

SOME RESULTS FOR GENERALIZED  
BERNOULLI, EULER, STIRLING NUMBERS

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SUMMARY

The present paper is a continuation of researches begun by the author in previous publications [3, 4, 5] on three classes of generalized Bernoulli, Euler, Stirling numbers. And here, of course, will be proved some additional interesting results.

1. GENERALIZED BERNOULLI, EULER NUMBERS AND POLYNOMIALS

The generalized Bernoulli, Euler numbers in question, and the related polynomials, are defined by the series

$$(1.1) \quad f(t; h, w) = \frac{ht}{(1+wt)^{h/w} - 1} = \sum_{n=0}^{\infty} B_{n;h,w} \frac{t^n}{n!},$$

$$(1.2) \quad \varphi(t; h, w) = \frac{2h(1+wt)^{1/w}}{(1+wt)^{2h/w} + 1} = \sum_{n=0}^{\infty} E_{n;h,w} \frac{t^n}{n!},$$

$$(1.3) \quad F(x, t; h, w) = \frac{ht(1+wt)^{x/w}}{(1+wt)^{h/w} - 1} = \sum_{n=0}^{\infty} B_{n;h,w}(x) \frac{t^n}{n!},$$

$$(1.4) \quad \phi(x, t; h, w) = \frac{2h(1+wt)^{x/w}}{(1+wt)^{h/w} + 1} = \sum_{n=0}^{\infty} E_{n;h,w}(x) \frac{t^n}{n!},$$

where  $h$  and  $w$  are real parameters.

These series, for a correct treatment, will be considered in the neighborhood of the origin.

The explicit expressions of  $B_{n;h,w}(x)$  for  $n = 0, 1, \dots, 5$  are

$$\begin{aligned} B_{0;h,w}(x) &= 1, \\ B_{1;h,w}(x) &= \frac{1}{2}(2x - h + w), \\ B_{2;h,w}(x) &= x(x - h) + \frac{1}{6}(h^2 - w^2), \\ B_{3;h,w}(x) &= \frac{1}{2}x(x - h)(2x - h - 3w) - \frac{1}{4}w(h^2 - w^2), \\ B_{4;h,w}(x) &= x(x - h)(x - 2w)(x - h - 2w) - \frac{1}{30}(h^2 - w^2)(h^2 - 19w^2) \\ B_{5;h,w}(x) &= x(x - h)[x^3 - \frac{3}{2}(h + 5w)x^2 + \frac{1}{6}(h^2 + 45hw + 110w^2)x + \frac{1}{6}h^3 - \frac{55}{6}hw^2 \\ &\quad - 15w^3] + \frac{1}{4}w(h^2 - w^2)(h^2 - 9w^2). \end{aligned}$$

And these can be deduced by the recurrent relation [4]

$$\sum_{r=0}^{n-1} \binom{n}{r} (-w)^{-r} \left(1 - \frac{h}{w}\right)_{n-r-1} B_{r;h,w}(x) = n(-x/w)_{n-1}, \quad n > 0,$$

where  $(a)_0 = 1$ ,  $(a)_r = a(a+1) \dots (a+r-1)$ .

The explicit expressions of  $E_{n;h,w}(x)$  for  $n = 0, 1, \dots, 5$  are

$$\begin{aligned} E_{0;h,w}(x) &= h, \\ E_{1;h,w}(x) &= \frac{1}{2}h(2x - h), \\ E_{2;h,w}(x) &= hx^2 - h(h+w)x + \frac{1}{2}h^2w, \\ E_{3;h,w}(x) &= hx^3 - \frac{3}{2}h(h+2w)x^2 + hw(3h+2w)x + \frac{1}{4}h^2(h^2 - 4w^2), \\ E_{4;h,w}(x) &= hx^4 - 2h(h+3w)x^3 + hw(9h+11w)x^2 + h(h^3 - 11hw^2 - 6w^3)x \\ &\quad - \frac{3}{2}h^2w(h^2 - 2w^2), \\ E_{5;h,w}(x) &= hx^5 - \frac{5}{2}h(h+4w)x^4 + 5hw(4h+7w)x^3 + \frac{5}{2}h(h^3 - 21hw^2 - 20w^3)x^2 \\ &\quad - 2hw(5h^3 - 25hw^2 - 12w^3)x - \frac{1}{2}h^2(h^4 - \frac{35}{2}h^2w^2 + 24w^4). \end{aligned}$$

And these can be deduced by the recurrent relation [4]

$$2E_{n;h,w}(x) + \sum_{r=0}^{n-1} \binom{n}{r} (-w)^{n-r} (-x/w)_{n-r} E_{r;h,w}(x) = 2h(-w)^n (-x/w)_n,$$

$n > 0$ . For relations with generalized Bernoulli, Euler, polynomials, it is easy to see that generalized Bernoulli, Euler, numbers can be derived by the formulas

$$B_{n;h,w} = B_{n;h,w}(0), \quad E_{n;h,w} = 2^n E_{n;h,w/2}(\frac{1}{2}).$$

The first six values of  $E_{n;h,w}$  are given by

$$\begin{aligned} E_{0;h,w} &= h, & E_{1;h,w} &= h(1-h), \\ E_{2;h,w} &= h(1-2h) + h(h-1)w, \\ E_{3;h,w} &= h(h-1)(2h^2+2h-1) + 3h(2h-1)w + 2h(1-h)w^2, \\ E_{4;h,w} &= h(2h-1)(4h^2+2h-1) + 6h(1-h)(2h^2+2h-1)w + 11h(1-2h)w^2 + 6h(h-1)w^3, \\ E_{5;h,w} &= 4h(1-h)(4h^4+4h^3-h^2-h-1) - 10(8h^4-4h^2+1)w + 35h(h-1)(2h^2+2h-1)w^2 \\ &\quad + 50h(2h-1)w^3 + 24h(1-h)w^4. \end{aligned}$$

Moreover, it will be useful to estimate also the expression

$$F_{n;h,w} = 2^n E_{n;h,w/2}(h/2).$$

And here, of course, we introduce the particular expressions for  $n = 0, 1, \dots, 5$ :

$$\begin{aligned} F_{0;h,w} &= h, & F_{1;h,w} &= 0, & F_{2;h,w} &= -h^3, \\ F_{3;h,w} &= 3h^3w, & F_{4;h,w} &= h^3(5h^2 - 11w^2), & F_{5;h,w} &= -50h^3w(h^2 - w^2). \end{aligned}$$

The theory of generalized Bernoulli, Euler, numbers and polynomials was first investigated by R. Lagrange [1], L. Tanzi Cattabianchi [2], and later extensively in the author's paper [4].

If  $h = 1$ ,  $w = 0$ , the numbers  $B_{n;h,w}$ ,  $E_{n;h,w}$ , and the polynomials  $B_{n;h,w}(x)$ ,  $E_{n;h,w}(x)$ , reduce to the ordinary Bernoulli, Euler, numbers and polynomials, generally defined by the generating expansions

$$(1.5) \quad \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \quad |t| < 2\pi,$$

$$(1.6) \quad \frac{2e^t}{e^{2t} + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \quad |t| < \pi/2,$$

$$(1.7) \quad \frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad |t| < 2\pi,$$

$$(1.8) \quad \frac{2e^{tx}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad |t| < \pi.$$

## 2. GENERALIZED STIRLING NUMBERS

The ordinary Stirling numbers of the first and second kind  $s_{n,r}$ ,  $S_{n,r}$ , are defined by the initial values

$$s_{n,1} = (-1)^{n-1}(n-1)!, \quad s_{n,n} = 1,$$

$$S_{n,1} = S_{n,n} = 1,$$

and the recurrences

$$s_{n,r} = s_{n-1,r-1} - (n-1)s_{n-1,r}, \quad 1 < r < n,$$

$$S_{n,r} = S_{n-1,r-1} + rS_{n-1,r}, \quad 1 < r < n,$$

with

$$\begin{aligned} s_{n,0} &= 0, & S_{n,0} &= 0, \\ s_{n,r} &= 0, & S_{n,r} &= 0, \text{ provided } r > n. \end{aligned}$$

In our paper [3], they have been generalized with the coefficients  $a_{n,r}^{(u)}$  satisfying the recurrence

$$a_{n,r}^{(u)} = a_{n-1,r-1}^{(u)} - [n+r(u-1)-1]a_{n-1,r}^{(u)}, \quad 1 < r < n,$$

with

$$\begin{aligned} a_{n,1}^{(u)} &= (-1)^{n-1}(u)_{n-1}, & a_{n,n}^{(u)} &= 1, \\ a_{n,0}^{(u)} &= 0, & a_{n,r}^{(u)} &= 0 \text{ provided } r > n. \end{aligned}$$

And the particular expressions of  $a_{n,r}^{(u)}$  for  $n = 1, 2, \dots, 5$ ,  $r = 1, 2, \dots, 5$ , are

$$\begin{aligned} a_{1,1}^{(u)} &= 1, & a_{2,1}^{(u)} &= -u, & a_{2,2}^{(u)} &= 1, \\ a_{3,1}^{(u)} &= (u)_2, & a_{3,2}^{(u)} &= -3u, & a_{3,3}^{(u)} &= 1, \\ a_{4,1}^{(u)} &= -(u)_3, & a_{4,2}^{(u)} &= u(7u+4), & a_{4,3}^{(u)} &= -6u, & a_{4,4}^{(u)} &= 1, \\ a_{5,1}^{(u)} &= (u)_4, & a_{5,2}^{(u)} &= -5u(u+1)(3u+2), & a_{5,3}^{(u)} &= 5u(5u+2), \\ a_{5,4}^{(u)} &= -10u, & a_{5,5}^{(u)} &= 1. \end{aligned}$$

Our paper [3] presents an extensive treatment of the coefficients  $a_{n,r}^{(u)}$ , and it is interesting to note here that

$$(2.1) \quad a_{n,r}^{(u)} = \frac{(-1)^{n-r}}{(u-1)^r r!} \Delta_{u-1}^r x(x)_n \text{ provided } x = 0,$$

where  $\Delta_x$  is the descending difference defined by the relation  $\Delta_x f(x) = f(x+v) - f(x)$ ,

$$(2.2) \quad a_{n,r}^{(u)} = \frac{(-1)^n}{(u-1)^r r!} \sum_{k=1}^r (-1)^k \binom{r}{k} (ku-k)_n.$$

$$(2.3) \quad (-1)^n(x)_n = \sum_{r=1}^n a_{n,r}^{(u)}(u-1)^r \left( \frac{x}{1-u} \right)_r,$$

$$(2.4) \quad (-1)^n(x)_n = \sum_{r=0}^n a_{n+1,r+1}^{(u)}(u-1)^r \left( \frac{x-u}{1-u} \right)_r,$$

$$(2.5) \quad a_{n,r}^{(u)} = \sum_{k=r}^n s_{n,k} S_{k,r} (1-u)^{k-r},$$

from which

$$(2.6) \quad a_{n,r}^{(1)} = s_{n,r},$$

$$(2.7) \quad \lim_{u \rightarrow \infty} [(1-u)^{r-n} a_{n,r}^{(u)}] = s_{n,r};$$

$$(2.8) \quad a_{n,r}^{(0)} = \begin{cases} 0, & r < n, \\ 1, & r = n, \end{cases}$$

$$(2.9) \quad a_{n,r}^{(2)} = (-1)^{n-r} \frac{n!}{r!} \binom{n-1}{r-1},$$

$$(2.10) \quad a_{n,r}^{(-1)} = \frac{1}{2^{n-r}} \cdot \frac{n!}{r!} \binom{r}{n-r}, \quad r \geq n/2,$$

$$(2.11) \quad a_{n,r}^{(1/2)} = \frac{(-1)^{n-r}}{2^{2n-2r}} \cdot \frac{(2n-r-1)!}{(n-1)!} \binom{n-1}{r-1}.$$

For references and applications of the coefficients  $a_{n,r}^{(u)}$  to the operators satisfying the condition of *permutableness of the second order*, see the more recent our paper [5].

### 3. PARTICULAR EXPANSIONS. $n^{\text{th}}$ DERIVATIVE OF

$$y(t) = \frac{(1+wt)^{h/w} + i}{(1+wt)^{h/w} - i}, \quad \text{where } i^2 = -1.$$

From (1.5), placing to the left member the term  $1, -t/2$  under the summation sign of the right member, we find, as it is well known, the expansion

$$(3.1) \quad \tan t = \sum_{n=0}^{\infty} (-1)^n 2^{2n+2} (2^{2n+2} - 1) B_{2n+2} \frac{t^{2n+1}}{(2n+2)!}, \quad |t| < \pi/2.$$

Now, an expansion analogous to (3.1) will be derived from (1.1), proceeding similarly. First of all we have

$$\begin{aligned} \frac{ht}{(1+wt)^{h/w} - 1} - 1 + \frac{1}{2}(h-w)t &= \frac{1}{2[(1+wt)^{h/w} - 1]} (2ht + [(h-w)t - 2][(1+wt)^{h/w} - 1]) \\ &= \sum_{r=2}^{\infty} B_{r;h,w} \frac{t^r}{r!}. \end{aligned}$$

At once, changing at first  $t$  with  $4t$ ,  $w$  with  $w/4$ , and after  $t$  with  $2t$ ,  $w$  with  $w/2$ , we obtain the expansions

$$\begin{aligned} \frac{1}{2[(1+wt)^{4h/w} - 1]} (8ht + [4ht - wt - 2][(1+wt)^{4h/w} - 1]) \\ = \sum_{r=2}^{\infty} 2^{2r} B_{r;h,w/4} \frac{t^r}{r!}. \end{aligned}$$

$$\frac{1}{2[(1+wt)^{2h/w} - 1]} (4ht + [2ht - wt - 2][(1+wt)^{2h/w} - 1]) = \sum_{r=2}^{\infty} 2^r B_{r;h,w/2} \frac{t^r}{r!},$$

from which it follows that

$$\begin{aligned} \sum_{r=2}^{\infty} (2^{2r} B_{r;h,w/4} - 2^r B_{r;h,w/2}) \frac{t^r}{r!} &= \frac{1}{2[(1+wt)^{4h/w} - 1]} (8ht + [4ht - wt - 2][(1+wt)^{4h/w} - 1] \\ &\quad - 4ht[(1+wt)^{2h/w} + 1] - [2ht - wt - 2][(1+wt)^{4h/w} - 1]) \\ &= \frac{ht[(1+wt)^{2h/w} - 1]^2}{(1+wt)^{4h/w} - 1} = \frac{ht[(1+wt)^{2h/w} - 1]}{(1+wt)^{2h/w} + 1}. \end{aligned}$$

$$(3.2) \quad \frac{h[(1+wt)^{2h/w} - 1]}{(1+wt)^{2h/w} + 1} = \sum_{n=0}^{\infty} 2^{n+2} (2^{n+2} B_{n+2;h,w/4} - B_{n+2;h,w/2}) \frac{t^{n+1}}{(n+2)!}.$$

Moreover by (1.4), replacing  $x$  by  $h/2$ ,  $w$  by  $w/2$ ,  $t$  by  $2t$ , we obtain the other expansion

$$(3.3) \quad \frac{2h(1+wt)^{h/w}}{(1+wt)^{2h/w} + 1} = \sum_{n=0}^{\infty} 2^n E_{n;h,w/2}(h/2) \frac{t^n}{n!},$$

and since

$$\begin{aligned} \frac{h}{i} \cdot \frac{(1+wt)^{2h/w} - 1}{(1+wt)^{2h/w} + 1} + \frac{2h(1+wt)^{h/w}}{(1+wt)^{2h/w} + 1} &= \frac{h[(1+wt)^{h/w} + i]^2}{i[(1+wt)^{2h/w} + 1]} \\ &= \frac{h}{i} \cdot \frac{(1+wt)^{h/w} + i}{(1+wt)^{h/w} - i}, \end{aligned}$$

we deduce thus the interesting expansion

$$(3.4) \quad \frac{h}{i} \cdot \frac{(1+wt)^{h/w} + i}{(1+wt)^{h/w} - i} = \sum_{n=0}^{\infty} \frac{2^{n+2}}{i} (2^{n+2} B_{n+2;h,w/4} - B_{n+2;h,w/2}) \frac{t^{n+1}}{(n+2)!} + \sum_{n=0}^{\infty} 2^n E_{n;h,w/2}(h/2) \frac{t^n}{n!}.$$

After this expansion and for the following, it will be to estimate the  $n^{\text{th}}$  derivative of

$$y(t) = \frac{(1+wt)^{h/w} + i}{(1+wt)^{h/w} - i}.$$

Now we consider two continuous and derivable functions  $y = f(u)$ ,  $u = \varphi(t)$ , and the formula for derivatives of a composite function

$$\frac{d^n y}{dt^n} = \sum_{r=1}^n \frac{(-1)^r}{r!} \cdot \frac{d^r y}{du^r} \sum_{k=1}^r (-1)^k \binom{r}{k} u^{r-k} \frac{d^k u}{dt^n}, \quad n > 0.$$

With the assumption

$$y = f(u) = \frac{2i}{u-i} + 1, \quad u = \varphi(t) = (1+wt)^{h/w},$$

we arrive at

$$\begin{aligned} \frac{d^n y}{dt^n} &= \sum_{r=1}^n \left[ \frac{(-1)^r}{r!} \cdot \frac{2i(-1)^r r!}{(u-i)^{r+1}} \cdot \sum_{k=1}^r (-1)^k \binom{r}{k} u^{r-k} (-w)^n \left( \frac{-kh}{w} \right)_n u^{k-nw/h} \right] \\ &= \frac{2i(-w)^n}{(1+wt)^n} \sum_{r=1}^n \left[ \frac{(1+wt)^{rh/w}}{[(1+wt)^{h/w} - i]^{r+1}} \cdot \sum_{k=1}^r (-1)^k \binom{r}{k} \left( \frac{-kh}{w} \right)_n \right]. \end{aligned}$$

Whence, by (2.2), we deduce, for  $n > 0$ ,

$$(3.5) \quad \frac{d^n}{dt^n} \frac{(1+wt)^{h/w} + i}{(1+wt)^{h/w} - i} = \frac{2iw^n}{(1+wt)^n} \sum_{r=1}^n \frac{(-1)^r r! h^r}{w^r} \cdot a_{n,r}^{(1-h/w)} \cdot \frac{(1+wt)^{rh/w}}{[(1+wt)^{h/w} - i]^{r+1}}.$$

Successively, putting  $t = 0$ , we have

$$\left[ \frac{d^n}{dt^n} \frac{(1+wt)^{h/w} + i}{(1+wt)^{h/w} - i} \right]_{t=0} = 2iw^n \sum_{r=1}^n \frac{(-1)^r r! h^r}{w^r} \cdot a_{n,r}^{(1-h/w)} \cdot \frac{1}{(1-i)^{r+1}}.$$

Moreover it is

$$\begin{aligned} \frac{1}{1-i} &= \frac{1+i}{2} = \frac{1}{\sqrt{2}} (\cos \pi/4 + i \sin \pi/4), \\ \frac{1}{(1-i)^{r+1}} &= \frac{1}{2^{(r+1)/2}} [\cos (r+1)\pi/4 + i \sin (r+1)\pi/4] \\ &= \frac{1}{2^{(r+1)/2}} [\cos (r-1)\pi/4 + i \sin (r-1)\pi/4], \end{aligned}$$

and, however, it follows that ( $n > 0$ )

$$(3.6) \quad \left[ \frac{d^n}{dt^n} \frac{(1+wt)^{h/w} + i}{(1+wt)^{h/w} - i} \right]_{t=0} = w^n \sum_{r=1}^n \frac{(-1)^{r-1} r! h^r}{2^{(r-1)/2} w^r} \cdot a_{n,r}^{(1-h/w)} [\cos (r-1)\pi/4 + i \sin (r-1)\pi/4].$$

#### 4. FORMULAS FOR THE GENERALIZED BERNOULLI, EULER, STIRLING NUMBERS

We now, by (3.6), deduce the expansion of the function  $\gamma(t)$  into a series of powers of  $t$ ,

$$\begin{aligned} \gamma(t) &= \sum_{n=0}^{\infty} \left[ \frac{d^n \gamma(t)}{dt^n} \right]_{t=0} \frac{t^n}{n!} \\ &= i + \sum_{n=1}^{\infty} \frac{(wt)^n}{n!} \sum_{r=1}^n \frac{(-1)^{r-1} r! h^r}{2^{(r-1)/2} w^r} a_{n,r}^{(1-h/w)} [\cos (r-1)\pi/4 + i \sin (r-1)\pi/4]. \end{aligned}$$

Hence, comparing with (3.4), we obtain the expansion

$$\begin{aligned} &\sum_{n=1}^{\infty} 2^{n+1} (2^{n+1} B_{n+1;h,w/4} - B_{n+1;h,w/2}) \frac{t^n}{(n+1)!} \\ &+ i E_{0;h,w/2}(h/2) + i \cdot \sum_{n=1}^{\infty} 2^n E_{n;h,w/2}(h/2) \frac{t^n}{n!} = ih \\ &+ \sum_{n=1}^{\infty} \frac{(wt)^n}{n!} \sum_{r=1}^n \frac{(-1)^{r-1} r! h^{r+1}}{2^{(r-1)/2} w^r} a_{n,r}^{(1-h/w)} [\cos (r-1)\pi/4 + i \sin (r-1)\pi/4], \end{aligned}$$

from which, separating real and imaginary parts, and equating the coefficients of  $t^n$  on both sides, we establish the interesting formulas

$$(4.1) \quad \begin{aligned} & 2^{n+1} (2^{n+1} B_{n+1;h,w/4} - B_{n+1;h,w/2}) \\ &= (n+1) \sum_{r=1}^n \frac{(-1)^{r-1} r! h^{r+1} w^{n-r}}{2^{(r-1)/2}} a_{n,r}^{(1-h/w)} \cos (r-1)\pi/4, \end{aligned}$$

$$(4.2) \quad \begin{aligned} & 2^n E_{n;h,w/2}(h/2) \\ &= \sum_{r=1}^n \frac{(-1)^{r-1} r! h^{r+1} w^{n-r}}{2^{(r-1)/2}} a_{n,r}^{(1-h/w)} \sin (r-1)\pi/4, \end{aligned}$$

both for  $n > 0$ . They realize the principal objective of the present paper.

### 5. PARTICULAR FORMULAS

In this section we shall indicate some special cases of (4.1), (4.2).

(a) If  $h = 1, w = 0$ , the generalized numbers  $B_{n;h,w}, E_{n;h,w}$ , reduce to the ordinary Bernoulli, Euler, numbers, while

$$\lim_{w \rightarrow \infty} [w^{n-r} a_{n,r}^{(1-1/w)}] = S_{n,r}.$$

Moreover, it is  $B_{2n+1} = 0$  for  $n > 0, E_{2n+1} = 0$ . Consequently, by (4.1), (4.2), we deduce the formulas

$$(5.1) \quad \frac{2^{2n+1} (2^{2n+2} - 1)}{n+1} B_{2n+2} = \sum_{r=1}^{2n+1} \frac{(-1)^{r-1} r!}{2^{(r-1)/2}} S_{2n+1,r} \cos (r-1)\pi/4,$$

$$(5.2) \quad \sum_{r=1}^{2n+1} \frac{(-1)^{r-1} r!}{2^{(r-1)/2}} S_{2n+1,r} \sin (r-1)\pi/4 = 0,$$

$$(5.3) \quad \sum_{r=1}^{2n} \frac{(-1)^{r-1} r!}{2^{(r-1)/2}} S_{2n,r} \cos (r-1)\pi/4 = 0, \quad n > 0,$$

$$(5.4) \quad E_{2n} = \sum_{r=2}^{2n} \frac{(-1)^{r-1} r!}{2^{(r-1)/2}} S_{2n,r} \sin (r-1)\pi/4, \quad n > 0.$$

Equations (5.1) and (5.4) are two additional formulas concerning ordinary Bernoulli, Euler, Stirling numbers.

(b) If  $w = h$ , we have (2.8)

$$a_{n,r}^{(1-h/w)} = \begin{cases} 0, & r < n, \\ 1, & r = n, \end{cases}$$

therefore, (4.1) reduces to

$$(5.5) \quad 2^{n+1} (2^{n+1} B_{n+1;h,h/4} - B_{n+1;h,h/2}) = \frac{(n+1)! (-h)^{n+1}}{2^{(n-1)/2}} \cos (n-1)\pi/4.$$

Moreover, by the recurrent relation for  $B_{n;h,h/2}$ , it follows that

$$B_{n;h,h/2} = -\frac{nh}{4} B_{n-1;h,h/2},$$

from which

$$(5.6) \quad B_{n;h,h/2} = \frac{n!(-h)^n}{2^{2n}}.$$

Consequently, by (5.5) we deduce

$$(5.7) \quad B_{n;h,h/4} = \frac{n!(-h)^n}{2^{3n}} [1 + 2^{(n+2)/2} \sin(n\pi/4)].$$

This formula can be derived by the recurrent relation

$$B_{n;h,h/4} + \frac{3nh}{8} B_{n-1;h,h/4} + \frac{n(n-1)h^2}{16} B_{n-2;h,h/4} \\ + \frac{n(n-1)(n-2)h^3}{256} B_{n-3;h,h/4} = 0, \quad n > 0,$$

easily transformable in other forms to constant coefficients.

(c) If  $w = -h$ , we have (2.9)

$$a_{n,r}^{(1-h/w)} = (-1)^{n-r} \frac{n!}{r!} \binom{n-1}{r-1},$$

and (4.1) reduces to

$$(5.8) \quad 2^{n+1} (2^{n+1} B_{n+1;h,-h/4} - B_{n+1;h,-h/2}) \\ = (n+1)! h^{n+1} \sum_{r=1}^n \frac{(-1)^{r-1} \binom{n-1}{r-1}}{2^{(r-1)/2}} \cos(r-1)\pi/4, \quad n > 0.$$

Moreover, it is [4]

$$B_{n;h,-w} = (-1)^n B_{n;h,w},$$

and comparing (5.5) with (5.8) we have the identity, for  $n > 0$ ,

$$(5.9) \quad \sum_{r=1}^n (-1)^{r-1} \binom{n-1}{r-1} 2^{(n-r)/2} \cos(r-1)\pi/4 = \cos(n-1)\pi/4.$$

Putting into (4.2) at first  $w = h$  and after  $w = -h$ , and remembering that [4]

$$E_{n;h,-w}(h/2) = (-1)^n E_{n;h,w}(h/2),$$

we prove the identity

$$(5.10) \quad \sum_{r=2}^n (-1)^r \binom{n-1}{r-1} 2^{(n-r)/2} \sin(r-1)\pi/4 = \sin(n-1)\pi/4,$$

for  $n > 1$ .

(d) If  $w = h/2$ , we have (2.10)

$$a_{n,r}^{(1-h/w)} = \frac{1}{2^{n-r}} \cdot \frac{n!}{r!} \binom{r}{n-r}, \quad r \geq n/2,$$

and (4.1) becomes

$$(5.11) \quad 2^{n+1} (2^{n+1} B_{n+1;h,h/8} - B_{n+1;h,h/4}) \\ = \frac{(n+1)! h^{n+1}}{2^{2n}} \sum_{r \geq n/2}^n (-1)^{r-1} \binom{r}{n-r} 2^{(3r+1)/2} \cos(r-1)\pi/4.$$

Consequently, returning to (5.7), it follows

$$(5.12) \quad 2^{2n+2} B_{n+1;h,h/8} = \frac{(n+1)!h^{n+1}}{2^{2n+2}} \left( (-1)^{n+1} [1 + 2^{(n+3)/2} \sin(n+1)\pi/4] \right. \\ \left. + \sum_{r \geq n/2}^n (-1)^{r-1} \binom{r}{n-r} 2^{(3r+5)/2} \cos(r-1)\pi/4 \right).$$

(e) If  $w = 2h$ , we have (2.11)

$$a_{n,r}^{(1-h/w)} = \frac{(-1)^{n-r}}{2^{2n-2r}} \cdot \frac{(2n-r-1)!}{(n-1)!} \cdot \binom{n-1}{r-1}.$$

Then (4.1), by (5.6) and the relation

$$B_{n;h,h} = 0 \quad \text{for } n > 0,$$

reduces to the identity, for  $n > 0$ ,

$$(5.13) \quad \sum_{r=1}^n r!(2n-r-1)! \binom{n-1}{r-1} 2^{(r+1)/2} \cos(r-1)\pi/4 = 2^n (n-1)!n!.$$

## 6. A DERIVATIVE FORMULA

Putting

$$P_{0;h,w} = i,$$

$$P_{n;h,w} = \frac{2^{n+1}}{(n+1)h} \cdot (2^{n+1} B_{n+1;h,w/4} - B_{n+1;h,w/2}) + \frac{i2^n}{h} E_{n;h,w/2}(h/2), \quad n > 0,$$

the expansion (3.4) can be written in the form

$$(6.1) \quad F(t) = \frac{(1+wt)^{h/w+j}}{(1+wt)^{h/w-i}} = \sum_{n=0}^{\infty} P_{n;h,w} \frac{t^n}{n!}.$$

Moreover, it is not difficult to show that the function  $F(t)$  satisfies the functional equation

$$(6.2) \quad \frac{2(1+wt)}{h} \cdot \frac{dF(t)}{dt} = 1 - F^2(t),$$

from which the recurrent relation follows,

$$(6.3) \quad 2P_{n+1;h,w} + 2nwP_{n;h,w} + h \sum_{r=0}^n \binom{n}{r} P_{r;h,w} P_{n-r;h,w} = 0, \quad n > 0.$$

If  $h = 1, w = 0$ , we have the interesting connections with the ordinary Bernoulli, Euler numbers

$$(6.4) \quad P_{2n;1,0} = iE_{2n}, \quad P_{2n+1;1,0} = \frac{2^{2n+1}(2^{2n+2}-1)}{n+1} B_{2n+2},$$

and from (6.3) we obtain, in conclusion, the special formulas

$$(6.5) \quad \sum_{r=0}^{n-1} \binom{2n}{2r+1} \frac{2^{2n}(2^{2r+2}-1)(2^{2n-2r}-1)}{(r+1)(n-r)} B_{2r+2} B_{2n-2r} - \sum_{r=0}^n \binom{2n}{2r} E_{2r} E_{2n-2r} \\ + \frac{2^{2n+2}(2^{2n+2}-1)}{n+1} B_{2n+2} = 0, \quad n > 0,$$

$$(6.6) \quad E_{2n+2} = \sum_{r=0}^n \binom{2n+1}{2r+1} \frac{2^{2r+1}(1-2^{2r+2})}{r+1} B_{2n+2} E_{2n-2r}, \quad n \geq 0.$$

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★★★★★

## EXPANSION

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As every science, save one, is modified and cast aside,  
While mathematics only is built upon and grows,  
So, too, my life's whole whims and whimsies pied  
See their demise, while my regard for you goes  
On. Like the Sieve of Eratosthenes, you sift  
My drifting days and sort the prime.

As determinants reflect a change, I mirror-image you  
And, palindromic, backward-forward go, from autumn into spring.  
Approaching the limit of joy, you bring a rate of change  
Which grows in my heart proportionate to you. Your range  
Is my domain. By you, my worthiness a proof shows,  
As solid as geometry, as crystalline as snows,  
As coming-now as spring.

I am subset of you.  
Happily, with you no negative numbers can deride  
My existence, that foolish enterprise of sensibility;  
Instead, a proper fraction of civility  
Is mine. By power of example, exponent of grace,  
You multiply and lace my life with life. The race  
Is mine! Cantor-like you lift  
Me to infinity sublime  
And grant me a number prime.