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SOME IDENTITIES FOR BERNOULLI AND EULER POLYNOMIALS

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1. Introduction

Let $\mathbb{N}=\{0,1,2,\ldots\}$. The Bernoulli polynomials $B_n(x)$ $(n\in\mathbb{N})$ and the Euler polynomials $E_n(x)$ $(n\in\mathbb{N})$ are defined by means of

$$\frac{ze^{xz}}{e^z-1}=\sum_{n=0}^{\infty}B_n(x)\frac{z^n}{n!}\quad\text{and}\quad\frac{2e^{xz}}{e^z+1}=\sum_{n=0}^{\infty}E_n(x)\frac{z^n}{n!}.$$

Those $B_n=B_n(0)$ and $E_n=2^nE_n(1/2)$ are called the Bernoulli numbers and the Euler numbers respectively. From the definitions we can easily deduce the following well known properties:

$$B_n(1-x)=(-1)^nB_n(x)\text{ and }B_n(x+1)-B_n(x)=nx^{n-1};$$
$$E_n(1-x)=(-1)^nE_n(x)\text{ and }E_n(x+1)+E_n(x)=2x^n.$$

In 1995 M. Kaneko [1] found that B_{2n} can be computed in terms of those B_i with $n\le i<2n$, namely he proved the formula

$$\sum_{i=0}^{n}\binom{n+1}{i}(n+i+1)B_{n+i}=0\quad\text{for }n=1,2,3,\ldots.$$

In 2001 H. Momiyama [2] extended the above result as follows: If $m,n\in\mathbb{N}$ and $m+n>0$, then

$$(-1)^m\sum_{i=0}^{m}\binom{m+1}{i}(n+i+1)B_{n+i}+(-1)^n\sum_{j=0}^{n}\binom{n+1}{j}(m+j+1)B_{m+j}=0.\quad(1)$$

In this paper we aim to make further extensions by a new method.

Now we state our main results.

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Theorem 1. *Let $\{f_k(x)\}_{k=0}^{\infty}$ be a sequence of polynomials given by*

$$\sum_{k=0}^{\infty} f_k(x)\frac{z^k}{k!} = e^{(x-1/2)z}F(z) \tag{2}$$

where $F(z)$ is a formal power series. Let $m, n \in \mathbb{N}$. If F is even, i.e. $F(-z) = F(z)$, then

$$(-1)^m \sum_{i=0}^{m} \binom{m}{i} f_{n+i}(x) = (-1)^n \sum_{j=0}^{n} \binom{n}{j} f_{m+j}(-x); \tag{3}$$

if F is odd, i.e. $F(-z) = -F(z)$, then

$$(-1)^m \sum_{i=0}^{m} \binom{m}{i} f_{n+i}(x) = -(-1)^n \sum_{j=0}^{n} \binom{n}{j} f_{m+j}(-x). \tag{4}$$

This general theorem will be proved in Section 2. Now we give a consequence of it.

Corollary 1. *Let $F(z)$ be an even or odd formal power series, and let $f_k(x)$ $(k \in \mathbb{N})$ be given by (2). Let $m, n \in \mathbb{N}$ and $\varepsilon = 1$ or -1 according to whether $F(z)$ is even or odd. Then*

$$\begin{aligned}&(-1)^m \sum_{i=0}^{m+1} \binom{m+1}{i} (n+i+1) f_{n+i}(x)\\ =& -\varepsilon(-1)^n \sum_{j=0}^{n+1} \binom{n+1}{j} (m+j+1) f_{m+j}(-x).\end{aligned} \tag{5}$$

Proof. Clearly $-zF(-z) = -\varepsilon zF(z)$ and

$$e^{(x-1/2)z} zF(z) = z\sum_{k=0}^{\infty} f_k(x)\frac{z^k}{k!} = \sum_{k=1}^{\infty} f_k^*(x)\frac{z^k}{k!}$$

where $f_k^*(x) = kf_{k-1}(x)$. In view of Theorem 1, we have

$$(-1)^{m+1} \sum_{i=0}^{m+1} \binom{m+1}{i} f_{n+1+i}^*(x) = -\varepsilon(-1)^{n+1} \sum_{j=0}^{n+1} \binom{n+1}{j} f_{m+1+j}^*(-x)$$

which is equivalent to (5). □

Observe that

$$\sum_{n=0}^{\infty} B_n(x)\frac{z^n}{n!} = e^{(x-1/2)z}\frac{z}{e^{z/2} - e^{-z/2}}$$

and

$$\sum_{n=0}^{\infty} E_n(x)\frac{z^n}{n!} = e^{(x-1/2)z}\frac{2}{e^{z/2}+e^{-z/2}}.$$

Also,

$$\begin{aligned} &B_{m+n+1}(x) + (-1)^{m+n}B_{m+n+1}(-x) \\ =&B_{m+n+1}(x) - B_{m+n+1}(1-(-x)) = -(m+n+1)x^{m+n} \end{aligned}$$

and

$$\begin{aligned} &E_{m+n+1}(x) + (-1)^{m+n}E_{m+n+1}(-x) \\ =&E_{m+n+1}(x) - E_{m+n+1}(1+x) = 2E_{m+n+1}(x) - 2x^{m+n+1}. \end{aligned}$$

So Theorem 1 and Corollary 1 imply the following result.

Theorem 2. *Let $m, n \in \mathbb{N}$. Then*

$$(-1)^m \sum_{i=0}^{m}\binom{m}{i}B_{n+i}(x) = (-1)^n \sum_{j=0}^{n}\binom{n}{j}B_{m+j}(-x) \tag{6}$$

and

$$(-1)^m \sum_{i=0}^{m}\binom{m}{i}E_{n+i}(x) = (-1)^n \sum_{j=0}^{n}\binom{n}{j}E_{m+j}(-x); \tag{7}$$

also

$$\begin{aligned} &(-1)^m \sum_{i=0}^{m}\binom{m+1}{i}(n+i+1)B_{n+i}(x) \\ &+ (-1)^n \sum_{j=0}^{n}\binom{n+1}{j}(m+j+1)B_{m+j}(-x) \\ =&(-1)^m(m+n+2)(m+n+1)x^{m+n} \end{aligned} \tag{8}$$

and

$$\begin{aligned} &(-1)^m \sum_{i=0}^{m}\binom{m+1}{i}(n+i+1)E_{n+i}(x) \\ &+ (-1)^n \sum_{j=0}^{n}\binom{n+1}{j}(m+j+1)E_{m+j}(-x) \\ =&(-1)^m 2(m+n+2)\left(x^{m+n+1} - E_{m+n+1}(x)\right). \end{aligned} \tag{9}$$

Clearly (8) in the case $x = 0$ yields Momiyama's formula (1), and (9) provides a recurrent formula for Euler polynomials.

Putting $x = 0$ in (6) and $x = 1/2$ in (7) we then get

Corollary 2. *For $m, n \in \mathbb{N}$, we have*

$$(-1)^m \sum_{i=0}^{m} \binom{m}{i} B_{n+i} = (-1)^n \sum_{j=0}^{n} \binom{n}{j} B_{m+j} \tag{10}$$

and

$$(-1)^m \sum_{i=0}^{m} \binom{m}{i} \frac{E_{n+i}}{2^{n+i}} = (-1)^n \sum_{j=0}^{n} \binom{n}{j} E_{m+j}\left(-\frac{1}{2}\right). \tag{11}$$

2. Proof of Theorem 1

Suppose that $F(-z) = \varepsilon F(z)$ for all z where $\varepsilon \in \{1, -1\}$. Consider the generating function

$$G(x,y,z) := \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left((-1)^m \sum_{i=0}^{m}\binom{m}{i} f_{n+i}(x)\right)\frac{y^m}{m!}\cdot\frac{z^n}{n!}.$$

What we have to show is the identity $G(x,y,z) = \varepsilon G(-x,z,y)$. Changing the order of summation, we obtain

$$\begin{aligned}
G(x,y,z) =& \sum_{n=0}^{\infty}\sum_{i=0}^{\infty}\sum_{m=i}^{\infty}(-1)^m\binom{m}{i} f_{n+i}(x)\frac{y^m}{m!}\cdot\frac{z^n}{n!}\\
=& \sum_{n=0}^{\infty}\sum_{i=0}^{\infty} f_{n+i}(x)\frac{z^n}{n!}\sum_{m=i}^{\infty}(-1)^m\binom{m}{i}\frac{y^m}{m!}\\
=& \sum_{n=0}^{\infty}\sum_{i=0}^{\infty} f_{n+i}(x)\frac{z^n}{n!}\cdot\frac{(-y)^i}{i!}e^{-y}\\
=& e^{-y}\sum_{k=0}^{\infty}\sum_{i=0}^{k}\frac{f_k(x)}{k!}\binom{k}{i}z^{k-i}(-y)^i\\
=& e^{-y}\sum_{k=0}^{\infty} f_k(x)\frac{(z-y)^k}{k!}\\
=& e^{-y}e^{(x-1/2)(z-y)}F(z-y)\\
=& e^{x(z-y)-(y+z)/2}F(z-y).
\end{aligned}$$

From this, we have

$$G(-x,z,y) = e^{-x(y-z)-(z+y)/2}F(y-z) = e^{x(z-y)-(y+z)/2}\varepsilon F(z-y) = \varepsilon G(x,y,z),$$

as desired.

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Added in Proof. The main results of this paper were further extended in [3] by the second author.

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