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Bernstein-type operators which preserve polynomials

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ABSTRACT

In this paper we present the sequence of linear Bernstein-type operators defined for $f \in C[0,1]$ by $B_n(f \circ \tau^{-1}) \circ \tau$, B_n being the classical Bernstein operators and τ being any function that is continuously differentiable ∞ times on $[0,1]$, such that $\tau(0) = 0$, $\tau(1) = 1$ and $\tau'(x) > 0$ for $x \in [0,1]$. We investigate its shape preserving and convergence properties, as well as its asymptotic behavior and saturation. Moreover, these operators and others of King type are compared with each other and with B_n. We present as an interesting byproduct sequences of positive linear operators of polynomial type with nice geometric shape preserving properties, which converge to the identity, which in a certain sense improve B_n in approximating a number of increasing functions, and which, apart from the constant functions, fix suitable polynomials of a prescribed degree. The notion of convexity with respect to τ plays an important role.

1. Introduction

Starting from the classical Bernstein operators B_n and from a sequence r_n of continuous functions defined on $[0,1]$ with $0 \le r_n(x) \le 1$ for each $x \in [0,1]$ and $n \in \mathbb{N} = \{1, 2, \ldots\}$, King introduced in [1] a sequence of positive linear operators defined for $f \in C[0,1]$ as

$$f \longmapsto (B_n f) \circ r_n. \tag{1}$$

In fact, even more general operators were considered as early as 1969 in [2], where the corresponding degree of approximation was investigated in terms of the first modulus of continuity. King focused on the particular case where

$$r_n(x) = \begin{cases} x^2 & n = 1, \\ -\dfrac{1}{2(n-1)} + \sqrt{\left(\dfrac{n}{n-1}\right)x^2 + \dfrac{1}{4(n-1)^2}} & n = 2, 3, \ldots \end{cases}$$

and proved that the corresponding operators hold fixed the functions $e_i(x) = x^i$ for $i = 0, 2$ and approximate each continuous function on $[0,1]$ with an order of approximation at least as good as that of $B_n f(x)$ whenever $0 \le x < 1/3$. This case was slightly extended in [3] by considering a family of sequences of operators $B_{n,\alpha}$ that preserve e_0 and $e_2 + \alpha e_1$ with $\alpha \in [0, +\infty)$ ($B_{n,0}$ is the aforesaid King's sequence). Moreover, a further extension appeared in [4]; here the authors considered (1) with $r_n = \tau_n := (B_n \tau)^{-1} \circ \tau$, τ being any continuous strictly increasing function defined on $[0,1]$ with $\tau(0) = 0$ and $\tau(1) = 1$, that is to say, they studied the sequence $V_n^\tau : C[0,1] \to C[0,1]$ defined by

$$V_n^\tau f := (B_n f) \circ \tau_n = (B_n f) \circ (B_n \tau)^{-1} \circ \tau. \tag{2}$$

Note that each V_n^τ fixes e_0 and τ. Note also that if $\tau = (e_2 + \alpha e_1)/(1+\alpha)$, then $V_n^\tau = B_{n,\alpha}$.

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On the other hand, in [5] Gonska and Piţul remarked that it is impossible to find *polynomial* operators of the form (1) that fix e_2, and raised the question of whether it is possible to find another type of positive linear polynomial operator fulfilling this property. While the reader can find an answer to a more ambitious question in the up-to-date paper [6] via the sequence $B_{n,0,j}$ (which fixes e_0 and e_j) defined, for $f \in C[0,1]$ and $x \in [0,1]$, as

$$B_{n,0,j}f(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\left(\frac{k(k-1)\cdots(k-j+1)}{n(n-1)\cdots(n-j+1)}\right)^{1/j}\right), \quad n \ge j,$$

a much simpler one is given by the sequence of operators defined by

$$\sum_{k=0}^{n} \binom{n}{k} x^{2k}(1-x^2)^{n-k} f\left(\sqrt{\frac{k}{n}}\right), \quad f \in C[0,1],\ 0 \le x \le 1,$$

which turns out to be a particular case (after taking $\tau = e_2$) of the sequence

$$B_n^\tau f := B_n(f \circ \tau^{-1}) \circ \tau. \tag{3}$$

In this note we study this sequence B_n^τ under certain general assumptions on the τ to be fixed. We investigate its shape preserving and convergence properties, as well as its asymptotic behavior and saturation. Moreover, these operators and those described in (2) are compared with each other and with B_n. We present as an interesting byproduct, according to the thread of this introduction, sequences of positive linear operators *of polynomial type* with nice geometric shape preserving properties, which converge to the identity, which in certain sense improve B_n in approximating a number of increasing functions, and which, apart from the constant functions, *fix suitable polynomials of a prescribed degree*.

Throughout the paper we shall assume that τ is any function on [0, 1] continuously differentiable ∞ times, such that $\tau(0) = 0$, $\tau(1) = 1$ and $\tau'(x) > 0$ for $x \in [0,1]$. We shall also make use of the following usual notation: $C^k[0,1]$ denotes the space of all functions continuously differentiable k times defined on [0, 1], and D^k denotes the usual kth differential operators, though we keep on using also the classical notation $f', f'', \ldots$ for low order derivatives of f.

2. Properties of B_n^τ

Firstly we detail the definition of the operators that we are concerned with, already defined in (3):

$$B_n^\tau f(x) = \sum_{k=0}^{n} \binom{n}{k} \tau(x)^k (1-\tau(x))^{n-k} (f \circ \tau^{-1})(k/n), \quad f \in C[0,1],\ x \in [0,1].$$

Now we are listing some basic properties of these operators which can be easily derived from well-known properties of B_n. Actually, if $\tau = e_1$, then $B_n^\tau = B_n$.

We begin with some easy to obtain identities:

$$B_n^\tau e_0 = e_0, \qquad B_n^\tau \tau = \tau, \qquad B_n^\tau \tau^2 = \left(1 - \frac{1}{n}\right)\tau^2 + \frac{\tau}{n}. \tag{4}$$

As for shape preserving properties, one can check that the B_n^τ are k-convex for $k = 0, 1$, i.e. whenever this makes sense, $D^k f \ge 0$ implies $D^k B_n^\tau f \ge 0$ for $k = 0, 1$; roughly speaking they are positive and increasing. Note that in general they are not convex: take for instance $\tau = (e_2 + e_1)/2$ and calculate the image of e_1 for low values of n.

Besides the previous classical k-convexity, we can consider the notion of τ-convexity: a function $f \in C^k[0,1]$ is said to be τ-convex of order $k \in \mathbb{N}$ whenever

$$D_\tau^k f := D^k(f \circ \tau^{-1}) \circ \tau \ge 0$$

(for further details see for instance [7,8]). Obviously, the operators B_n^τ map τ-convex functions of order k onto τ-convex functions of order k, so they are said to be τ-convex of order k. As a direct consequence of this fact, the operators B_n^τ do not increase the degree of the so called τ-polynomials. To be more specific, if we use the notation $\mathbb{P}_{\tau,k} = \langle \tau^i : i = 0, 1, \ldots, k\rangle$, then

$$B_n^\tau(\mathbb{P}_{\tau,k}) \subset \mathbb{P}_{\tau,k}.$$

Passing on to convergence properties, if we use (4) and the basic fact that $\{e_0, \tau, \tau^2\}$ is an extended complete Tchebychev system on [0, 1], then the famous Korovkin theorem (see [9] or the excellent monograph [10]) tells us that $B_n^\tau f$ converges uniformly to $f \in C[0,1]$. Moreover, the next proposition contains a quantitative version that can be derived from [11, Theorem 4]. We make use of the following τ-polynomial considered in that paper:

$$F(t,x) = \begin{vmatrix} 1 & \tau(x) & \tau^2(x) \\ 0 & \tau'(x) & (\tau^2)'(x) \\ 1 & \tau(t) & \tau^2(t) \end{vmatrix} = \tau'(x)(\tau(t) - \tau(x))^2.$$

We also point out that in [12] Freud proved the existence of a constant $K > 0$ such that

$$K(t-x)^2 \le F(t,x) \quad \text{for all } t, x \in [0,1]. \tag{5}$$

Proposition 1. *Let $K > 0$ be a constant satisfying (5). Suppose that $f \in C[0,1]$ with modulus of continuity $\omega(f,\cdot)$, that $x \in [0,1]$ and that $\delta > 0$; then*

$$|B_n^\tau f(x) f(x)| \leq \omega(f,\delta)\left(1 + \frac{\tau'(x)\tau(x)(1-\tau(x))}{nK\delta^2}\right). \tag{6}$$

Note that if we apply the previous proposition with $\tau = e_1$, then (6) becomes the well-known estimate for the Bernstein operators, namely,

$$|B_n f(x) - f(x)| \leq \omega(f,\delta)\left(1 + \frac{x(1-x)}{n\delta^2}\right).$$

The following propositions provide us with further information about this approximation process. Specifically, they deal with some aspects of the asymptotic behavior, monotonic convergence, and saturation of the sequence B_n^τ.

The propositions are stated without proofs as they are more or less direct consequences of the corresponding results for the Bernstein operators B_n, that the reader may recover just by taking $\tau = e_1$ in these propositions, and that can be seen for instance by consulting [13–15] (these last two papers as regards Propositions 4 and 5 respectively). We merely mention that an important role is played here by the notion of convexity with respect to a function. In this respect, it is important to recall that $f \in C[0,1]$ is said to be convex with respect to τ whenever

$$\begin{vmatrix} 1 & 1 & 1 \\ \tau(x_0) & \tau(x_1) & \tau(x_2) \\ f(x_0) & f(x_1) & f(x_2) \end{vmatrix} \geq 0, \quad 0 \leq x_0 < x_1 < x_2 \leq 1.$$

We also observe that a function f is convex with respect to τ if and only if $f \circ \tau^{-1}$ is convex in the classical sense.

Proposition 2. *Suppose that we have $f \in C[0,1]$ and $x \in (0,1)$ such that $f''(x)$ exists. Then*

$$\lim_{n\to+\infty} 2n(B_n^\tau f(x) - f(x)) = \tau(x)(1-\tau(x))D_\tau^2 f(x),$$

or equivalently

$$\lim_{n\to+\infty} 2n(B_n^\tau f(x) - f(x)) = \tau(x)(1-\tau(x))\left(-\frac{\tau''(x)f'(x)}{\tau'(x)^3} + \frac{f''(x)}{\tau'(x)^2}\right). \tag{7}$$

Proposition 3. *Suppose that $f \in C[0,1]$. Then f is convex with respect to τ if and only if*

$$\lim\sup_{n\to+\infty} 2n(B_n^\tau f(x) - f(x)) \geq 0, \quad x \in (0,1).$$

Proposition 4. *Suppose that $f \in C[0,1]$. Then the following items are equivalent:*

(i) *f is convex with respect to τ,*
(ii) *$B_n^\tau f \geq f$ for $n \in \mathbb{N}$,*
(iii) *$B_n^\tau f \geq B_{n+1}^\tau f$ for $n \in \mathbb{N}$.*

Proposition 5. *Suppose that $f \in C[0,1]$ and let ψ be a finitely valued Lebesgue-integrable function on $(0,1)$ such that for each $x \in (0,1)$*

$$\lim\inf_{n\to\infty} 2n(B_n^\tau f(x) - f(x)) \leq \psi(x) \leq \lim\sup_{n\to\infty} 2n(B_n^\tau f(x) - f(x)).$$

Then, almost everywhere on $[0,1]$,

$$\psi = \tau(1-\tau)\left(-\frac{\tau''}{(\tau')^3}f' + \frac{1}{(\tau')^2}f''\right).$$

Proposition 6. *Suppose that $f \in C[0,1]$. Then*

$$2n(B_n^\tau f(x) - f(x)) = o(1), \quad x \in (0,1),$$

if and only if $f \in \mathbb{P}_{\tau,1}$.

Proposition 7. *Suppose that $f \in C[0,1]$. Then*

$$2n|B_n^\tau f(x) - f(x)| \leq M, \quad x \in (0,1),$$

if and only, almost everywhere on $[0, 1]$,

$$\left|\tau(1-\tau)\left(-\frac{\tau''}{(\tau')^3}f' + \frac{1}{(\tau')^2}f''\right)\right| \le M.$$

We end this section by going into the aforementioned question raised in [5] related to the existence of polynomial operators that fix polynomials.

Let us fix an integer number $m \ge 2$, a real number $\alpha > 0$ and consider the function τ defined as

$$\tau(x) = \frac{x^m + \alpha x}{1+\alpha}. \tag{8}$$

Then the corresponding operators V_n^τ and B_n^τ fix e_0 and the polynomial τ, but only the B_n^τ are of polynomial type.

3. Comparing B_n, V_n^τ and B_n^τ

We devote this section to a comparative study of the approximation of functions which are increasing and fulfill certain additional convexity assumptions, by means of the sequences of operators B_n, V_n^τ and B_n^τ. We shall see that in a certain sense some improvements of the classical approximation by Bernstein polynomials are obtained. Note that if $f \in C^1[0, 1]$, then the convexity of f with respect to τ amounts to the fact that f'/τ' is increasing.

Theorem 8. *Let $f \in C[0, 1]$ be increasing and convex with respect to τ. Assume also that τ is convex. Then*

$$f(x) \le V_n^\tau f(x) \le B_n f(x), \quad 0 \le x \le 1.$$

Proof. The fact that $f(x) \le V_n^\tau f(x)$, $0 \le x \le 1$, follows from [14, Theorem 2]. For the other inequality, from the convexity of τ, $B_n\tau \ge \tau$. Now, $(B_n\tau)^{-1}$ is increasing, so

$$(B_n\tau)^{-1} \circ B_n\tau \ge (B_n\tau)^{-1} \circ \tau.$$

This implies that for $x \in [0, 1]$, $x \ge ((B_n\tau)^{-1} \circ \tau)(x)$ and directly

$$V_n^\tau f(x) \le B_n f(x). \quad \square$$

Theorem 9. *Suppose that $f \in C^2[0, 1]$. Suppose that there exists $n_0 \in \mathbb{N}$ such that*

$$f(x) \le B_n^\tau f(x) \le B_n f(x), \quad \textit{for all } n \ge n_0,\ x \in (0, 1). \tag{9}$$

Then

$$f''(x) \ge \frac{\tau''(x)}{\tau'(x)} f'(x) \ge \left(1 - \frac{x(1-x)\tau'(x)^2}{\tau(x)(1-\tau(x))}\right) f''(x), \quad x \in (0, 1). \tag{10}$$

In particular, $f''(x) \ge 0$.

Conversely, if (10) holds with strict inequalities at a given point $x_0 \in (0, 1)$, then there exists $n_0 \in \mathbb{N}$ such that for $n \ge n_0$

$$f(x_0) < B_n^\tau f(x_0) < B_n f(x_0).$$

Proof. From (9) we have that

$$0 \le 2n\left(B_n^\tau f(x) - f(x)\right) \le 2n\left(B_n f(x) - f(x)\right), \quad n \ge n_0,\ x \in (0, 1).$$

Then, using (7) (recall the classical Voronovskaya formula for B_n and the fact that $B_n^\tau = B_n$ if $\tau = e_1$),

$$0 \le \tau(x)(1-\tau(x))\left(-\frac{\tau''(x)f'(x)}{\tau'(x)^3} + \frac{f''(x)}{\tau'(x)^2}\right) \le x(1-x)f''(x)$$

from which (10) follows directly.

Conversely, if (10) holds with strict inequalities for a given $x_0 \in (0, 1)$, then directly

$$0 < \tau(x_0)(1-\tau(x_0))\left(-\frac{\tau''(x_0)f'(x_0)}{\tau'(x_0)^3} + \frac{f''(x_0)}{\tau'(x_0)^2}\right) < x_0(1-x_0)f''(x_0),$$

and using (7) again, the proof follows. $\square$

The next result is stated without proof since it follows the pattern given in the proof of Theorem 9. It suffices to use the asymptotic formula for the operators V_n^τ stated in [4, Theorem 5.1] which reads as follows:

$$\lim_{n\to\infty} n\left(V_n^\tau f(x) - f(x)\right) = \frac{x(1-x)}{2}\tau'(x)\left(\frac{f'}{\tau'}\right)'(x).$$

Theorem 10. *Suppose that $f \in C^2[0,1]$. Suppose that there exists $n_0 \in \mathbb{N}$ such that*

$$V_n^\tau f(x) \le B_n^\tau f(x), \quad n \ge n_0,\ x \in (0,1).$$

Then for all $x \in (0,1)$ the following inequality holds:

$$\left(1 - \frac{x(1-x)\tau'(x)^2}{\tau(x)(1-\tau(x))}\right) f''(x) \le \left(\frac{\tau''(x)}{\tau'(x)} - \frac{\tau'(x)\tau''(x)x(1-x)}{\tau(x)(1-\tau(x))}\right) f'(x). \tag{11}$$

Conversely, if (11) holds with strict inequalities at a given point $x_0 \in (0,1)$, then there exists $n_0 \in \mathbb{N}$ such that for $n \ge n_0$

$$V_n^\tau f(x_0) < B_n^\tau f(x_0).$$

4. A particular case: $\tau = (e_m + \alpha e_1)/(1+\alpha)$

Let us take an integer $m \ge 2$, a real $\alpha > 0$ and τ as in (8), i.e.

$$\tau(x) = \frac{x^m + \alpha x}{1+\alpha}.$$

This takes us to the setting in which the operators B_n^τ, V_n^τ fix polynomials of degree m.

Let us also assume that a function $f \in C^2[0,1]$ satisfies the hypothesis of Theorem 8, namely, f is increasing and convex with respect to τ (note that τ is obviously convex in the classical sense), which amounts to the fact that f'/τ' is increasing as well, which is equivalent to

$$f''(x) \ge \frac{m(m-1)x^{m-2}}{mx^{m-1}+\alpha} f'(x), \quad 0 \le x \le 1. \tag{12}$$

Thus the first inequality in (10) is satisfied and it is easy to see that the inequality (11) holds as well.

The second inequality in (10) is obviously true for those x for which the factor accompanying $f''(x)$ is non-positive, which after some calculations is equivalent to

$$w(x) := (x^m + \alpha x)(1 + \alpha - x^m - \alpha x) - x(1-x)(mx^{m-1} + \alpha)^2 \le 0.$$

By elementary considerations, it can be proved that w has a single root $\theta = \theta(m,\alpha) \in (0,1)$, $w > 0$ in $[0,\theta)$ and $w(x) < 0$ for each $x \in (\theta, 1]$. If we further restrict our attention to the case $m = 2$, then

$$\theta = \theta(2,\alpha) = \frac{1 - 2\alpha + \sqrt{4\alpha^2 + 8\alpha + 1}}{6},$$

which increases with α and satisfies $\theta(2,\alpha) \longrightarrow 1/3$ as $\alpha \to 0$ and $\theta(2,\alpha) \longrightarrow 1/2$ as $\alpha \to +\infty$.

Finally, we are going a bit further with the following result, which should be compared with [3, Theorem 1]:

Corollary 11. *Let $f \in C^2[0,1]$ be increasing and strictly convex. Then there exists $\alpha > 0$ such that the operators B_n^τ and V_n^τ for the function*

$$\tau(x) = \frac{x^2 + \alpha x}{1+\alpha}$$

satisfy the following properties:

(i) *For each $x \in [0,1]$*

$$f(x) \le V_n^\tau f(x) \le B_n f(x).$$

(ii) *For each $x \in (0,1)$ there exists $n_0 \in \mathbb{N}$ such that*

$$V_n^\tau f(x) < B_n^\tau f(x), \quad n \ge n_0.$$

(iii) *For each*

$$x \in \left(\frac{1 - 2\alpha + \sqrt{4\alpha^2 + 8\alpha + 1}}{6}, 1\right],$$

there exists $n_0 \in \mathbb{N}$ such that

$$f(x) < B_n^\tau f(x) < B_n f(x), \quad n \ge n_0.$$

Proof. Let M be a lower bound of f'' on [0, 1] and let us take $\alpha > 2f'(1)/M$. Thus,

$$\alpha > \frac{2f'(x)}{f''(x)} - 2x, \quad 0 < x < 1,$$

or equivalently

$$f''(x) - \frac{2}{2x+\alpha}f'(x) > 0, \quad 0 < x < 1,$$

which allows to end the proof after considering the results and remarks of the present section. □

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