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AN EXPLICIT FORMULA FOR HIGHER ORDER BERNOULLI POLYNOMIALS OF THE SECOND KIND

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Abstract

In this paper, the authors establish a formula expressing the Bernoulli polynomials of the second kind and general order k, $b_n^{(k)}(x)$, in terms of those of first order, $b_n(x) = b_n^{(1)}(x)$.

1. Introduction and Results

The Bernoulli polynomials $b_n^{(k)}(x)$ of the second kind and of order k, for any integer k, may be defined by (see [2,4,10])

$$\left(\frac{t}{\log(1+t)}\right)^k (1+t)^x = \sum_{n=0}^{\infty} b_n^{(k)}(x)t^n, \quad |t| < 1. \tag{1}$$

The numbers $b_n^{(k)} = b_n^{(k)}(0)$ are the Bernoulli numbers of the second kind and of order k; $b_n^{(1)} = b_n$, $b_n^{(1)}(x) = b_n(x)$ are the ordinary Bernoulli numbers and polynomials of the second kind (see [1,4,5, 10-13]), and $C_n = n!b_n$ are the Cauchy numbers of the first kind (see [8, 13]). By (1.1), we have

$$b_n^{(k)}(x) = \sum_{\substack{v_1,\cdots,v_k \in \mathbb{N}_0 \\ v_1+\cdots+v_k=n}} b_{v_1}(x/k)b_{v_2}(x/k)\cdots b_{v_k}(x/k), \tag{2}$$

$$b_n^{(k)} = \sum_{\substack{v_1,\cdots,v_k \in \mathbb{N}_0 \\ v_1+\cdots+v_k=n}} b_{v_1}b_{v_2}\cdots b_{v_k}, \tag{3}$$

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and

$$b_n^{(k)}(x) = b_0^{(k)}\binom{x}{n} + b_1^{(k)}\binom{x}{n-1} + \cdots + b_{n-1}^{(k)}\binom{x}{1} + b_n^{(k)}. \tag{4}$$

where $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{N}$ being the set of positive integers.

The numbers b_n satisfy the recurrence relation (see [1, 5])

$$b_0 = 1, \quad \sum_{j=0}^{n} \frac{(-1)^j}{n-j+1} b_j = 0 \quad (n \geq 1), \tag{5}$$

so we find $b_1 = \frac{1}{2}, b_2 = -\frac{1}{12}, b_3 = \frac{1}{24}, b_4 = -\frac{19}{720}, b_5 = \frac{3}{160}, b_6 = -\frac{863}{60480}$.

The numbers b_n satisfy many interesting relations. For example (see [1,5,8])

$$b_n = \int_0^1 \binom{x}{n} dx, \quad 1 - \log 2 = \sum_{n=1}^{\infty} \frac{|b_n|}{n+1}, \quad \gamma = \sum_{n=1}^{\infty} \frac{|b_n|}{n}, \quad 1 + \sum_{n=1}^{\infty} \frac{|b_n| H_n}{n} = \frac{\pi^2}{6}, \tag{6}$$

where $\binom{x}{n} = \frac{x(x-1)(x-2)\cdots(x-n+1)}{n!}$, γ is the Euler constant and $H_n = \sum_{k=1}^{n} \frac{1}{k}$ is the n-th harmonic number.

The Bernoulli polynomials $B_n^{(k)}(x)$ of order k, for any integer k, may be defined by (see [3,4,6,7,10])

$$\left(\frac{t}{e^t - 1}\right)^k e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!}, \quad |t| < 2\pi. \tag{7}$$

The numbers $B_n^{(k)} = B_n^{(k)}(0)$ are the Bernoulli numbers of order k, and $B_n^{(1)} = B_n$ are the ordinary Bernoulli numbers. The numbers $B_n^{(n)}$ are called Nörlund numbers (see [3]), or Cauchy numbers of the second kind (see [8, 13]). Nörlund found the exponential generating function (see [10, p.150])

$$\frac{t}{(1+t)\log(1+t)} = \sum_{n=0}^{\infty} B_n^{(n)} \frac{t^n}{n!} \quad (|t| < 1). \tag{8}$$

These numbers b_n, $b_n^{(k)}$, $B_n^{(n)}$ and $B_n^{(k)}$ satisfy various identities. For example (see [1-4])

$$n! b_n = B_n^{(n)} + n B_{n-1}^{(n-1)}, \quad B_n^{(n)} = n! \sum_{j=0}^{n} (-1)^{n-j} b_j, \quad \text{and} \quad n! b_n^{(k)} = \frac{k}{k-n} B_n^{(n-k)}. \tag{9}$$

The paper's central result is a formula expressing the Bernoulli polynomials of the second kind and general order k, $b_n^{(k)}(x)$, in terms of those of first order, $b_n(x) = b_n^{(1)}(x)$. That is, we shall prove the following main conclusion.

Theorem. *Let $n, k \in \mathbb{N}$ and $n \geq k-1$. Then*

$$(-1)^{k-1}(k-1)!(n-k)!b_n^{(k)}(x) = \sum_{j=0}^{k-1}(n-1-j)!$$
$$\times \sum_{\substack{v_1,\cdots,v_{k-j}\in\mathbb{N}_0 \\ v_1+\cdots+v_{k-j}=j}} (n-j-1-x)^{v_1}(n-j-2-x)^{v_2}\cdots(n-k-x)^{v_{k-j}}b_{n-j}(x). \quad (10)$$

By taking $x=0$ in Equation (10), we can deduce the following.

Corollary 1. *Let $n, k \in \mathbb{N}$ and $n \geq k-1$. Then*

$$(-1)^{k-1}(k-1)!(n-k)!b_n^{(k)}$$
$$= \sum_{j=0}^{k-1}(n-1-j)! \sum_{\substack{v_1,\cdots,v_{k-j}\in\mathbb{N}_0 \\ v_1+\cdots+v_{k-j}=j}} (n-j-1)^{v_1}(n-j-2)^{v_2}\cdots(n-k)^{v_{k-j}}b_{n-j}. \quad (11)$$

Taking $k=2,3,4$ in (10) and (11), we immediately deduce the following expressions for the first few higher order Bernoulli polynomials and numbers of the second kind:

$$b_n^{(2)}(x) = (1-n)b_n(x) + (x+2-n)b_{n-1}(x) \quad (n \geq 1);$$
$$b_n^{(3)}(x) = \frac{1}{2}(n-1)(n-2)b_n(x)$$
$$+\frac{1}{2}(n-2)(2n-5-2x)b_{n-1}(x) + \frac{1}{2}(n-3-x)^2b_{n-2}(x) \quad (n \geq 2);$$
$$b_n^{(4)}(x) = -\frac{1}{6}(n-1)(n-2)(n-3)b_n(x) - \frac{1}{6}(n-2)(n-3)(3n-9-3x)b_{n-1}(x)$$
$$-\frac{1}{6}(n-3)\left((n-3-x)^2 + (n-3-x)(n-4-x) + (n-4-x)^2\right)b_{n-2}(x)$$
$$-\frac{1}{6}(n-4-x)^3b_{n-3}(x) \quad (n \geq 3);$$

$$b_n^{(2)} = (1-n)b_n + (2-n)b_{n-1} \quad (n \geq 1);$$
$$b_n^{(3)} = \frac{1}{2}(n-1)(n-2)b_n + \frac{1}{2}(n-2)(2n-5)b_{n-1} + \frac{1}{2}(n-3)^2b_{n-2} \quad (n \geq 2);$$
$$b_n^{(4)} = -\frac{1}{6}(n-1)(n-2)(n-3)b_n - \frac{1}{6}(n-2)(n-3)(3n-9)b_{n-1}$$
$$-\frac{1}{6}(n-3)\left((n-3)^2 + (n-3)(n-4) + (n-4)^2\right)b_{n-2} - \frac{1}{6}(n-4)^3b_{n-3}$$
$$(n \geq 3).$$

By (11), (3), and noting that $C_n = n!b_n$, we obtain an explicit formula for the sum involving Cauchy numbers of the first kind:

$$(-1)^{k-1}(k-1)!(n-k)! \sum_{\substack{v_1,\cdots,v_k \in \mathbb{N}_0 \\ v_1+\cdots+v_k=n}} \frac{C_{v_1}C_{v_2}\cdots C_{v_k}}{v_1!v_2!\cdots v_k!}$$
$$= \sum_{j=0}^{k-1} \sum_{\substack{v_1,\cdots,v_{k-j} \in \mathbb{N}_0 \\ v_1+\cdots+v_{k-j}=j}} (n-j-1)^{v_1}(n-j-2)^{v_2}\cdots(n-k)^{v_{k-j}} \frac{C_{n-j}}{n-j}. \quad (12)$$

Corollary 2. *Let $n, k \in \mathbb{N}_0$ and $m \in \mathbb{N}$. Then*

$$(-1)^k k! \int_0^m b_n^{(k)}(x)dx = \sum_{i=1}^{n+1} \binom{m}{i} \sum_{j=0}^{k} \frac{(n-i-j)!}{(n-i-k)!}$$
$$\times \sum_{\substack{v_1,\cdots,v_{k+1-j} \in \mathbb{N}_0 \\ v_1+\cdots+v_{k+1-j}=j}} (n-i-j)^{v_1}(n-i-j-1)^{v_2}\cdots(n-i-k)^{v_{k+1-j}} b_{n+1-i-j}. \quad (13)$$

By taking $m = 1$ in (13), we can deduce the following:

$$(-1)^k k!(n-k-1)! \int_0^1 b_n^{(k)}(x)dx = \sum_{j=0}^{k} (n-1-j)!$$
$$\times \sum_{\substack{v_1,\cdots,v_{k+1-j} \in \mathbb{N}_0 \\ v_1+\cdots+v_{k+1-j}=j}} (n-j-1)^{v_1}(n-j-2)^{v_2}\cdots(n-k-1)^{v_{k+1-j}} b_{n-j}. \quad (14)$$

Taking $k = 0, 1$; $m = 1, 2, 3$ in (13) and noting that $b_n^{(0)}(x) = \binom{x}{n}$, we have

$$\int_0^1 \binom{x}{n} dx = b_n, \quad \int_0^2 \binom{x}{n} dx = 2b_n + b_{n-1}, \quad \int_0^3 \binom{x}{n} dx = 3b_n + 3b_{n-1} + b_{n-2}.$$

and

$$\int_0^1 b_n(x)dx = (1-n)b_n + (2-n)b_{n-1},$$
$$\int_0^2 b_n(x)dx = 2(1-n)b_n + 3(2-n)b_{n-1} + (3-n)b_{n-2},$$
$$\int_0^3 b_n(x)dx = 3(1-n)b_n + 6(2-n)b_{n-1} + 4(3-n)b_{n-2} + (4-n)b_{n-3}.$$

2. Proof of Theorem

In this section, we shall complete the proof of the theorem. First, the following lemma (see [2,10]) is crucial to the proof of the theorem. To be more self-contained, we present a simpler proof here.

Lemma. *Let $n, k \in \mathbb{N}_0$ and $m \in \mathbb{N}$. Then*

$$b_n^{(k+1)}(x) = \frac{n-k}{-k} b_n^{(k)}(x) + \frac{n-k-1-x}{-k} b_{n-1}^{(k)}(x). \tag{15}$$

Proof. By (1), we have

$$\begin{aligned}\sum_{n=0}^{\infty}(n-k)b_n^{(k)}(x)t^{n-k-1} &= \frac{d}{dt}\left(\frac{1}{\log(1+t)}\right)^k (1+t)^x \\ &= -k\left(\frac{1}{\log(1+t)}\right)^{k+1}(1+t)^{x-1} + x\left(\frac{1}{\log(1+t)}\right)^k (1+t)^{x-1}, \end{aligned} \tag{16}$$

i.e.,

$$\begin{aligned}&\sum_{n=1}^{\infty}(n-k-1)b_{n-1}^{(k)}(x)t^{n-k-1} \\ &= -kt\left(\frac{1}{\log(1+t)}\right)^{k+1}(1+t)^{x-1} + xt\left(\frac{1}{\log(1+t)}\right)^k (1+t)^{x-1}. \end{aligned} \tag{17}$$

By (16) and (17), we have

$$\begin{aligned}&\sum_{n=0}^{\infty}(n-k)b_n^{(k)}(x)t^{n-k-1} + \sum_{n=1}^{\infty}(n-k-1)b_{n-1}^{(k)}(x)t^{n-k-1} \\ &= -k\left(\frac{1}{\log(1+t)}\right)^{k+1}(1+t)^{x} + x\left(\frac{1}{\log(1+t)}\right)^k (1+t)^{x} \\ &= -k\sum_{n=0}^{\infty} b_n^{(k+1)}(x)t^{n-k-1} + x\sum_{n=0}^{\infty} b_n^{(k)}(x)t^{n-k}. \end{aligned} \tag{18}$$

Comparing the coefficient of t^{n-k-1} on both sides of (18), we get

$$b_n^{(k+1)}(x) = \frac{n-k}{-k} b_n^{(k)}(x) + \frac{n-k-1-x}{-k} b_{n-1}^{(k)}(x).$$

This proves the lemma. □

Now we complete the proof of the theorem by using mathematical induction and the method of coefficients (see [9]).

Proof of theorem. First note that (10) holds for $k = 1, 2$, by (15). Now suppose (10) is true for some natural number k and all $n \geq k-1$. By superposition of (15), we have

$$
\begin{aligned}
&(-1)^k k!(n-k-1)! b_n^{(k+1)}(x) \\
&= (-1)^{k-1}(k-1)!(n-k)! b_n^{(k)}(x) + (-1)^{k-1}(k-1)!(n-k-1)!(n-k-1-x) b_{n-1}^{(k)}(x) \\
&= \sum_{j=0}^{k-1}(n-1-j)! \\
&\quad \times \sum_{\substack{v_1,\cdots,v_{k-j}\in\mathbb{N}_0 \\ v_1+\cdots+v_{k-j}=j}} (n-j-1-x)^{v_1}(n-j-2-x)^{v_2}\cdots(n-k-x)^{v_{k-j}} b_{n-j}(x) \\
&\quad +(n-k-1-x)\sum_{j=0}^{k-1}(n-2-j)! \\
&\quad \times \sum_{\substack{v_1,\cdots,v_{k-j}\in\mathbb{N}_0 \\ v_1+\cdots+v_{k-j}=j}} (n-j-2-x)^{v_1}(n-j-3-x)^{v_2}\cdots(n-k-1-x)^{v_{k-j}} b_{n-1-j}(x) \\
&= \sum_{j=0}^{k-1}(n-1-j)! \\
&\quad \times \sum_{\substack{v_1,\cdots,v_{k-j}\in\mathbb{N}_0 \\ v_1+\cdots+v_{k-j}=j}} (n-j-1-x)^{v_1}(n-j-2-x)^{v_2}\cdots(n-k-x)^{v_{k-j}} b_{n-j}(x) \\
&\quad +(n-k-1-x)\sum_{j=1}^{k}(n-1-j)! \\
&\quad \times \sum_{\substack{v_1,\cdots,v_{k+1-j}\in\mathbb{N}_0 \\ v_1+\cdots+v_{k+1-j}=j-1}} (n-j-1-x)^{v_1}(n-j-2-x)^{v_2}\cdots(n-k-1-x)^{v_{k+1-j}} b_{n-j}(x) \\
&= \sum_{j=0}^{k}(n-1-j)! \\
&\quad \times \sum_{\substack{v_1,\cdots,v_{k+1-j}\in\mathbb{N}_0 \\ v_1+\cdots+v_{k+1-j}=j}} (n-j-1-x)^{v_1}(n-j-2-x)^{v_2}\cdots(n-k-1-x)^{v_{k+1-j}} b_{n-j}(x),
\end{aligned}
$$

which shows that (10) is also true for the natural number $k+1$. The theorem follows by induction. □

Now we complete the proof of Corollary 2.

Proof of Corollary 2. By (11), (4), and noting that $\frac{d}{dx} b_n^{(k)}(x) = b_{n-1}^{(k-1)}(x)$ (see [11]), we have

$$(-1)^k k! \int_0^m b_n^{(k)}(x)dx = (-1)^k k! \left(b_{n+1}^{(k+1)}(m) - b_{n+1}^{(k+1)} \right) = (-1)^k k! \sum_{i=1}^{n+1} \binom{m}{i} b_{n+1-i}^{(k+1)}$$

$$= \sum_{i=1}^{n+1} \binom{m}{i} \sum_{j=0}^{k} \frac{(n-i-j)!}{(n-i-k)!}$$

$$\times \sum_{\substack{v_1,\cdots,v_{k+1-j}\in\mathbb{N}_0 \\ v_1+\cdots+v_{k+1-j}=j}} (n-i-j)^{v_1}(n-i-j-1)^{v_2}\cdots(n-i-k)^{v_{k+1-j}} b_{n+1-i-j}.$$

This completes the proof. □

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