstein polynomials, q-Stirling numbers and q-Bernoulli polynomials

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Abstract : In this paper, we give new identities involving Phillips q-Bernstein polynomials and we derive some interesting properties of q-Bernstein polynomials associated with q-Stirling numbers and q-Bernoulli polynomials.

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1. Introduction

When one talks of q-extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, then we always assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we usually assume that $|1-q|_p < 1$. Here, the symbol $|\cdot|_p$ stands for the p-adic absolute value on $\mathbb{C}_p$ with $|p|_p \le 1/p$. For each x, the q-basic numbers are defined by

$$[x]_q = \frac{1-q^x}{1-q}, \text{ and } [n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q, n \in \mathbb{N}, \text{ (see [1-17])}.$$

Throughout this paper we assume that $q \in \mathbb{C}$ with $|q| < 1$ and we use the notation of Gaussian binomial coefficient in the form

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} = \frac{[n]_q[n-1]_q \cdots [n-k+1]_q}{[k]_q!}, n, k \in \mathbb{N}.$$

Note that

$$\lim_{q \to 1} \binom{n}{k}_q = \binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}, \text{ (see [4-12])}.$$

The Gaussian binomial coefficient satisfies the following recursion formula:

$$\binom{n+1}{k}_q = \binom{n}{k-1}_q + q^k \binom{n}{k}_q = q^{n-k}\binom{n}{k-1}_q + \binom{n}{k}_q, \text{ (see [7, 8])}. \tag{1}$$

The q-binomial formulae are known as

$$(1-b)_q^n = (b:q)_n = \prod_{i=1}^{n}(1-bq^{i-1}) = \sum_{i=0}^{n}\binom{n}{i}_q q^{\binom{i}{2}}(-1)^i b^i, \tag{2}$$

and

$$\frac{1}{(1-b)_q^n} = \frac{1}{(b:q)_n} = \frac{1}{\prod_{i=1}^{n}(1-bq^{i-1})} = \sum_{i=0}^{\infty}\binom{n+i-1}{i}_q b^i, \text{ (see [7, 8])}.$$

Now, we define the q-exponential function as follows:

$$\lim_{n\to\infty}\frac{1}{(x:q)_n}=\lim_{n\to\infty}\sum_{k=0}^{\infty}\binom{n+k-1}{k}_q x^k=\sum_{k=0}^{\infty}\frac{x^k(1-q)^k}{[k]_q!}=e_q(x(1-q)). \tag{3}$$

A Bernoulli trial involves performing an experiment once and noting whether a particular event A occurs. The outcome of Bernoulli trial is said to be "success" if A occurs and a "failure" otherwise. Let k be the number of successes in n independent Bernoulli trials, the probabilities of k are given by the binomial probability law:

$$p_n(k)=\binom{n}{k}p^k(1-p)^{n-k},\ \text{for } k=0,1,\cdots,n,$$

where $p_n(k)$ is the probability of k successes in n trials. For example, a communication system transmit binary information over channel that introduces random bit errors with probability $\xi=10^{-3}$. The transmitter transmits each information bit three times, an a decoder takes a majority vote of the received bits to decide on what the transmitted bit was. The receiver can correct a single error, but it will make the wrong decision if the channel introduces two or more errors. If we view each transmission as a Bernoulli trial in which a "success" corresponds to the introduction of an error, then the probability of two or more errors in three Bernoulli trial is

$$p(k\geq 2)=\binom{3}{2}(0.001)^2(0.999)+\binom{3}{3}(0.001)^3\approx 3(10^{-6}),\ \text{see [18]}.$$

Let $C[0,1]$ denote the set of continuous function on $[0,1]$. For $f\in C[0,1]$, Bernstein introduced the following well known linear operator in [2]:

$$B_n(f|x)=\sum_{k=0}^{n}f(\frac{k}{n})\binom{n}{k}x^k(1-x)^{n-k}=\sum_{k=0}^{n}f(\frac{k}{n})B_{k,n}(x).$$

Here $B_n(f|x)$ is called the Bernstein operator of order n for f. For $k,n\in\mathbb{Z}_+$, the Bernstein polynomials of degree n is defined by

$$B_{k,n}(x)=\binom{n}{k}x^k(1-x)^{n-k}.$$

By the definition of Bernstein polynomials, we can see that Bernstein basis is the probability mass function of binomial distribution. Based on the q-integers Phillips introduced the q-analogue of well known Bernstein polynomials (see [15, 16]). For $f\in C[0,1]$, Phillips introduced the q-extension of $\mathbb{B}_n(f|x)$ as follows:

$$\mathbb{B}_{n,q}(f\mid x)=\sum_{k=0}^{n}B_{k,n}(x,q)f\left(\frac{[k]_q}{[n]_q}\right)=\sum_{k=0}^{n}f\left(\frac{[k]_q}{[n]_q}\right)\binom{n}{k}_q x^k(1-x)_q^{n-k}. \tag{4}$$

Here $\mathbb{B}_{n,q}(f\mid x)$ is called the q-Bernstein operator of order n for f. For $k,n\in\mathbb{Z}_+$, the q-Bernstein polynomial of degree n is defined by

$$B_{k,n}(x,q)=\binom{n}{k}_q x^k(1-x)_q^{n-k}, x\in[0,1]. \tag{5}$$

For example, $B_{0,1}(x,q)=1-x, B_{1,1}(x,q)=x$, and $B_{0,2}(x,q)=1-[2]_q x+qx^2,\cdots$. Also $B_{k,n}(x,q)=0$ for $k>n$, because $\binom{n}{k}_q=0$. The q-binomial distribution associated with the q-boson oscillator are introduced in [19, 20]. For $n,k\in\mathbb{Z}_+$, its probabilities are given by

$$p(X=k)=\binom{n}{k}_q x^k(1-x)_q^{n-k}, \text{ where } x\in[0,1].$$

This distributions are studied by several authors and has applications in physics as well as in approximation theory due to the q-Bernstein polynomials and the q-Bernstein operators (see [16, 19, 20]). From the definition of q-Bernstein polynomials, we note that the q-Bernstein basis is the probability mass function of q-binomial distribution. Recently, several authors have studied the analogs of Bernstein polynomials (see [1, 2, 5, 8, 9, 10, 15, 16, 17]). In [5], Gupta-Kim-Choi-Kim gave the generating function of Phillips q-Bernstein polynomials as follows:

$$\begin{aligned}\frac{x^kt^k}{[k]_q!}e_q((1-x)_qt)&=\frac{x^kt^k}{[k]_q!}\sum_{n=0}^{\infty}\frac{(1-x)_q^nt^n}{[n]_q!}\\&=\sum_{n=k}^{\infty}\binom{n}{k}_q\frac{x^k(1-x)_q^{n-k}}{[n]_q!}t^n\\&=\sum_{n=k}^{\infty}B_{k,n}(x,q)\frac{t^n}{[n]_q!}.\end{aligned}$$

Because $B_{k,0}(x,q)=B_{k,1}(x,q)=B_{k,2}(x,q)=\cdots=B_{k,k-1}(x,q)=0$, we obtain the generating function for $B_{k,n}(x,q)$ as follows:

$$F_q^{(k)}(t,x)=\frac{x^kt^k}{[k]_q!}e_q((1-x)_qt)=\sum_{n=0}^{\infty}B_{k,n}(x,q)\frac{t^n}{[n]_q!}, \text{ see [5]},$$

where $n,k\in\mathbb{Z}_+$ and $x\in[0,1]$.

Notice that

$$B_{k,n}(x,q)=\begin{cases}\binom{n}{k}_q x^k(1-x)_q^{n-k}, & \text{if } n\ge k\\ 0, & \text{if } n<k,\end{cases}$$

for $n,k\in\mathbb{Z}_+$ (see [5, 15, 16]).

In this paper we study the generating function of Phillips q-Bernstein polynomial and give some identities on the Phillips q-Bernstein polynomials. From the generating function of q-Bernstein polynomial, we derive recurrence relation and derivative of the Phillips q-Bernstein polynomials. Finally, we investigate some interesting properties of q-Bernstein polynomials related to q-Stirling numbers and q-Bernoulli polynomials.

2. q-Bernstein polynomials, q-Stirling numbers and q-Bernoulli polynomials

Let

$$F_q^{(k)}(t,x)=\sum_{n=0}^{\infty}B_{k,n}(x,q)\frac{t^n}{[n]_q!}.$$

From (5) and (3), we note that

$$\begin{aligned}
F_q^{(k)}(t,x) &= \sum_{n=0}^{\infty} \binom{n}{k}_q x^k (1-x)_q^{n-k} \frac{t^n}{[n]_q!} \\
&= \sum_{n=0}^{\infty} \binom{n+k}{k}_q \frac{x^k(1-x)_q^n}{[n+k]_q!} t^{n+k} \\
&= \frac{x^k t^k}{[k]_q!} \sum_{n=0}^{\infty} \frac{(1-x)_q^n}{[n]_q!} t^n \\
&= \frac{x^k t^k}{[k]_q!} e_q((1-x)_q t),
\end{aligned}$$

where $n, k \in \mathbb{Z}_+$ and $x \in [0,1]$.

Note that

$$\lim_{q\to 1} F_q^{(k)}(t,x) = \frac{x^k t^k}{k!} e^{(1-x)t} = \sum_{n=0}^{\infty} B_{k,n}(x) \frac{t^n}{n!},$$

where $B_{k,n}(x)$ are the Bernstein polynomial of degree n.

The q-derivative $D_q f$ of function f is defined by

$$(D_q f)(x) = \frac{df(x)}{d_q x} = \frac{f(x) - f(qx)}{(1-q)x}, \text{ (see [6]).} \tag{7}$$

From (7), we note that

$$D_q(fg)(x) = g(x) D_q f(x) + f(qx) D_q g(x), \text{ (see [6]).} \tag{8}$$

The q-Bernstein operator is given by

$$\mathbb{B}_{n,q}(f \mid x) = \sum_{k=0}^{n} B_{k,n}(x,q) f\left(\frac{[k]_q}{[n]_q}\right), \text{ (see Eq. (4)).}$$

Thus, we have

$$\mathbb{B}_{n,q}(1 \mid x) = \sum_{k=0}^{n} B_{k,n}(x,q) = \sum_{k=0}^{n} \binom{n}{k}_q x^k (1-x)_q^{n-k} = 1,$$

and

$$\mathbb{B}_{n,q}(x \mid x) = \sum_{k=0}^{n} \left(\frac{[k]_q}{[n]_q}\right) B_{k,n}(x,q) = x \sum_{k=0}^{n-1} \binom{n-1}{k}_q x^k (1-x)_q^{n-k} = x,$$

where $x \in [0,1]$ and $n, k \in \mathbb{Z}_+$.

For $f \in C[0,1]$, we have

$$\begin{aligned}
\mathbb{B}_{n,q}(f \mid x) &= \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) B_{k,n}(x,q) \\
&= \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) \binom{n}{k}_q x^k (1-x)_q^{n-k} \\
&= \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) x^k \binom{n}{k}_q \sum_{j=0}^{n-k} \binom{n-k}{j}_q (-1)^j q^{\binom{j}{2}} x^j.
\end{aligned}$$

It is easy to show that

$$\binom{n}{k}_q \binom{n-k}{j}_q = \binom{n}{k+j}_q \binom{k+j}{k}_q .$$

Let $k+j=m$. Then we have

$$\binom{n}{k}_q \binom{n-k}{j}_q = \binom{n}{m}_q \binom{m}{k}_q . \tag{10}$$

By (9) and (10), we easily get

$$\mathbb{B}_{n,q}(f \mid x) = \sum_{m=0}^{n} \binom{n}{m}_q x^m \sum_{k=0}^{m} \binom{m}{k}_q q^{\binom{m-k}{2}} (-1)^{m-k} f\left(\frac{[k]_q}{[n]_q}\right). \tag{11}$$

Therefore, we obtain the following proposition.

Proposition 1. For $f \in C[0,1]$ and $n \in \mathbb{Z}_+$, we have

$$\mathbb{B}_{n,q}(f \mid x) = \sum_{m=0}^{n} \binom{n}{m}_q x^m \sum_{k=0}^{m} \binom{m}{k}_q q^{\binom{m-k}{2}} (-1)^{m-k} f\left(\frac{[k]_q}{[n]_q}\right). \tag{11}$$

It is well known that the second kind Stirling numbers are defined by

$$\frac{(e^t-1)^k}{k!} = \frac{1}{k!} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} e^{lt} = \sum_{n=0}^{\infty} S(n,k) \frac{t^n}{n!}, \tag{12}$$

where $k \in \mathbb{N}$ (see [7, 8, 9, 10, 17]).

Let Δ be the shift difference operator with $\Delta f(x) = f(x+1) - f(x)$. By iterative process, we see that

$$\Delta^n f(0) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(k), \text{ for } n \in \mathbb{N}. \tag{13}$$

From (12) and (13), we have

$$\begin{aligned} \sum_{n=0}^{\infty} S(n,k) \frac{t^n}{n!} &= \frac{1}{k!} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} e^{lt} \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{k!} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} l^n \right) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{\Delta^k 0^n}{k!} \frac{t^n}{n!}, \text{ (see [7, 8, 9])}. \end{aligned} \tag{14}$$

By comparing the coefficients on the both sides of (14), we get

$$S(n,k) = \frac{\Delta^k 0^n}{k!}, \text{ for } n,k \in \mathbb{Z}_+. \tag{15}$$

Now, we consider the q-extension of (13). Let $(Eh)(x) = h(x+1)$ be the shift operator. Then the q-difference operator is defined by

$$\Delta_q^n := (E-I)_q^n = \prod_{i=1}^{n}(E - Iq^{i-1}), \text{ (see [7])},$$

where I is an identity operator.

For $f \in C[0,1]$ and $n \in \mathbb{N}$, we have

$$\Delta_q^n f(0) = \sum_{k=0}^{n} \binom{n}{k}_q (-1)^k q^{\binom{k}{2}} f(n-k) = \sum_{k=0}^{n} \binom{n}{k}_q (-1)^{n-k} q^{\binom{n-k}{2}} f(k). \tag{16}$$

Note that (16) is exactly q-extension of (13). That is, $\lim_{q\to 1} \Delta_q^n f(0) = \Delta^n f(0)$.

As the q-extension of (12), the second kind q-Stirling numbers are defined by

$$\frac{q^{-\binom{k}{2}}}{[k]_q!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j}_q q^{\binom{k-j}{2}} e^{[j]_q t} = \sum_{n=0}^{\infty} S(n,k:q) \frac{t^n}{n!}, \text{ (see [7, 8])}. \quad (17).$$

By (16), we obtain the following theorem.

Theorem 2. For $f \in C[0,1]$ and $n \in \mathbb{Z}_+$, we have

$$\mathbb{B}_{n,q}(f \mid x) = \sum_{k=0}^{n} \binom{n}{k}_q x^k \Delta_q^k f\left(\frac{0}{[n]_q}\right).$$

In the special case, $f(x) = x^m (m \in \mathbb{Z}_+)$, we obtain the following corollary.

Corollary 3. For $x \in [0,1]$ and $m, n \in \mathbb{Z}_+$, we have

$$[n]_q^m \mathbb{B}_{n,q}(x^m \mid x) = \sum_{k=0}^{n} \binom{n}{k}_q x^k \Delta_q^k 0^m.$$

By (17), we easily get

$$\begin{aligned} S(n,k:q) &= \frac{q^{-\binom{k}{2}}}{[k]_q!} \sum_{j=0}^{k} (-1)^j q^{\binom{j}{2}} \binom{k}{j}_q [k-j]_q^n \\ &= \frac{q^{-\binom{k}{2}}}{[k]_q!} \sum_{j=0}^{k} (-1)^{k-j} q^{\binom{k-j}{2}} \binom{k}{j}_q [j]_q^n \\ &= \frac{q^{-\binom{k}{2}}}{[k]_q!} \Delta_q^k 0^m. \end{aligned} \tag{18}$$

The equation (18) seems to be the q-extension of the equation (15). That is, $\lim_{q\to 1} S(n,k:q) = S(n,k)$.

By Corollary 3 and (18), we obtain the following corollary.

Corollary 4. For $x \in [0,1]$ and $m, n \in \mathbb{Z}_+$, we have

$$[n]_q^m \mathbb{B}_{n,q}(x^m \mid x) = \sum_{k=0}^{n} \binom{n}{k}_q x^k [k]_q! q^{\binom{k}{2}} S(m, k : q).$$

From (1) and (5), for $0 \le k \le n$, we have

$$\begin{aligned} & q^k(1-q^{n-k-1}x)B_{k,n-1}(x,q) + xB_{k-1,n-1}(x,q) \\ &= q^k(1-q^{n-k-1}x)\binom{n-1}{k}_q x^k(1-x)_q^{n-1-k} + x\binom{n-1}{k-1}_q x^{k-1}(1-x)_q^{n-k} \\ &= q^k\binom{n-1}{k}_q x^k(1-x)_q^{n-k} + \binom{n-1}{k-1}_q x^k(1-x)_q^{n-k} \\ &= \binom{n}{k}_q x^k(1-x)_q^{n-k}. \end{aligned} \tag{19}$$

By (2), (7) and (8), we get

$$\frac{dB_{k,n}(x,q)}{d_q x} = -\binom{n}{k}_q x^k[n-k]_q(1-qx)_q^{n-k-1} + \binom{n}{k}_q [k]_q x^{k-1}(1-qx)_q^{n-k}. \tag{20}$$

From the definition of Gaussian binomial coefficient (= q-binomials coefficient) and (2), we note that

$$\binom{n}{k}_q [k]_q x^{k-1}(1-qx)_q^{n-k} = q^{-(k-1)}[n]_q B_{k-1,n-1}(qx,q), \tag{21}$$

and

$$\binom{n}{k}_q x^k[n-k]_q(1-qx)_q^{n-k-1} = [n]_q q^{-k} B_{k,n-1}(qx,q).$$

By (20) and (21), we see that

$$\frac{dB_{k,n}(x,q)}{d_q x} = [n]_q q^{-k}(qB_{k-1,n-1}(qx,q) - B_{k,n-1}(qx,q)). \tag{22}$$

Thus, we note that the q-derivative of the q- Bernstein polynomials of degree n are also polynomial of degree $n-1$. Therefore, by (19) and (22), we obtain the following recurrence formulae:

Theorem 5(Recurrence formulae for $B_{k,n}(x,q)$). For $k, n \in \mathbb{Z}_+$ and $x \in [0,1]$, we have

$$q^k(1-q^{n-k-1}x)B_{k,n-1}(x,q) + xB_{k-1,n-1}(x,q) = B_{k,n}(x,q),$$

and

$$\frac{dB_{k,n}(x,q)}{d_q x} = [n]_q q^{-k}(qB_{k-1,n-1}(qx,q) - B_{k,n-1}(qx,q)).$$

We also get from (5) and (6) that

$$\begin{aligned}
&\frac{[n-k]_q}{[n]_q}B_{k,n}(x,q)+\frac{[k+1]_q}{[n]_q}B_{k+1,n}(x,q)\\
&=(1-xq^{n-k-1})\binom{n-1}{k}_q x^k(1-x)_q^{n-k-1}+x\binom{n-1}{k}_q x^k(1-x)_q^{n-k-1}\\
&=(1-xq^{n-k-1})B_{k,n-1}(x,q)+xB_{k,n-1}(x,q)\\
&=B_{k,n-1}(x,q)+x[n-k-1]_q(1-q)B_{k,n-1}(x,q).
\end{aligned} \tag{23}$$

By (23), we obtain the following theorem.

Theorem 6. For $k,n\in\mathbb{Z}_+$ and $x\in[0,1]$, we have

$$\frac{[n-k]_q}{[n]_q}B_{k,n}(x,q)+\frac{[k+1]_q}{[n]_q}B_{k+1,n}(x,q)=B_{k,n-1}(x,q)+x[n-k-1]_q(1-q)B_{k,n-1}(x,q).$$

From Theorem 6 we note that q-Bernstein polynomials can be written as a linear combination of polynomials of higher order.

For $k,n\in\mathbb{N}$, we easily get from (5) that q-Bernstein polynomials can be expressed in the form

$$\begin{aligned}
&\frac{[n-k+1]_q}{[k]_q}\left(\frac{x}{1-xq^{n-k}}\right)x^{k-1}(1-x)_q^{n-k+1}\binom{n}{k-1}_q\\
&=\frac{[n]_q!}{[k]_q![n-k]_q!}x^k(1-x)_q^{n-k}\\
&=\binom{n}{k}_q x^k(1-x)_q^{n-k}\\
&=B_{k,n}(x,q).
\end{aligned} \tag{24}$$

By (24), we obtain the following proposition.

Proposition 7. For $n,k\in\mathbb{N}$ and $x\in[0,1]$, we have

$$B_{k,n}(x,q)=\frac{[n-k+1]_q}{[k]_q}\left(\frac{x}{1-xq^{n-k}}\right)B_{k-1,n}(x,q).$$

The q-Bernstein polynomials of degree n can be written in terms of power basis $\{1,x,x^2,\cdots,x^n\}$. By using the definition of q-Bernstein polynomial and q-binomial theorem, we get

$$\begin{aligned}
B_{k,n}(x,q)&=\binom{n}{k}_q x^k(1-x)_q^{n-k}=\binom{n}{k}_q x^k\sum_{i=0}^{n-k}\binom{n-k}{i}_q(-1)^i q^{\binom{i}{2}}x^i\\
&=\sum_{i=0}^{n-k}\binom{n-k}{i}_q\binom{n}{k}_q(-1)^i q^{\binom{i}{2}}x^{i+k}\\
&=\sum_{i=k}^{n}\binom{n-k}{i-k}_q\binom{n}{k}_q(-1)^{i-k} q^{\binom{i-k}{2}}x^{i}.
\end{aligned} \tag{25}$$

By simple calculation, we easily see that

$$\binom{n}{k}_q \binom{n-k}{i-k}_q = \binom{n}{i}_q \binom{i}{k}_q. \tag{26}$$

Therefore, by (25) and (26), we obtain the following theorem.

Theorem 8. For $k, n \in \mathbb{Z}_+$ and $x \in [0,1]$, we have

$$B_{k,n}(x,q) = \sum_{i=k}^{n} \binom{n}{i}_q \binom{i}{k}_q (-1)^{i-k} q^{\binom{i-k}{2}} x^i.$$

We get from the properties of q-Bernstein polynomials that

$$\begin{aligned}
\sum_{k=1}^{n} \frac{\binom{k}{1}_q}{\binom{n}{1}_q} B_{k,n}(x,q) &= \sum_{k=1}^{n} \frac{[k]_q}{[n]_q} \binom{n}{k}_q x^k (1-x)_q^{n-k} \\
&= \sum_{k=1}^{n} \binom{n-1}{k-1}_q x^k (1-x)_q^{n-k} \\
&= x \sum_{k=0}^{n-1} \binom{n-1}{k}_q x^k (1-x)_q^{n-k-1} = x,
\end{aligned}$$

and that

$$\begin{aligned}
\sum_{k=2}^{n} \frac{\binom{k}{2}_q}{\binom{n}{2}_q} B_{k,n}(x,q) &= \sum_{k=2}^{n} \binom{n-2}{k-2}_q x^k (1-x)_q^{n-k} \\
&= x^2 \sum_{k=0}^{n-2} \binom{n-2}{k}_q x^k (1-x)_q^{n-k-2} = x^2.
\end{aligned}$$

Continuing this process, we obtain

$$\sum_{k=i}^{n} \frac{\binom{k}{i}_q}{\binom{n}{i}_q} B_{k,n}(x,q) = x^i.$$

Therefore, we obtain the following theorem.

Theorem 9. For $k, i \in \mathbb{Z}_+$ and $x \in [0,1]$, we have

$$\sum_{k=i}^{n} \frac{\binom{k}{i}_q}{\binom{n}{i}_q} B_{k,n}(x,q) = x^i.$$

Now we define q-Bernoulli polynomials of order k as follows:

$$\left(\frac{z}{e^z - 1}\right)^k e_q(zx) = \sum_{n=0}^{\infty} \beta_n^{(k)}(x,q) \frac{z^n}{[n]_q!}, \quad k \in \mathbb{N}. \tag{27}$$

From the generating function (27) of q-Bernoulli polynomials and (3), we derive

$$
\begin{aligned}
\left(\frac{z}{e^z-1}\right)^k e_q(zx) &= \left(\sum_{m=0}^{\infty} B_m^{(k)} \frac{z^m}{m!}\right)\left(\sum_{l=0}^{\infty} \frac{x^l z^l}{[l]_q!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} \frac{B_m^{(k)} x^{n-m}[n]_q!}{m![n-m]_q!}\right)\frac{z^n}{[n]_q!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} \frac{[m]_q!}{m!} B_m^{(k)} \binom{n}{m}_q x^{n-m}\right)\frac{z^n}{[n]_q!},
\end{aligned} \tag{28}
$$

where $B_m^{(k)}$ are the n-th Bernoulli numbers of order k(see [6]).

From (27) and (28), we easily get

$$
\beta_n^{(k)}(x,q) = \sum_{m=0}^{n} \binom{n}{m}_q \frac{[m]_q!}{m!} x^{n-m} B_m^{(k)}, \tag{29}
$$

where $B_m^{(k)}$ are the m-th ordinary Bernoulli numbers of order k.

From (26) and (27), we note that

$$
\begin{aligned}
\frac{(tx)^k}{[k]_q!} e_q((1-x)_q t) &= \frac{x^k(e^t-1)^k}{[k]_q!}\left(\frac{t}{e^t-1}\right)^k e_q((1-x)_q t) \\
&= \frac{k!}{[k]_q!} x^k \left(\sum_{m=0}^{\infty} S(m,k)\frac{t^m}{m!}\right)\left(\sum_{n=0}^{\infty} \beta_n^{(k)}((1-x)_q,q)\frac{t^n}{[n]_q!}\right) \\
&= \frac{k!}{[k]_q!} x^k \sum_{l=0}^{\infty}\left(\sum_{m=0}^{l} \frac{[m]_q!}{m!} S(m,k) \binom{l}{m}_q \beta_{l-m}^{(k)}((1-x)_q,q)\right)\frac{t^l}{[l]_q!}.
\end{aligned} \tag{30}
$$

Therefore, by (6) and (30), we obtain the following theorem,

Theorem 10. For $k, l \in \mathbb{Z}_+$ and $x \in [0,1]$, we have

$$
B_{k,l}(x,q) = \frac{k!}{[k]_q!} x^k \sum_{m=0}^{l} \frac{[m]_q!}{m!} S(m,k) \beta_{l-m}^{(k)}((1-x)_q,q) \binom{l}{m}_q,
$$

where $\beta_l^{(k)}((1-x)_q,q)$ are called the l-th q-Bernoulli polynomials.

From (15) and Theorem 10, we have the following corollary.

Corollary 11. For $k, l \in \mathbb{Z}_+$ and $x \in [0,1]$, we have

$$
B_{k,l}(x,q) = \frac{x^k}{[k]_q!} \sum_{m=0}^{l} \frac{[m]_q!}{m!} \binom{l}{m}_q \beta_{l-m}^{(k)}((1-x)_q,q) \Delta^k 0^m.
$$

It is well known that

$$
x^n = \sum_{k=0}^{n} \binom{x}{k} k! S(n,k), \ \text{(see [7])}. \tag{31}
$$

By (31) and Theorem 9, we easily see that

$$\sum_{k=0}^{i} \binom{x}{k} k! S(i,k) = \sum_{k=i}^{n} \frac{\binom{k}{i}_q}{\binom{n}{i}_q} B_{k,n}(x,q).$$

3. A matrix representation for q-Bernstein polynomials

Given a polynomial is written as a linear combination of q-Bernstein basis functions:

$$B_q(x) = C_0^q B_{0,n}(x,q) + C_1^q B_{1,n}(x,q) + \cdots + C_n^q B_{n,n}(x,q). \tag{32}$$

It is easy to write (32) as a dot product of two vectors:

$$B_q(x) = \begin{pmatrix} B_{0,n}(x,q), B_{1,n}(x,q), \ldots, B_{n,n}(x,q) \end{pmatrix} \begin{pmatrix} C_0^q \\ C_1^q \\ \vdots \\ C_n^q \end{pmatrix}. \tag{33}$$

Now, we can convert (33) to

$$B_q(x) = \begin{pmatrix} 1, x, \ldots, x^n \end{pmatrix} \begin{pmatrix} b_{0,0}^q & 0 & \cdots & 0 \\ b_{1,0}^q & b_{1,1}^q & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,0}^q & b_{n,1}^q & \cdots & b_{n,n}^q \end{pmatrix} \begin{pmatrix} C_0^q \\ C_1^q \\ \vdots \\ C_n^q \end{pmatrix},$$

where $b_{i,j}^q$ are the coefficients of the power basis that are used to determine the respective q-Bernstein polynomials.

From (5) and (6), we note that

$$\begin{aligned} B_{0,2}(x,q) &= (1-x)_q^2 = \sum_{l=0}^{2} \binom{2}{l}_q (-1)^l q^{\binom{l}{2}} = 1 - [2]_q x + q x^2 \\ B_{1,2}(x,q) &= \binom{2}{1}_q x(1-x)_q = [2]_q x(1-x) = [2]_q x - [2]_q x^2 \\ B_{2,2}(x,q) &= x^2. \end{aligned}$$

In the quadratic case($n = 2$), the matrix can be represented by

$$B_q(x) = \begin{pmatrix} 1, x, x^2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -[2]_q & [2]_q & 0 \\ q & -[2]_q & 1 \end{pmatrix} \begin{pmatrix} C_0^q \\ C_1^q \\ C_2^q \end{pmatrix}.$$

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