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On the Theory of the Bernoulli Polynomials and Numbers

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1. NEW REPRESENTATIONS OF THE BERNOULLI POLYNOMIALS

1.1. *Explicit Formulas for the* nth *Derivative of the Generating Function of the Bernoulli Numbers.*

As is known (see, for example, Gould [1, p. 46] and Comtet [2, p. 48]) the Bernoulli numbers B_n are generated by the Taylor expansion

$$g(t) = \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n \qquad (|t| < 2\pi). \tag{1}$$

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In consequence of the parity of the function $g(t)+t/2$ ($B_1=-\frac{1}{2}$) the numbers $B_n=0$ for all odd $n \geqslant 3$. As is indicated in [2, p. 220], if $g(t)$ is expanded into the powers of e^t-1 ($t=\ln[1+(e^t-1)]$), and after applying the Newton binomial formula and the Taylor expansions of the exponential functions, all numbers B_n will be represented by the classical formula

$$B_n=\sum_{k=0}^{n}(-1)^k\frac{k!S(n,k)}{k+1} \qquad (n=0,1,2,\ldots) \tag{2}$$

(concerning the priority of the this formula it is noted in [1, p. 44] that it is very old and it is difficult to tell who its first author is). Here $S(n,k)$ are the Stirling numbers of the second kind, generated by the Newton expansions [2, p.207]:

$$x^n=\sum_{k=0}^{n}S(n,k)(x)_k \qquad (n=0,1,2,\ldots), \tag{3}$$

where by $(x)_k$ ($k=0,1,2,\ldots$) for an arbitrary x here and below we shall mean the product

$$(x)_k=x(x-1)\cdots(x-k+1), \qquad (x)_0=1. \tag{4}$$

The numbers $S(n,k)$ satisfy the "triangular" recurrence relation [2, p. 208]

$$\begin{gathered}S(n,k)=S(n-1,k-1)+kS(n-1,k), \qquad n,k\geqslant 1;\\ S(n,0)=S(0,k)=0, \qquad S(0,0)=1; \qquad S(n,k)=0 \qquad (k>n)\end{gathered} \tag{5}$$

and in an explicit from they will be represented by the formula proved in [2, p. 204],

$$S(n,k)=\frac{1}{k!}\Delta^kO^n=\frac{1}{k!}\sum_{\nu=0}^{k}(-1)^{k-\nu}\binom{k}{\nu}\nu^n \qquad (n,k\geqslant 0), \tag{6}$$

where Δ^kO^n is the finite difference of the kth order of the function x^n at the point $x=0$. Here and further on in our work we take into consideration this difference operator Δ (see, for example, [2, p. 13]) which generates successively from each function $f(x)$, determined at the points x, $x+1,\ldots,x+k,\ldots$, the functions

$$\Delta^kf(x)\equiv\Delta(\Delta^{k-1}f(x))=\Delta^{k-1}f(x+1)-\Delta^{k-1}f(x) \qquad (k=1,2,\ldots;\Delta^0f(x)=f(x)). \tag{7}$$

The explicit form of the functions $\Delta^k f(x)$ is given by the well-known formula [2, p. 13]

$$\Delta^k f(x) = \sum_{\nu=0}^{k} (-1)^{k-\nu} \binom{k}{\nu} f(x+\nu) \qquad (k=0, 1, 2,...) \tag{8}$$

(here we shall note that formula (6) follows very simply if we apply the operator Δ^k to (3) (see below the formula (51) in Section 1.4), set $x=0$ and take into consideration (8) for $f(x)=x^n$ and $x=0$).

Although a vast literature is devoted to the Bernoulli numbers, in comparison the generating function itself, $g(t)$, is studied to a lesser extent. For example, until now the explicit form of the nth derivative $g^{(n)}(t)$ for an arbitrary positive integer n was unknown, from which, in particular, for $t=0$ the formula (2) for the Bernoulli numbers $B_n = g^{(n)}(0)$ would follow immediately. First, we shall solve this problem.

Since the function $g(t)$ is meromorphic in the finite t-plane with simple poles $2m\pi i$, $m=\pm 1, \pm 2,...$, and corresponding residues $2m\pi i$, $m=\pm 1, \pm 2,...$, we shall consider this function in the corresponding punctured t-plane. Further, $D=d/dt$ denotes the operator for derivation:

THEOREM 1. *In the finite t-plane punctured at the points $2m\pi i$, $m=\pm 1, \pm 2,...$ we have the explicit formula*

$$D^n \left(\frac{t}{e^t-1}\right) = \sum_{k=0}^{n} (-1)^k k! S(n,k)\, G_k(t) \qquad (n=0, 1, 2,...), \tag{9}$$

where the functions $G_k(t)$ are regular for the considered t and have the representation

$$\begin{aligned} G_0(t) &\equiv g(t) = \frac{t}{e^t-1}, \\ G_k(t) &= \frac{e^{-t}}{(1-e^{-t})^{k+1}} \left[t - \sum_{\nu=1}^{k} \frac{(1-e^{-t})^\nu}{\nu} \right] \qquad (1 \leqslant k \leqslant n; n \geqslant 1), \end{aligned} \tag{10}$$

and, in particular, for $t=0$ have values

$$G_k(0) = \frac{1}{k+1} \qquad (0 \leqslant k \leqslant n;\ n \geqslant 0) \tag{11}$$

by which the formula (9) is reduced to the classical formula (2) for the Bernoulli numbers $B_n = D^n_{t=0}[t/(e^t-1)]$, $n=0, 1, 2,...$.

Proof. For $n=0$ the theorem is evidently true and therefore we shall consider $n \geqslant 1$. For finite $t \neq 2m\pi i$ $(m = \pm 1, \pm 2, \ldots)$ if we set

$$z = f(t) = e^t \tag{12}$$

and

$$w = \rho(z) = \frac{\operatorname{Ln} z}{z-1}, \tag{13}$$

where

$$\operatorname{Ln} z = \ln |z| + i \operatorname{Arg} z \qquad (\operatorname{Arg} z = \operatorname{Im} t), \tag{14}$$

we obtain the composite function

$$w = g(t) \equiv \rho[f(t)] = \frac{t}{e^t - 1}, \tag{15}$$

i.e., the generating function $g(t)$ from (1). Hence, according to our work [3, pp. 220–221, Theorem 1], its nth derivative $g^{(k)}(t)$ will have the form

$$g^{(n)}(t) = \sum_{k=1}^{n} A_{nk}(t) \cdot \rho^{(k)}(z) \qquad (n = 1, 2, \ldots), \tag{16}$$

where

$$A_{nk}(t) = \frac{1}{k!} \sum_{\nu=1}^{k} (-1)^{k-\nu} \binom{k}{\nu} f^{k-\nu} D^n f^{\nu} \qquad (1 \leqslant k \leqslant n). \tag{17}$$

Now, if we take into consideration (6) from (12) and (17), we obtain

$$A_{nk}(t) = S(n, k)\, e^{kt} \qquad (1 \leqslant k \leqslant n). \tag{18}$$

Further, for $z \neq 1$ the Leibniz formula applied to the function (13) yields

$$\begin{aligned} \rho^{(k)}(z) &= \sum_{\nu=0}^{k} \binom{k}{\nu} D^{k-\nu} \operatorname{Ln} z \cdot D^{\nu}(z-1)^{-1} \\ &= \frac{(-1)^k k!}{(z-1)^{k+1}} \left[\operatorname{Ln} z - \sum_{\nu=1}^{k} \frac{1}{\nu} \left(\frac{z-1}{z} \right)^{\nu} \right] \qquad (1 \leqslant k \leqslant n). \end{aligned}$$

Hence, if we set (18) and (19) (for $z = e^t$ and $t \neq 2m\pi i$, $m = 0, \pm 1, \pm 2, \ldots$) in (16), we shall obtain the explicit form of the nth derivative of the function (15) for $t \neq 2m\pi i$, $m = 0, \pm 1, +2, \ldots$ in the form (9)–(10) (we can add the term for $k = 0$ in (9) since $S(n, 0) = 0$ for $n \geqslant 1$ according to (5)).

In order to extend the formula (9) for $t=0$ as well, we shall turn to the functions (10). In the finite t-plane the functions $G_k(t)$, $0 \leqslant k \leqslant n$, are meromorphic with poles $t = 2m\pi i$ ($m = \pm 1, \pm 2, \ldots$) of multiplicity $k+1$. At the point $t=0$ the functions $G_k(t)$, $0 \leqslant k \leqslant n$, are regular although their analytic representation (10) has the indeterminate form 0/0. For $G_0(t)$ this is evident, where $G_0(0) = 1$. In Section 2 (see Section 2.8, Theorem 18, formula (169)) we shall prove this for all functions (10) finding their power expansions in the neighborhood of the point $t=0$. Now, we shall prove the regularity at the point $t=0$ in the following way. The derivatives of the functions

$$g_0(t) = t, \qquad g_k(t) = t - \sum_{\nu=1}^{k} \frac{(1-e^{-t})^\nu}{\nu} \qquad (k = 1, 2, \ldots) \tag{20}$$

have the following expansion in the finite t-plane:

$$g_k'(t) = (1-e^{-t})^k = k! \sum_{\alpha=k}^{\infty} (-1)^{\alpha-k} S(\alpha, k) \frac{t^\alpha}{\alpha!} \qquad (k = 0, 1, 2, \ldots) \tag{21}$$

(on the basis of the formula $D_{t=0}^{\alpha}(1-e^{-t})^k = (-1)^{\alpha-k} k! S'(\alpha, k)$ following from the Newton binomial formula, from (6), and the last formula from (5); in [2, p. 206], the expansion (21) is obtained in an equivalent, but longer, method applied for the function $(e^t - 1)^k/k!$ as a generating function of the Stirling numbers of the second kind). If we integrate (21) along some rectifiable curve between 0 and t, we shall obtain the following expansion everywhere in the finite t-plane:

$$g_k(t) = k! \sum_{\alpha=k}^{\infty} (-1)^{\alpha-k} S(\alpha, k) \frac{t^{\alpha+1}}{(\alpha+1)!} \qquad (k = 0, 1, 2, \ldots). \tag{22}$$

With the help of (22) and (21) (if in (21) we replace k with $k+1$) the regularity of all functions (10) at the point $t=0$ and their values (11) at this point become evident. Hence, according to the analytic continuation principle, the formula (9) holds for $t=0$ as well. This denotes that our formula (9), in particular, for $t=0$ is reduced to the classical formula (2) for the Bernoulli numbers $B_n = D_{t=0}^{n}[t/(e^t - 1)]$, $n = 0, 1, 2, \ldots$.

This completes the proof of Theorem 1.

Now, we shall give a representation of the functions $G_k(t)$, $0 \leqslant k \leqslant n$, by means of a definite integral which has the advantage that the regularity at the point $t=0$ and the values (11) are evident:

THEOREM 2. *Let the finite t-plane be divided into the bands $H_m(t)$: $\{t \mid (2m-1)\pi < \operatorname{Im} t < (2m+1)\pi\}$, $m = 0, \pm 1, \pm 2, \ldots$. Then in each band*

$H_m(t)$, $m = 0, \pm 1, \pm 2, \ldots$, from which for $m = \pm 1, \pm 2, \ldots$ the point $t = 2m\pi i$ is extracted, all functions (10) *have integral representation*

$$G_k(t) = e^{-t}\left[\frac{2m\pi i}{(1-e^{-t})^{k+1}} + \int_0^1 \frac{x^k\,dx}{1-(1-e^{-t})x}\right] \qquad (0 \leqslant k \leqslant n), \quad (23)$$

where the integration on the x-plane is performed along the finite segment $[0, 1]$ *from the real axis.*

Proof. If we integrate (21) in the band $H_m(t)$ ($m = 0, \pm 1, \pm 2, \ldots$) along an arbitrary rectifiable curve which connects the points $2m\pi i$ and t ($t \neq 2m\pi i$), and take into consideration that $g_k(2m\pi i) = 2m\pi i$, then we obtain

$$g_k(t) = 2m\pi i + \int_0^t (1-e^{-\tau})^k\,d\tau \qquad (0 \leqslant k \leqslant n). \qquad (24)$$

For an arbitrary fixed $t \in H_m(t)$, $t \neq 2m\pi i$, the transformation

$$x = \frac{1-e^{-\tau}}{1-e^{-t}} \qquad (25)$$

lays the band $H_m(\tau)$ from the τ-plane on the x-plane with an infinite cut along the ray starting from the point $x = 1/(1-e^{-t})$ radially with respect to the origin $x = 0$. In addition to that, when the point t describes the band $H_m(t) - \{2m\pi i\}$ the point $x = 1/(1-e^{-t})$ describes the entire x-plane with a cut along the segment $[0, 1]$ from the real axis. Hence, our transformation (25) reduces the integral (24) to the form

$$g_k(t) = 2m\pi i + (1-e^{-t})^{k+1}\int_0^1 \frac{x^k\,dx}{1-(1-e^{-t})x} \qquad (0 \leqslant k \leqslant n), \quad (26)$$

where the integration in the x-plane is performed along the segment $[0, 1]$ from the real axis. Now, we can eliminate the restriction $t \neq 2m\pi i$. Indeed, the integral in (26) is a regular function of t at the point $t = 2m\pi i$ as well and at this point it has the value $1/(k+1)$. Hence, according to the analytic continuation principle, the formula (26) holds everywhere in an arbitrary band $H_m(t)$, $m = 0, \pm 1, \pm 2, \ldots$.

Now, entering in (10) with (26), we shall obtain the representation (23) according to the assertion of the theorem. From this representation the regularity and the values (11) of the functions (23) at the point $t = 0$ are evident, noting that this point belongs to the band $H_0(t)$.

This completes the proof of Theorem 2.

Remark 1. Conversely, from (23) we can obtain the representation (10) in the band $H_m(t)$ with the help of the formula

$$\int_0^1 \frac{x^k\,dx}{1-(1-e^{-t})x} = \frac{g_k(t) - 2m\pi i}{(1-e^{-t})^{k+1}}$$

$$(0 \leqslant k \leqslant n;\ t \in H_m(t), m = 0 \pm 1, \pm 2, \ldots), \tag{27}$$

ensuing from (26). From the band $H_m(t)$ the representation (10) is transferred everywhere in the finite t-plane punctured at the points $2m\pi i$ $(m = \pm 1, \pm 2, \ldots)$ by analytic continuation.

Remark 2. When t tends to any of the boundary straight lines $\operatorname{Im} t = (2m \pm 1)\pi$ $(m = 0, \pm 1, \pm 2, \ldots)$ within the band $H_m(t)$, the limit of the integral in (27) exists and it is equal to the values of the right-hand side of (27) on the corresponding straight line $\operatorname{Im} t = (2m \pm 1)\pi$. Hence, the formula (23) will be valid on the straight lines $\operatorname{Im} t = (2m \pm 1)\pi$ $(m = 0, \pm 1, \pm 2, \ldots)$ as well if instead of the integral we consider its corresponding one-sided limits.

1.2. *Explicit Formula for the* nth *Derivative of the Generating Function of the Bernoulli Polynomials. Corollary: Transfer of the Classical Representation of the Bernoulli Numbers to the Bernoulli Polynomials.*

Now we shall obtain a more general analogy of Theorem 1 as we find the nth derivative with respect to t of the generating function (see, for example, [2, p. 48])

$$g(t, x) = \frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \qquad (|t| < 2\pi) \tag{28}$$

of the Bernoulli polynomials

$$B_n(x) = \sum_{\nu=0}^{n} \binom{n}{\nu} B_\nu x^{n-\nu} \qquad (n = 0, 1, 2, \ldots), \tag{29}$$

which yields the possibility of transferring the classical representation (2) of the Bernoulli numbers to these polynomials:

THEOREM 3. *In the finite t-plane punctured at the points* $2m\pi i$, $m = \pm 1, \pm 2, \ldots$, *we have the explicit formula*

$$\frac{\partial^n g(t, x)}{\partial t^n} = e^{tx} \sum_{k=0}^{n} (-1)^k \cdot \Delta^k x^n \cdot G_k(t) \qquad (n = 0, 1, 2, \ldots). \tag{30}$$

In particular, for $x = 0$ *we have the formula* (9).

In particular, for $t=0$ *we obtain the following new and remarkable representation of the Bernoulli polynomials* (29) (*compare with* [4, pp. 235–236, *formula* (8) *and its corollaries*),

$$B_n(x)=\sum_{k=0}^{n}(-1)^k\frac{\Delta^k x^n}{k+1}\qquad (n=0,1,2,\ldots), \tag{31}$$

which generalizes the classical representation (2) *for the Bernoulli numbers* $B_n=B_n(0)$, $n=0,1,2,\ldots$.

Proof. The Leibniz formula applied for the product $g(t,x)=g(t)\,e^{tx}$ and our formula (9) yield the following generalization of the formula (29):

$$\frac{\partial^n g(t,x)}{\partial t^n}=e^{tx}\sum_{\nu=0}^{n}\binom{n}{\nu}x^{n-\nu}D^\nu g(t)\qquad (n=0,1,2,\ldots), \tag{32}$$

If we replace the order of summation, we shall obtain

$$\begin{aligned}\frac{\partial^n g(t,x)}{\partial t^n}&=e^{tx}\sum_{\nu=0}^{n}\binom{n}{\nu}x^{n-\nu}\sum_{k=0}^{\nu}(-1)^k\,k!\,S(\nu,k)\,G_k(t)\\&=e^{tx}\sum_{k=0}^{n}(-1)^k k!\,G_k(t)\sum_{\nu=k}^{n}S(\nu,k)\binom{n}{\nu}x^{n-\nu}\\&=e^{tx}\sum_{k=0}^{n}(-1)^k\cdot\Delta^k x^n\cdot G_k(t).\end{aligned} \tag{30'}$$

Indeed, according to (8), the Newton binomial formula, (6) and the last formula in (5) the finite difference $\Delta^k x^n$ as a polynomial in the usual powers of x is equal to

$$\begin{aligned}\Delta^k x^n&=\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(x+j)^n\\&=\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\sum_{\nu=0}^{n}\binom{n}{\nu}j^\nu x^{n-\nu}\\&=\sum_{\nu=0}^{n}\binom{n}{\nu}x^{n-\nu}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^\nu\\&=k!\sum_{\nu=0}^{n}\binom{n}{\nu}S(\nu,k)\,x^{n-\nu}=k!\sum_{\nu=k}^{n}\binom{n}{\nu}S(\nu,k)\,x^{n-\nu}.\end{aligned} \tag{33}$$

By this, the formula (30) is proved. For $x=0$ it is reduced to the formula (9) since $\Delta^k O^n=k!S(n,k)$ according to (6). For $t=0$ the formula (30) on the basis of (11) yields the representation (31) for the Bernoulli polynomials

$B_n(x) = [\partial^n g(t, x)/\partial t^n]_{t=0}$. For $x = 0$ the representation (31) of the Bernoulli polynomials $B_n(x)$ immediately generates the classical formula (2) for the Bernoulli numbers $B_n = B_n(0)$.

This completes the proof of Theorem 3.

1.3. *Transfer of the Kronecker–Bergmann Formula for the Bernoulli Numbers to the Bernoulli Polynomials.*

The representation (31) of the Bernoulli polynomials (29) yields the possibility of substituting the Kronecker–Bergmann formula [5, 6] for the Bernoulli numbers for these polynomials (in a modification of Gould [1, pp. 46–47] and Comtet [2, p. 221]),

$$B_n = \sum_{k=0}^{n} (-1)^k \binom{n+1}{k+1} \frac{Z(k, n)}{k+1} \qquad (n = 0, 1, 2, \ldots), \tag{34}$$

where

$$Z(k, n) = \sum_{j=0}^{k} j^n \qquad (0 \leqslant k \leqslant n;\ Z(0, 0) = 1). \tag{35}$$

THEOREM 4. *The Bernoulli polynomials* (29) *have representation*

$$B_n(x) = \sum_{k=0}^{n} (-1)^k \binom{n+1}{k+1} \frac{Z_{kn}(x)}{k+1} \qquad (n = 0, 1, 2, \ldots), \tag{36}$$

where

$$Z_{kn}(x) = \sum_{j=0}^{k} (x+j)^n \qquad (0 \leqslant k \leqslant n;\ Z_{00}(x) = 1). \tag{37}$$

In particular, for $x = 0$ *we have the Kronecker–Bergmann formula* (34)–(35) *for the Bernoulli numbers* $B_n = B_n(0)$, *where* $Z(k, n) \equiv Z_{kn}(0)$.

Proof. From (31) and the first formula in (33) we obtain the following representation of the Bernoulli polynomials,

$$\begin{aligned} B_n(x) &= \sum_{k=0}^{n} (-1)^k \frac{1}{k+1} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (x+j)^n \\ &= \sum_{j=0}^{n} C(n, j)(x+j)^n \qquad (n = 0, 1, 2, \ldots), \end{aligned} \tag{38}$$

where

$$C(n, j) = (-1)^j \sum_{k=j}^{n} \binom{k}{j} \frac{1}{k+1} \qquad (0 \leqslant j \leqslant n). \tag{39}$$

We can give another form to the sums (39), establishing for $1 \leqslant j \leqslant n$ $(n \geqslant 1)$ the recurrence formula

$$C(n, j) - C(n, j-1) = (-1)^j \sum_{k=j-1}^{n} \binom{k+1}{j} \frac{1}{k+1}$$
$$= (-1)^j \frac{1}{j} \sum_{k=j-1}^{n} \binom{k}{j-1} = (-1)^j \frac{1}{j} \binom{n+1}{j}, \quad (40)$$

where

$$C(n, 0) = \sum_{k=0}^{n} \frac{1}{k+1} \equiv H_{n+1} \qquad (n = 0, 1, 2, ...) \quad (41)$$

are the harmonic numbers (in our case $n = 1, 2, ...$). If in the recurrence formula (40) we replace j with $k+1$ and sum from $k = 0$ to $k = j-1$, we obtain the formula

$$C(n, j) = H_{n+1} + \sum_{k=0}^{j-1} (-1)^{k+1} \binom{n+1}{k+1} \frac{1}{k+1} \qquad (1 \leqslant j \leqslant n; n \geqslant 1). \quad (42)$$

For $j = n$ $(n \geqslant 1)$ from (39) and (42) we obtain the well-known representation of the harmonic numbers

$$H_{n+1} = \sum_{k=0}^{n} (-1)^k \binom{n+1}{k+1} \frac{1}{k+1} \qquad (n = 0, 1, 2, ...), \quad (43)$$

evidently true for $n = 0$ as well. With the help of (43) the formula (42) obtains the form

$$C(n, j) = \sum_{k=j}^{n} (-1)^k \binom{n+1}{k+1} \frac{1}{k+1}, \quad (44)$$

valid in general for $0 \leqslant j \leqslant n$, $n = 0, 1, 2, ...$. Entering in (38) with this representation and replacing the order of summation, we obtain the formula (36)–(37).

This completes the proof of Theorem 4.

1.4. *A Linear Combination of the Bernoulli Polynomials with the Stirling Numbers of the First Kind.*

The representation (31) also yields the possibility of substituting for the Bernoulli polynomials (29) the classical formula (see, for example, [2, pp. 220–221])

$$\sum_{n=0}^{m} s(m, n) B_n = (-1)^m \frac{m!}{m+1} \qquad (m = 0, 1, 2, ...), \quad (45)$$

linearly connecting the Bernoulli numbers B_n with the Stirling numbers of the first kind $s(m, n)$, generated by the Taylor expansion [2, p. 213]

$$(x)_m = \sum_{n=0}^{m} s(m, n)\, x^n \qquad (m = 0, 1, 2, \ldots). \tag{46}$$

The Stirling numbers of the first kind $s(m, n)$ satisfy the "triangular" recurrence relation [2, p. 214]

$$\begin{aligned} s(m, n) &= s(m-1, n-1) - (m-1)\, s(m-1, n), \qquad m, n \geqslant 1; \\ s(m, 0) = s(0, n) &= 0, \qquad s(0, 0) = 1; \qquad s(m, n) = 0 \qquad (n > m) \end{aligned} \tag{47}$$

and for $0 \leqslant n \leqslant m$ they have an explicit representation by means of the Stirling numbers of the second kind (6) by the Schlömilch formula (for an elegant derivation of this formula see [2, p. 216]:

$$s(m, n) = \sum_{h=0}^{m-n} (-1)^h \binom{m-1+h}{m-n+h} \binom{2m-n}{m-n-h} S(m-n+h, h). \tag{48}$$

THEOREM 5. *The linear combination of the Bernoulli polynomials* (29) *composed of the Stirling numbers of the first kind has the Newton expansion*

$$\sum_{n=0}^{m} s(m, n)\, B_n(x) = \sum_{k=0}^{m} (-1)^{m-k} \frac{(m)_{m-k}}{m-k+1} (x)_k \quad (m = 0, 1, 2, \ldots) \tag{49}$$

In particular, for $x = 0$ we have the classical formula (45) *for the Bernoulli numbers $B_n = B_n(0)$.*

Proof. If we multiply (31) by $s(m, n)$ and sum from $n = 0$ to $n = m$, we shall obtain

$$\begin{aligned} \sum_{n=0}^{m} s(m, n)\, B_n(x) &= \sum_{n=0}^{m} s(m, n) \sum_{k=0}^{n} (-1)^k \frac{\Delta^k x^n}{k+1} \\ &= \sum_{k=0}^{m} \frac{(-1)^k}{k+1} \sum_{n=k}^{m} s(m, n)\, \Delta^k x^n. \end{aligned} \tag{50}$$

Applying the operator Δ^k on (3), we obtain the Newton expansion of the finite difference

$$\Delta^k x^n = \sum_{\alpha=k}^{n} S(n, \alpha)(\alpha)_k (x)_{\alpha-k} \qquad (0 \leqslant k \leqslant n;\ n \geqslant 0), \tag{51}$$

with the help of which we obtain

$$\sum_{n=k}^{m} s(m, n)\,\Delta^k x^n = \sum_{\alpha=k}^{m} (\alpha)_k (x)_{\alpha-k}\,\delta(m, \alpha), \tag{52}$$

where

$$\delta(m, \alpha) = \sum_{n=\alpha}^{m} s(m, n)\, S(n, \alpha) \tag{53}$$

is the Kronecker symbol which is equal to zero if $\alpha \neq m$ and to unity if $\alpha = m$ (the formula (53) follows if, to the equality obtained by the combination of the two formulas (46) and (3),

$$\begin{aligned}(x)_m &= \sum_{n=0}^{m} s(m, n)\, x^n = \sum_{n=0}^{m} s(m, n) \sum_{\alpha=0}^{n} S(n, \alpha)(x)_\alpha \\ &= \sum_{\alpha=0}^{m} (x)_\alpha \sum_{n=\alpha}^{m} s(m, n)\, S(n, \alpha),\end{aligned}$$

we apply successively the operator Δ^α $(\alpha = 0, 1,..., m)$ and each time set $x = 0$). Hence, from (52) we obtain the formula

$$\sum_{n=k}^{m} s(m, n)\,\Delta^k x^n = (m)_k (x)_{m-k} \qquad (0 \leqslant k \leqslant m;\ m \geqslant 0) \tag{54}$$

which reduces (50) to the form (49) after replacing k with $m - k$.

This completes the proof of Theorem 5.

The following method yields a set of new explicit formulas for the Bernoulli numbers B_n.

2. New Representations of the Bernoulli Numbers

2.1. *The Functions $T_n(z)$*

Instead of the sum (2) we shall consider the class of rational functions

$$T_n(z) = \sum_{k=0}^{n} (-1)^k \frac{k!\, S(n, k)}{z + k} \qquad (n = 0, 1, 2,...), \tag{55}$$

which in the z-plane have simple poles $z = -k$ and corresponding residues $(-1)^k k!\, S(n, k)$ $(0 \leqslant k \leqslant n)$; for $n > 0$ the pole $z = 0$ vanishes and the functions $T_n(z)$ are rational of corresponding order n $(n = 1, 2,...)$; the function $T_0(z) = 1/z$ is of order 1 as is the function $T_1(z) = -1/(z + 1)$. Then the Bernoulli numbers B_n are particular values of the functions $T_n(z)$ for $z = 1$, i.e.,

$$B_n = T_n(1) \qquad (n = 0, 1, 2, \ldots). \tag{56}$$

From here, in particular, for odd $n \geqslant 3$ it follows that $T_n(1) = 0$.

The following property of the functions $T_n(z)$ has a direct relation to the theory of the linear-fractional functions:

THEOREM 6. *The linear combination of rational functions* (55) *composed of the Stirling numbers of the first kind yields the linear-fractional function*

$$\sum_{n=0}^{m} s(m, n)\, T_n(z) = (-1)^m \frac{m!}{z+m} \qquad (m = 0, 1, 2, \ldots). \tag{57}$$

In particular, for $z = 1$ we have the classical formula (45).

Proof. Indeed, from (55) by formula (53) we have

$$\sum_{n=0}^{m} s(m, n)\, T_n(z) = \sum_{n=0}^{m} s(m, n) \sum_{k=0}^{n} (-1)^k \frac{k!\, S(n, k)}{z+k}$$
$$= \sum_{k=0}^{m} (-1)^k \frac{k!}{z+k}\, \delta(m, k) = (-1)^m \frac{m!}{z+m}. \tag{57'}$$

2.2. *Generation of an Arbitrary Function $T_n(z)$ by the Action of Any of the Operators $(z\Delta)^v$, $v = 0, 1, 2, \ldots, n$, on the Function $T_{n-v}(z)$, respectively.*

Further, when we study the functions (55), we encounter the operator $z\Delta$, where Δ is the difference operator determined in (7); i.e., we have the action $z\Delta f(z) = z(f(z+1) - f(z))$ on a suitable function $f(z)$. On the basis of the action of the operator Δ on the product of two suitable functions $f(z)$ and $g(z)$, namely,

$$\Delta f(z)\, g(z) = g(z)\, \Delta f(z) + f(z+1)\, \Delta g(z), \tag{58}$$

we can determine more generally for $n = 1, 2, \ldots$ the operator $(z\Delta)^n$ of the nth order by the recurrence formula $(z\Delta)^n = z\Delta (z\Delta)^{n-1}$, where $(z\Delta)^0 = 1$; i.e., the operator $(z\Delta)^n$ has the following expansion in the powers of the operator Δ,

$$(z\Delta)^n = \sum_{k=0}^{n} S(n, k) \langle z \rangle_k\, \Delta^k \qquad (n = 0, 1, 2, \ldots), \tag{59}$$

where $S(n, k)$ are the Stirling numbers of the second kind and the symbol $\langle z \rangle_k$, $k = 0, 1, 2, \ldots$, for an arbitrary z here and everywhere below denotes the product

$$\langle z \rangle_k = z(z+1) \cdots (z+k-1) \qquad (k = 0, 1, 2, \ldots;\ \langle z \rangle_0 = 1). \tag{60}$$

The operator $(z\Delta)^n$, $n=0, 1, 2, \ldots$, as well as the formula (59), is well known in the theory of finite differences and can be found (with other notation), for example, in the book by Ch. Jordan [4, pp. 199–200].

Remark 3. We show here how the formula (59) can be obtained by a direct induction (compare with the proof of the formula (59) in the book by Ch. Jordan [4, pp. 199–200, formula (2)]). For $n=1$ $(n=0)$ the formula (59) is true. If we assume that it is true to some $n \geqslant 1$ inclusively, then at the next step we shall obtain

$$\begin{aligned}(z\Delta)^{n+1} &= z\Delta(z\Delta)^n = z \sum_{k=0}^{n} S(n,k)\,\Delta[\langle z\rangle_k \Delta^k] \\ &= z \sum_{k=0}^{n} S(n,k)[\Delta\langle z\rangle_k \cdot \Delta^k + \langle z+1\rangle_k \Delta^{k+1}] \\ &= \sum_{k=0}^{n} kS(n,k)\langle z\rangle_k \Delta^k + \sum_{k=0}^{n} S(n,k)\langle z\rangle_{k+1} \Delta^{k+1} \\ &= \sum_{k=1}^{n+1} [S(n,k-1) + kS(n,k)]\langle z\rangle_k \Delta^k \\ &= \sum_{k=0}^{n+1} S(n+1,k)\langle z\rangle_k \Delta^k, \end{aligned} \tag{59'}$$

by which the formula (59) is proved.

Now we discover the following genesis of the functions $T_n(z)$:

THEOREM 7. *Each function $T_n(z)$ $(n=0, 1, 2, \ldots)$ of the class* (55) *is generated by the action of any of the operators $(z\Delta)^\nu$, $\nu=0, 1, 2, \ldots, n$, on the function $T_{n-\nu}(z)$ of the same class, respectively, of order not higher than that of the former one*; *i.e., we have the following operator formula for reduction of the order*

$$T_n(z) = (z\Delta)^\nu T_{n-\nu}(z) \qquad (\nu = 0, 1, 2, \ldots, n, n \geqslant 0), \tag{61}$$

where $z \neq -1, -2, \ldots, -n$ for $\nu < n$ $(n \geqslant 1)$ and *$z \neq 0, -1, -2, \ldots, -n$ for $\nu = n$* $(n \geqslant 1)$.

Proof. For $\nu = 0$ the formula (61) is evidently true. For $\nu = 1$ we shall verify the basic formula

$$T_n(z) = z\Delta T_{n-1}(z) \qquad (n \geqslant 1) \tag{62}$$

for $z \neq -1, -2, \ldots, -n$ if $n > 1$ and for $z \neq 0, -1, -2, \ldots, -n$ if $n = 1$. For $n \geqslant 1$ from (55) by means of (5) we obtain

$$T_n(z) = \sum_{k=1}^{n} (-1)^k \frac{k!\, S(n-1, k-1)}{z+k} + \sum_{k=0}^{n-1} (-1)^k \frac{k!\, kS(n-1, k)}{z+k},$$

or, replacing in the first sum k with $k+1$, we obtain the formula

$$T_n(z) = z \sum_{k=0}^{n-1} (-1)^{k+1} \frac{k!\, S(n-1, k)}{(z+k)(z+k+1)} \qquad (n \geqslant 1). \tag{63}$$

On the other hand, the right-hand side of (63) is equal to the expression $z(T_{n-1}(z+1) - T_{n-1}(z)]$ calculated directly by (55). Hence, the formula (62) holds. From here by induction we obtain the general formula (61) as well.

This completes the proof of Theorem 7.

From the formula (61) (or (63)) it follows that for $n \geqslant 2$ all functions $T_n(z)$ vanish at the point $z = 0$; i.e., we have the equality

$$T_n(0) = 0 \qquad (n \geqslant 2). \tag{64}$$

The equality (64) permits extracting the factor z from the right-hand side of (55) for $n \geqslant 2$, without changing the Stirling numbers and the denominator in the general term. Noting the identity

$$\frac{1}{z+k} = \frac{1}{k}\left(1 - \frac{z}{z+k}\right)$$

from (55) we obtain the identity

$$T_n(z) = T_n(0) - z \sum_{k=1}^{n} (-1)^k \frac{(k-1)!\, S(n, k)}{z+k} \qquad (n \geqslant 1), \tag{65}$$

which for $n \geqslant 2$ on the basis of (64) yields the representation

$$T_n(z) = z \sum_{k=1}^{n} (-1)^{k+1} \frac{(k-1)!\, S(n, k)}{z+k} \qquad (n \geqslant 2). \tag{66}$$

From the formulas (63) and (66), in particular, for $z = 1$ on the basis of (56) we obtain the following formulas for the Bernoulli numbers B_n:

$$B_n = \sum_{k=0}^{n-1} (-1)^{k+1} \frac{k!\, S(n-1, k)}{(k+1)(k+2)} \qquad (n \geqslant 1) \tag{63'}$$

and

$$B_n = \sum_{k=1}^{n} (-1)^{k+1} \frac{(k-1)!\, S(n,k)}{k+1} \qquad (n \geqslant 2). \tag{66'}$$

2.3. *Analytic Expression of an Arbitrary Function $T_n(z)$ by Means of Any of the Functions $T_{n-\nu}(z)$, $\nu = 0, 1, 2,\dots, n$, and as a Corollary a Series of New Representations of the Bernoulli Numbers $B_n = T_n(1)$*

The actions of the operators $(z\Delta)^\nu$, $\nu = 0, 1, 2,\dots, n$, on the corresponding functions $T_{n-\nu}(z)$ by the formula (61) yield the following series of analytic representations of the function $T_n(z)$:

THEOREM 8. *Each function $T_n(z)$ $(n = 0, 1, 2,\dots)$ of the class (55) is expressed by means of any function $T_{n-\nu}(z)$. $\nu = 0, 1, 2,\dots, n$, of the same class of order not higher than that of the former one by the analytic formula*

$$T_n(z) = \sum_{r=0}^{\nu} (-1)^{\nu-r} \cdot \frac{\langle z\rangle_r}{r!} \cdot \Delta^r z^\nu \cdot T_{n-\nu}(z+r). \tag{67}$$

In particular, the nth Bernoulli number $B_n = T_n(1)$ $(n = 0, 1, 2,\dots)$ has the series of representations

$$B_n = \sum_{r=0}^{\nu} (-1)^{\nu-r} r!\, S(\nu+1, r+1)\, T_{n-\nu}(r+1) \qquad (\nu = 0, 1, 2,\dots, n). \tag{68}$$

Proof. From (61) and (59) the expansion

$$T_n(z) = \sum_{\alpha=0}^{\nu} S(\nu, \alpha) \langle z\rangle_\alpha\, \Delta^\alpha T_{n-\nu}(z) \tag{69}$$

follows.

By the formula (8) we have

$$\Delta^\alpha T_{n-\nu}(z) = \sum_{r=0}^{\alpha} (-1)^{\alpha-r} \binom{\alpha}{r} T_{n-\nu}(z+r). \tag{70}$$

Entering in (69) with (70) and replacing the order of summation, we obtain the formula

$$T_n(z) = \sum_{r=0}^{\nu} (-1)^{\nu-r} \frac{\langle z\rangle_r}{r!}\, p_{\nu-r,r}(z)\, T_{n-\nu}(z+r), \tag{71}$$

where $p_{\nu-r,r}(z)$ denotes the polynomial of degree $n-\nu$ with respect to z:

$$p_{\nu-r,r}(z) = \sum_{\alpha=r}^{\nu} (-1)^{\nu-\alpha} S(\nu, \alpha)(\alpha)_r \langle z+r\rangle_{\alpha-r}. \tag{72}$$

We shall show that the polynomial $p_{\nu-r,r}(z)$ is identical with the polynomial generated by the finite difference $\Delta^r z^\nu$. Indeed, according to the formula

$$\Delta^r \langle z\rangle_\alpha = (\alpha)_r \langle z+r\rangle_{\alpha-r} \tag{73}$$

we can transform the sum (72) in the following way:

$$\begin{aligned} p_{\nu-r,r}(z) &= \sum_{\alpha=r}^{\nu} (-1)^{\nu-\alpha} S(\nu,\alpha)\, \Delta^r \langle z\rangle_\alpha \\ &= (-1)^\nu \Delta^r \sum_{\alpha=0}^{\nu} (-1)^\alpha S(\nu,\alpha) \langle z\rangle_\alpha \\ &= (-1)^\nu \Delta^r \sum_{\alpha=0}^{\nu} S(\nu,\alpha)(-z)_\alpha \\ &= (-1)^\nu \Delta^r (-z)^\nu = \Delta^r z^\nu. \end{aligned} \tag{74}$$

Here we have taken into consideration the formula (3) for the last sum. Hence, the formula (71) can be written in the form (67).

In particular, for $z=1$ from (67) and (56) we obtain the formula (68) for the Bernoulli numbers. Indeed, the value of the polynomial $\Delta^r z^\nu$ for $z=1$ is obtained in a simple form from (51) with the help of (5):

$$\Delta^r 1^\nu = r!\, S(\nu,r) + (r+1)!\, S(\nu,r+1) = r!\, S(\nu+1,r+1). \tag{75}$$

This completes the proof of Theorem 8.

In particular, for $\nu=n$ the formula (68) yields the following representation of the Bernoulli numbers:

$$B_n = \sum_{r=0}^{n} (-1)^{n-r} \frac{r!\, S(n+1,r+1)}{r+1} \qquad (n=0,1,2,\ldots). \tag{76}$$

On the other hand, the right-hand side of (76) to the factor $(-1)^n$ is equal to the value of the Bernoulli polynomial $B_n(x)$ for $x=1$ calculated by the formula (31) with the help of (75). This gives the classical equality

$$B_n(1) = (-1)^n B_n \qquad (n=0,1,2,\ldots), \tag{77}$$

which in the literature is proved in a different way.

Formula (75) can be generalized:

THEOREM 9. *The value of the polynomial* $\Delta^r z^\nu$ $(r=0,1,2,\ldots,\nu,$ $\nu=0,1,2,\ldots)$ *for any positive integer* $z=l=1,2,3,\ldots$ *is equal to*

$$\Delta^r l^\nu = \sum_{\alpha=0}^{l-1} \binom{l-1}{\alpha} (r+\alpha)!\, S(\nu+1, r+\alpha+1). \tag{78}$$

Proof. We shall apply induction. For $l=1$ the equality (78) is reduced to (75). If we assume that the equality (78) is true to some $l \geqslant 1$ inclusively and for arbitrary nonnegative integers r and ν ($r \leqslant \nu$), then at the next step on the basis of (7) we shall have

$$\begin{aligned}
\Delta^r (l+1)^\nu &= \Delta^r l^\nu + \Delta^{r+1} l^\nu \\
&= \sum_{\alpha=0}^{l-1} \binom{l-1}{\alpha} (r+\alpha)!\, S(\nu+1, r+\alpha+1) \\
&\quad + \sum_{\alpha=0}^{l-1} \binom{l-1}{\alpha} (r+\alpha+1)!\, S(\nu+1, r+\alpha+2) \\
&= \sum_{\alpha=0}^{l} \left[\binom{l-1}{\alpha} + \binom{l-1}{\alpha-1} \right] (r+\alpha)!\, S(\nu+1, r+\alpha+1) \\
&= \sum_{\alpha=0}^{l} \binom{l}{\alpha} (r+\alpha)!\, S(\nu+1, r+\alpha+1), \qquad (78')
\end{aligned}$$

by which the formula (78) is proved.

If we express $\Delta^r l^\nu$ by any of the formulas (8), (33), (51) and (72) as well and equate to (78), we shall obtain corresponding equalities. For our purposes we shall further note two equalities of this kind in the simplest case, namely, by equating (75) to the last sum in (33) for $x=1$ and (72) for $z=1$ ($r=0, 1, 2, \ldots, \nu$, $\nu = 0, 1, 2, \ldots$), respectively:

$$S(\nu+1, r+1) = \sum_{\alpha=r}^{\nu} S(\alpha, r) \binom{\nu}{\alpha}, \qquad (79)$$

$$r!\, S(\nu+1, r+1) = \sum_{\alpha=r}^{\nu} (-1)^{\nu-\alpha} \alpha!\, S(\nu, \alpha) \binom{\alpha}{r}. \qquad (80)$$

The equality (79) is obtained in [2, p. 209, formula (3c)] in another way.

2.4. *Another Series of Analytic Representations of the Functions $T_n(z)$*

For further application of the formula (67) it is expedient to keep in mind the following equalities:

THEOREM 10. *We have the formulas*

$$\begin{aligned}
(z\Delta)^n \frac{1}{z+\alpha} &= (-1)^\alpha \frac{\langle z \rangle_\alpha}{\alpha!} \Delta^\alpha T_n(z) \\
&= \sum_{r=0}^{n} (-1)^{n-r} \frac{\langle z \rangle_r}{r!} \cdot \frac{\Delta^r z^n}{z+\alpha+r} \qquad (\alpha, n = 0, 1, 2, \ldots). \quad (81)
\end{aligned}$$

In particular, for $z = 1$ we have the formula

$$\Delta^\alpha T_n(1) = (-1)^{n+\alpha} \frac{1}{\alpha} T_{n+1}(\alpha) \qquad (\alpha, n \geqslant 1). \tag{82}$$

Proof. From the verifiable by induction formula

$$\Delta^\alpha \frac{1}{z+k} = (-1)^\alpha \frac{\alpha!}{\langle z+k\rangle_{\alpha+1}} \tag{83}$$

and the verifiable by the basic property of the proportion formula

$$\frac{\langle z\rangle_\alpha}{\langle z+k\rangle_{\alpha+1}} = \frac{\langle z\rangle_k}{\langle z+\alpha\rangle_{k+1}} \tag{84}$$

the formula

$$(-1)^\alpha \frac{\langle z\rangle_\alpha}{\alpha!} \Delta^\alpha \frac{1}{z+k} = (-1)^k \frac{\langle z\rangle_k}{k!} \Delta^k \frac{1}{z+\alpha} \tag{85}$$

follows.

Then by the formula (59) on the one hand we have

$$\begin{aligned}(z\Delta)^n \frac{1}{z+\alpha} &= \sum_{k=0}^{n} S(n,k)\langle z\rangle_k \Delta^k \frac{1}{z+\alpha}\\ &= (-1)^\alpha \frac{\langle z\rangle_\alpha}{\alpha!} \sum_{k=0}^{n} (-1)^k k!\, S(n,k)\, \Delta^\alpha \frac{1}{z+k}\\ &= (-1)^\alpha \frac{\langle z\rangle_\alpha}{\alpha!} \Delta^\alpha T_n(z),\end{aligned} \tag{81$'$}$$

and on the other hand if we express $\Delta^k(1/(z+\alpha))$ by the formula (8) we have

$$\begin{aligned}(z\Delta)^n \frac{1}{z+\alpha} &= \sum_{k=0}^{n} S(n,k)\langle z\rangle_k \Delta^k \frac{1}{z+\alpha}\\ &= \sum_{k=0}^{n} S(n,k)\langle z\rangle_k \sum_{r=0}^{k} (-1)^{k-r} \binom{k}{r} \frac{1}{z+\alpha+r}\\ &= \sum_{r=0}^{n} (-1)^{n-r} \frac{\langle z\rangle_r}{r!} \cdot \frac{1}{z+\alpha+r} \sum_{k=r}^{n} (-1)^{n-k} S(n,k)(k)_r \langle z+r\rangle_{k-r}\\ &= \sum_{r=0}^{n} (-1)^{n-r} \frac{\langle z\rangle_r}{r!} \cdot \frac{\Delta^r z^n}{z+\alpha+r},\end{aligned} \tag{81$''$}$$

where for the internal sum obtained we have taken into consideration (72)–(74).

By this, the equalities (81) are proved.

From (81), in particular, for $z = 1$ $(\alpha, n \geqslant 1)$ on the basis of (75) and (66) we obtain

$$(-1)^{\alpha} \Delta^{\alpha} T_n(1) = \sum_{r=0}^{n} (-1)^{n-r} \frac{r! S(n+1, r+1)}{\alpha + r + 1} = (-1)^n \frac{1}{\alpha} T_{n+1}(\alpha). \qquad (82')$$

This completes the proof of Theorem 10 and its corollary.

Theorem 10 is a bridge for a reciprocal transition between Theorem 8 and the following theorem:

THEOREM 11. *Each function $T_n(z)$ $(n = 0, 1, 2, \ldots)$ of the class (55) has the series of representations*

$$T_n(z) = \sum_{k=0}^{n-\nu} (-1)^{\nu+k} \frac{k!\, S(n-\nu, k)\, P_\nu(z, k)}{\langle z + k \rangle_{\nu+1}} \qquad (\nu = 0, 1, 2, \ldots, n), \quad (86)$$

where

$$P_\nu(z, k) = \sum_{\alpha=0}^{\nu} (-1)^{\nu-\alpha} \alpha!\, S(\nu, \alpha) \langle z \rangle_\alpha \langle z + k + \alpha + 1 \rangle_{\nu-\alpha}. \qquad (87)$$

In particular, each Bernoulli number $B_n = T_n(1)$ $(n = 0, 1, 2, \ldots)$ has the series of representations

$$B_n = \sum_{k=0}^{n-\nu} (-1)^{k+1} \frac{k!\, S(n-\nu, k)\, p_{\nu-1}(k)}{\langle k + 1 \rangle_{\nu+1}} \qquad (\nu = 0, 1, 2, \ldots, n), \quad (88)$$

where

$$p_{\nu-1}(k) \equiv (-1)^{\nu+1} P_\nu(1, k) = \sum_{\alpha=0}^{\nu} (-1)^{\alpha+1} (\alpha!)^2 S(\nu, \alpha) \langle k + \alpha + 2 \rangle_{\nu-\alpha}. \quad (89)$$

Proof. If in (67) we express $T_{n-\nu}(z + r)$ by (55), replace the order of summation and take into consideration (81), we obtain the formula

$$T_n(z) = \sum_{k=0}^{n-\nu} (-1)^k k!\, S(n-\nu, k) (z\Delta)^\nu \frac{1}{z+k}, \qquad (90)$$

i.e., the same formula which follows from (61) and (55). From (90) by the inverse procedure with the help of (81) the formula (67) follows.

By means of (59) and (83) we obtain a third representation of

$$(z\Delta)^{\nu} \frac{1}{z+k} = (-1)^{\nu} \frac{P_{\nu}(z, k)}{\langle z+k \rangle_{\nu+1}}, \tag{91}$$

by which the formula (90) obtains the form (86).

This completes the proof of Theorem 11.

The polynomials (87) are of degree ν with respect to z for $\nu \geqslant 0$ and of degree $\nu - 1$ with respect to k for $\nu \geqslant 1$. The coefficient of z^{ν} $(\nu \geqslant 0)$ in (87) is equal to unity on the basis of the equality

$$\sum_{\alpha=0}^{\nu} (-1)^{\alpha} \alpha!\, S(\nu, \alpha) = (-1)^{\nu} \qquad (\nu \geqslant 0), \tag{92}$$

following from (3) for $x = -1$. The coefficient of $k^{\nu-1}$ $(\nu \geqslant 1)$ in (87) is evidently equal to $(-1)^{\nu-1}$; i.e., in the polynomials (89) the leading coefficient is equal to unity. The polynomials (87) satisfy the recurrence formula

$$P_{\nu}(z, k) = z[(z+k+\nu)\, P_{\nu-1}(z, k) - (z+k)\, P_{\nu-1}(z+1, k)]$$
$$(\nu = 1, 2, \ldots;\ P_0(z, k) = 1), \tag{93}$$

obtained if in (91) we replace ν with $\nu - 1$, apply to both sides the operator $z\Delta$ and again identify with (91). From (93) for $\nu = 0, 1, 2, 3, 4, \ldots$ we obtain the table

$$P_0(z, k) = 1, \tag{93_0}$$

$$P_1(z, k) = z, \tag{93_1}$$

$$P_2(z, k) = z(z-k), \tag{93_2}$$

$$P_3(z, k) = z[z^2 - (4k+1)z + k(k-1)], \tag{93_3}$$

$$P_4(z, k) = z(z-k)[z^2 - 5(2k+1)z + k(k-5)], \tag{93_4}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

with the help of which the corresponding representations (86) are written immediately. In particular, for $z = 1$ the representations (88) are also written, as well as the table of the polynomials $p_{\nu-1}(k)$ according to the first formula in (89) (see Section 2.5, where we give the independent recurrence formula (105) for calculus of the polynomials $p_{\nu-1}(k)$ with the help of which their table (113) and the table (114) of the representations of the Bernoulli numbers by the formula (88) are composed).

From the table it is also seen that for even $\nu \geqslant 2$ the polynomials $P_{\nu}(z, k)$ vanish for $z = k$. In Section 2.9 we shall prove rigorously this assertion (see Theorem 23, formula (190)).

2.5. *Representation of the Functions $T_n(z)$ as a Quotient of Two Relatively Prime Polynomials*

For $\nu = n$ the formula (86)–(87) yields

$$T_n(z) = (-1)^n \frac{P_n(z, 0)}{\langle z \rangle_{n+1}} \qquad (n = 0, 1, 2, \ldots), \tag{94}$$

where

$$P_n(z, 0) = \sum_{\alpha=0}^{n} (-1)^{n-\alpha} \alpha!\, S(n, \alpha) \langle z \rangle_\alpha \langle z + \alpha + 1 \rangle_{n-\alpha}. \tag{95}$$

For $n \geqslant 1$ we can set

$$P_n(z, 0) = z P_{n-1}(z) \qquad (n = 1, 2, \ldots; P_0(z) = 1), \tag{96}$$

where $P_{n-1}(z)$ denote the polynomials

$$P_{n-1}(z) = \sum_{k=1}^{n} (-1)^{n-k} k!\, S(n, k) \langle z + 1 \rangle_{k-1} \langle z + k + 1 \rangle_{n-k}, \tag{97}$$

which on the basis of (93) (for $\nu = n$ and $k = 0$) satisfy the recurrence formula

$$P_{n-1}(z) = z[(z + n\, P_{n-1}(z) - (z + 1)\, P_{n-2}(z + 1)] \qquad (n \geqslant 2; P_0(z) = 1). \tag{98}$$

Hence, the formula (94) takes the form

$$T_n(z) = (-1)^n \frac{P_{n-1}(z)}{\langle z + 1 \rangle_n} \qquad (n = 1, 2, \ldots), \tag{99}$$

which represents the functions (55) as a ratio of two polynomials. These polynomials are relatively prime since (97) for $z = -\alpha$, $\alpha = 1, 2, \ldots, n$, is reduced only to the term for $k = \alpha$ which is different from zero, i.e.,

$$P_{n-1}(-\alpha) = (-1)^{n-1} \alpha!\, S(n, \alpha)(\alpha - 1)!(n - \alpha)! \qquad (n \geqslant 1). \tag{100}$$

Remark 4. Independently of the above method the formula (99) together with (97) can be obtained directly from (55), reducing the right-hand side to the common denominator. After that the recurrence formula (98) can be obtained entering in (62) with (99).

Further, for $z = 1$ from (99) we obtain the Bernoulli numbers as a quotient of two integers

$$B_n = (-1)^n \frac{P_{n-1}(1)}{(n + 1)!} \qquad (n = 1, 2, \ldots). \tag{101}$$

From here for odd $n \geqslant 3$ it follows that $P_{n-1}(1)=0$. In addition, from (98) it follows that $P_{n-1}(0)=0$ for all $n \geqslant 2$. Hence, from the polynomials $P_{n-1}(z)$ the factor z can be extracted, which is not evident from their representation (97). Now we shall solve this problem, finding for the numerator in (99) a representation simpler than (97):

THEOREM 12. *For $n=2, 3, 4,\ldots$ each function $T_n(z)$ of the class* (55) *has the representation*

$$T_n(z)=z\sum_{k=1}^{n-1}(-1)^{n-1-k}\frac{(k!)^2\,S(n-1,k)}{\langle z+1\rangle_{k+1}}; \tag{102}$$

i.e., it has the following representation as a ratio of two relatively prime polynomials,

$$T_n(z)=(-1)^n\frac{zp_{n-2}(z)}{\langle z+1\rangle_n}\qquad (n=2,3,4,\ldots), \tag{103}$$

where $p_{n-2}(z)$ are the polynomials of $(n-2)$th *degree*

$$p_{n-2}(z)=\sum_{k=1}^{n-1}(-1)^{k+1}(k!)^2\,S(n-1,k)\langle z+k+2\rangle_{n-1-k}\qquad (n\geqslant 2) \tag{104}$$

which satisfy the recurrence formula

$$p_{n-2}(z)=z(z+n)\,p_{n-3}(z)-(z+1)^2\,p_{n-3}(z+1)\quad (n\geqslant 3;\ p_0(z)=1). \tag{105}$$

Proof. For arbitrary positive integers $n\geqslant 2$ and $\alpha\geqslant 1$ from (82) with the help of (55), (83) and (84) we obtain

$$\begin{aligned}T_n(\alpha)&=(-1)^{n-1+\alpha}\alpha\,\Delta^\alpha T_{n-1}(\alpha)\\&=(-1)^{n-1+\alpha}\alpha\sum_{k=1}^{n-1}(-1)^k k!\,S(n-1,k)\,\Delta^\alpha_{z=1}\frac{1}{z+k}\\&=(-1)^{n-1+\alpha}\alpha\sum_{k=1}^{n-1}(-1)^k k!\,S(n-1,k)\frac{(-1)^\alpha\alpha!}{\langle k+1\rangle_{\alpha+1}}\\&=(-1)^n\alpha\sum_{k=1}^{n-1}(-1)^{k+1}k!\,S(n-1,k)\frac{k!}{\langle\alpha+1\rangle_{k+1}}\\&=(-1)^n\alpha\sum_{k=1}^{n-1}(-1)^{k+1}\frac{(k!)^2\,S(n-1,k)}{\langle\alpha+1\rangle_{k+1}}.\end{aligned} \tag{106}$$

If we reduce to the common denominator, we obtain

$$T_n(\alpha) = (-1)^n \frac{\alpha p_{n-2}(\alpha)}{\langle \alpha + 1 \rangle_n} \qquad (n \geqslant 2), \tag{107}$$

where $p_{n-2}(\alpha)$ denotes the value of the polynomial (104) for $z = \alpha$.

On equating the right-hand sides of (107) and (99) for $z = \alpha$ $(n \geqslant 2)$ we obtain the equality

$$P_{n-1}(\alpha) = \alpha p_{n-2}(\alpha) \qquad (n \geqslant 2; \alpha = 1, 2, \ldots); \tag{108}$$

i.e., the polynomials (97) and (104) (multiplied by z) have equal values for infinite values of the argument $z = \alpha = 1, 2, \ldots$. Hence, the two polynomials are identical and for an arbitrary z we have the identity

$$P_{n-1}(z) = z p_{n-2}(z) \qquad (n = 2, 3, \ldots). \tag{109}$$

Now, setting (109) in (99) we obtain the formula (103), from which, if we perform the division of the sum (104) term by term by $\langle z + 1 \rangle_n$, we obtain the formula (102) (formally this procedure is the same as the one for inverse transition from (107) to the last equality in (106)).

Finally, entering in (98) with (109) we obtain the recurrence formula (105).

This completes the proof of Theorem 12.

Remark 5. For $z = 1$ the formula (102) is reduced to the formula (63′) for the Bernoulli numbers—the multiplier $(-1)^n$ can be omitted (for even n this is evident, and for odd $n \geqslant 3$ since $B_n = 0$).

Comparing (104), after replacing n with $n + 1$, and (89), considering k as an arbitrary independent variable, we see that the two polynomials are identical. Hence, by means of the formula (103) the two series of representations (88) and (68) for the Bernoulli numbers B_n are equivalent.

For $\nu = n$ and $\nu = n - 1$ from (88) we obtain

$$B_n = -\frac{p_{n-1}(0)}{(n+1)!} \qquad (n \geqslant 0; p_{-1}(0) = -1) \tag{110}$$

and

$$B_n = \frac{p_{n-2}(1)}{(n+1)!} \qquad (n \geqslant 1; p_{-1}(1) = -1), \tag{111}$$

respectively. Since $B_n = 0$ for odd $n \geqslant 3$, from (110)-(111) it follows that the polynomials $p_n(z)$, determined from (105) after replacing n with $n + 2$, always have a root $z=0$ for even $n \geqslant 2$ and a root $z=1$ for odd $n \geqslant 1$. From (110)–(111) it also follows that

$$p_{n-1}(0) = -p_{n-2}(1) \qquad (n \geqslant 1;\ p_{-1}(1) = -1); \tag{112}$$

i.e., for $n \geqslant 1$ the free term of the polynomial $p_{n-1}(z)$ is equal to the sum of the coefficients of the preceding polynomial $p_{n-2}(z)$ taken with the opposite sign.

By the formula (105) we obtain the table of the polynomials $p_n(z)$, $n=-1, 0, 1, 2,\ldots$ (for $n=-1$ on the basis of (89) for $\nu=0$):

$$\begin{aligned}
p_{-1}(z) &= -1,\\
p_0(z) &= 1,\\
p_1(z) &= z-1,\\
p_2(z) &= z(z-5),\\
p_3(z) &= (z-1)(z^2-15z-4),\\
p_4(z) &= z(z^3-42z^2+119z+42),\\
p_5(z) &= (z-1)(z^4-98z^3+659z^2+518z+120),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned} \tag{113}$$

From this table by the formula (103) the tables of the functions (55) as a quotient of two relatively prime polynomials are written immediately. From table (113) (setting $z=k$) the table for the representations of the Bernoulli numbers by the formula (88) for $\nu=0, 1, 2, 3, 4, 5,\ldots$ is written as well (for $\nu=0$ and $\nu=1$ we have the classical formula (2) and the formula (63′) already obtained by us, respectively):

$$\begin{aligned}
B_n &= \sum_{k=0}^{n-2} (-1)^{k+1} \frac{k!\, S(n-2,k)(k-1)}{\langle k+1\rangle_3} && (n \geqslant 2),\\
B_n &= \sum_{k=0}^{n-3} (-1)^{k+1} \frac{k!\, S(n-3,k)\, k(k-5)}{\langle k+1\rangle_4} && (n \geqslant 3),\\
B_n &= \sum_{k=0}^{n-4} (-1)^{k+1} \frac{k!\, S(n-4,k)(k-1)(k^2-15k-4)}{\langle k+1\rangle_5} && (n \geqslant 4),\\
B_n &= \sum_{k=0}^{n-5} (-1)^{k+1} \frac{k!\, S(n-5,k)\, k(k^3-42k^2+119k+42)}{\langle k+1\rangle_6} && (n \geqslant 5),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned} \tag{114}$$

For nonnegative integers $z=\alpha$ ($\alpha=0, 1, 2,\ldots$) the representation (102) by means of (84) can be rewritten in the form obtained in the third equality of (106):

$$T_n(\alpha)=\alpha!\alpha\sum_{k=1}^{n-1}(-1)^{n-1-k}\frac{k!\,S(n-1,k)}{\langle k+1\rangle_{\alpha+1}}\qquad(n\geqslant 2).\tag{115}$$

Another application of this formula for the proof of Theorem 19 below can be seen in Section 2.9.

By the aid of (83) and (59) the representation (102) is written in the operator form

$$\begin{aligned}T_n(z)&=(-1)^{n-1}z\sum_{k=1}^{n-1}k!\,S(n-1,k)\,\Delta^k\frac{1}{z+1}\\&=(-1)^{n-1}z(x\Delta)_{x=1}^{n-1}\frac{1}{z+x}\qquad(n\geqslant 2;\ z\neq -1,-2,\ldots,-n).\end{aligned}\tag{116}$$

If we equate (116) and (61), we obtain the series of identities ($\nu=0,1,2,\ldots,n;\ n\geqslant 2$),

$$(z\Delta)^\nu T_{n-\nu}(z)=(-1)^{n-1}z(x\Delta)_{x=1}^{n-1}\frac{1}{z+x},\tag{117}$$

for $z\neq -1,-2,\ldots,-n$ if $\nu<n$ and for $z\neq 0,-1,-2,\ldots,-n$ if $\nu=n$. In particular, for $\nu=n$ and $\nu=n-1$ we have

$$(z\Delta)^n\frac{1}{z}=(-1)^{n-1}z(x\Delta)_{x=1}^{n-1}\frac{1}{z+x}\qquad(n\geqslant 2;\ z\neq 0,-1,-2,\ldots,-n),\tag{118}$$

$$(z\Delta)^{n-1}\frac{1}{z+1}=(-1)^{n}z(x\Delta)_{x=1}^{n-1}\frac{1}{z+x}\qquad(n\geqslant 2;\ z\neq -1,-2,\ldots,-n).\tag{119}$$

For $z=1$ from (116) we have the following operator representation of the Bernoulli numbers:

$$B_n=(-1)^{n-1}(x\Delta)_{\substack{x=1\\z=1}}^{n-1}\frac{1}{z+x}\qquad(n\geqslant 2).\tag{120}$$

If in (102) we replace n with $n-\nu$ ($\nu=1,2,\ldots,n-2;\ n\geqslant 3$) and apply the operation (61), we shall obtain another series of representations of the functions (55) as a ratio of two relatively prime polynomials. In this case in order to apply the operator (59) it is necessary to keep in mind the following formula for the higher differences of the product of two suitable functions (see, for example, [4, p. 97, formula (10)]):

$$\Delta^n f(z)\, g(z) = \sum_{k=0}^{n} \binom{n}{k} \Delta^{n-k} f(z+k)\, \Delta^k g(z) \qquad (n = 0, 1, 2, \ldots). \tag{121}$$

(The formula (121) follows by induction from (58) and is an analogue of the classic Leibniz formula from the analysis for the higher derivatives of the product of two differentiable functions.) In addition, we need the formula

$$\Delta^q \frac{1}{\langle z+k \rangle_{\alpha+1}} = (-1)^q \frac{\langle \alpha+1 \rangle_q}{\langle z+k \rangle_{\alpha+1+q}} \tag{122}$$

as well, obtained if to (83) we apply the operator Δ^q. For example, in the basic case $\nu = 1$ we have the representation:

THEOREM 13. *For $n = 2, 3, 4, \ldots$ each function $T_n(z)$ of the class* (55) *has a representation*

$$\begin{aligned} T_n(z) &= z \sum_{k=0}^{n-2} (-1)^{n-1-k} \frac{(k!)^2\, S(n-2, k)(kz-1)}{\langle z+1 \rangle_{k+2}} \\ &\equiv (-1)^n \frac{z p_{n-2}(z)}{\langle z+1 \rangle_n}, \end{aligned} \tag{123}$$

where $p_{n-2}(z)$ are the same polynomials from Theorem 12 (*see* (104)) *which here emerge in the form*

$$p_{n-2}(z) = \sum_{k=0}^{n-2} (-1)^{k+1} (k!)^2\, S(n-2, k)(kz-1) \langle z+k+3 \rangle_{n-2-k} \qquad (n \geqslant 2). \tag{124}$$

Proof. For $n = 2$ the formula (123) is evidently true. For $n \geqslant 3$ applying the formula (62) to (102) on the basis of (58) we obtain

$$\begin{aligned} T_n(z) &= z\Delta T_{n-1}(z) = z\Delta \left[z \sum_{k=0}^{n-2} (-1)^{n-k} \frac{(k!)^2 S(n-2, k)}{\langle z+1 \rangle_{k+1}} \right] \\ &= z \left[z \sum_{k=0}^{n-2} (-1)^{n-1-k} \frac{(k!)^2 S(n-2, k)(k+1)}{\langle z+1 \rangle_{k+2}} \right. \\ &\quad \left. + \sum_{k=0}^{n-2} (-1)^{n-k} \frac{(k!)^2 S(n-2, k)}{\langle z+2 \rangle_{k+1}} \right] \\ &= z \sum_{k=0}^{n-2} (-1)^{n-1-k} \frac{(k!)^2 S(n-2, k)(kz-1)}{\langle z+1 \rangle_{k+2}}, \end{aligned} \tag{123'}$$

by which Theorem 13 is proved.

For nonnegative integers $z=\alpha$ ($\alpha=0, 1, 2,...$) the formula (123) by means of (84) can also be written as

$$T_n(\alpha)=\alpha!\alpha \sum_{k=0}^{n-2} (-1)^{n-1-k} \frac{k!\, S(n-2,k)(\alpha k-1)}{\langle k+1\rangle_{\alpha+2}} \qquad (n\geqslant 2), \quad (125)$$

which, in particular, for $\alpha=1$ yields the first formula of the series (114).

The different identical representations of the polynomials $p_{n-2}(z)$ can serve as formulas for summation and as the origin of many useful equalities. Here we shall note one typical series of this kind:

THEOREM 14. *For an arbitrary* $n=0, 1, 2,...$ *we have the series of equalities* ($\alpha=0, 1, 2,..., n$):

$$\sum_{k=\alpha}^{n} (-1)^{n-k} k!\, kS(n,k) \binom{k}{\alpha}$$
$$= \alpha!\, \alpha S(n+1, \alpha+1)+(\alpha+1)!\, (\alpha+1)\, S(n+1, \alpha+2). \quad (126)$$

In particular, for $\alpha=0$ *and* $\alpha=1$ *we have*

$$\sum_{k=0}^{n} (-1)^{n-k} k!\, kS(n,k)=2^n-1 \qquad (n=0, 1, 2,...) \qquad (126_0)$$

and

$$\sum_{k=1}^{n} (-1)^{n-k} k!\, k^2S(n,k)=2\cdot 3^n-3\cdot 2^n+1 \qquad (n=1, 2,...), \quad (126_1)$$

respectively.

Proof. For nonnegative integers $z=-\alpha$, $2\leqslant\alpha\leqslant n$ ($n\geqslant 2$) in the sum (124) the terms for $k\geqslant\alpha-2$ only will be different from zero, i.e.,

$$p_{n-2}(-\alpha)=(n-\alpha)! \sum_{k=\alpha-2}^{n-2} (-1)^k k!\, S(n-2,k)(\alpha k+1)(k)_{\alpha-2}. \quad (127)$$

By equating (127) and (100) by means of (109) we obtain the equality

$$\sum_{k=\alpha-2}^{n-2} (-1)^{n-k} k!\, S(n-2,k)(\alpha k+1)\binom{k}{\alpha-2}=(\alpha-1)!(\alpha-1)\, S(n,\alpha), \quad (128)$$

or, by replacing n with $n+2$ and α with $\alpha+2$ the formula

$$\sum_{k=\alpha}^{n} (-1)^{n-k}k!\, S(n,k)[(\alpha+2)\,k+1]\binom{k}{\alpha}$$
$$= (\alpha+1)!\,(\alpha+1)\,S(n+2,\alpha+2) \qquad (0 \leqslant \alpha \leqslant n;\ n \geqslant 0) \qquad (129)$$

is obtained.

By the aid of (80) the formula (129) is written as

$$(\alpha+2)\sum_{k=\alpha}^{n} (-1)^{n-k}k!\, kS(n,k)\binom{k}{\alpha}$$
$$= \alpha!\,[(\alpha+1)^2 S(n+2,\alpha+2) - S(n+1,\alpha+1)], \qquad (130)$$

whence if we express $S(n+2,\alpha+2)$ by (5) we obtain the formula (126). The particular cases are obtained on the basis of (6).

This completes the proof of Theorem 14.

2.6. *Extraction of the Multiplier $T_2(z)$ from the Functions $T_n(z)$.*

For $n \geqslant 2$ from Theorem 12 (or Theorem 13) it is seen that each function $T_n(z)$ has the multiplier $T_2(z) = z/(z+1)(z+2)$:

THEOREM 15. *For $n \geqslant 2$ each function $T_n(z)$ of the class* (55) *has a representation*

$$T_n(z) = (-1)^n T_2(z)\, R_{n-2}(z), \qquad (131)$$

where $R_{n-2}(z)$ is a rational function of order $n-2$ having the following expansion in elementary fractions:

$$R_{n-2}(z) = 1 + \sum_{k=0}^{n-2} (-1)^{n-k} \frac{(k+2)!\, S(n,k+3)(k+2)(k+1)}{z+k+3}. \qquad (132)$$

In particular, for $n \geqslant 2$ the Bernoulli numbers $B_n = T_n(1)$ have the representation

$$B_n = (-1)^n B_2 R_{n-2}(1) \qquad (B_2 = \tfrac{1}{6}), \qquad (133)$$

where

$$R_{n-2}(1) = 1 + \sum_{k=0}^{n-2} (-1)^{n-k} \frac{(k+2)!\, S(n,k+3)(k+2)(k+1)}{k+4}. \qquad (134)$$

Proof. For $n \geqslant 2$ from (55) we have

$$\frac{T_n(z)}{T_2(z)} = \sum_{k=1}^{n} (-1)^k k!\, S(n,k) \frac{(z+1)(z+2)}{z(z+k)}$$
$$= \sum_{k=1}^{n} (-1)^k k!\, S(n,k) \left[1 + \frac{2}{kz} - \frac{(k-1)(k-2)}{k(z+k)}\right]$$
$$= (-1)^n - \sum_{k=1}^{n} (-1)^k \frac{(k-1)!\, S(n,k)(k-1)(k-2)}{z+k}, \quad (135)$$

according to (92) and (64). From here, replacing k with $k+3$, we obtain the theorem.

From the comparison of (131) with (102) and (123) we obtain the following representations of the functions $R_{n-2}(z)$,

$$R_{n-2}(z) = \sum_{k=1}^{n-1} (-1)^{k+1} \frac{(k!)^2 S(n-1,k)}{\langle z+3\rangle_{k-1}} \qquad (n \geqslant 2), \quad (136)$$

$$R_{n-2}(z) = \sum_{k=0}^{n-2} (-1)^{k+1} \frac{(k!)^2 S(n-2,k)(kz-1)}{\langle z+3\rangle_{k}} \qquad (n \geqslant 2), \quad (137)$$

which are equivalent to the following representations as a ratio of two relatively prime polynomials

$$R_{n-2}(z) = \frac{p_{n-2}(z)}{\langle z+3\rangle_{n-2}} \qquad (n \geqslant 2), \quad (138)$$

where $p_{n-2}(z)$ are expressed by (104) and (124), respectively.

For $z=0$ from (136) on the basis of (55) we obtain

$$R_{n-2}(0) = -2B_{n-1} \qquad (n \geqslant 2). \quad (139)$$

Hence, from (132) we have another representation,

$$-2B_{n-1} = 1 + \sum_{k=0}^{n-2} (-1)^{n-k} \frac{(k+2)!\, S(n,k+3)(k+2)(k+1)}{k+3} \qquad (n \geqslant 2). \quad (140)$$

2.7. *Conversion of the Basic Formula for the Bernoulli Numbers*

The conversion of the formula (68) expressed the values of the functions $T_n(z)$ for natural values of the argument by the Bernoulli numbers. In order to solve this problem we shall need the absolute Stirling numbers $\sigma(n,k)$ of the first kind [2, p. 213],

$$\sigma(n,k) = |s(n,k)| = (-1)^{n+k} s(n,k), \quad (141)$$

where $s(n, k)$ are the algebraic Stirling numbers of the first kind, (46)–(48). The numbers $\sigma(n, k)$ are generated by the expansion (46) if we replace x with $-z$ [2, p. 213],

$$\langle z\rangle_n = \sum_{k=0}^{n} \sigma(n, k)\, z^k \qquad (n = 0, 1, 2,\ldots), \tag{142}$$

and they satisfy the recurrence formula

$$\begin{aligned} \sigma(n, k) &= \sigma(n-1, k-1) + (n-1)\,\sigma(n-1, k), \qquad n, k \geqslant 1; \\ \sigma(n, 0) = \sigma(0, k) &= 0, \qquad \sigma(0, 0) = 1; \qquad \sigma(n, k) = 0 \qquad (k > n) \end{aligned} \tag{143}$$

(the explicit form of the numbers $\sigma(n, k)$ is given by the formulas (48) and (141)).

THEOREM 16. *For natural values of the argument the values of each function $T_q(z)$ $(q = 0, 1, 2,\ldots)$ of the class* (55) *are expressed by the Bernoulli numbers by the formula*

$$n!\, T_q(n+1) = \sum_{\nu=0}^{n} \sigma(n+1, \nu+1)\, B_{q+\nu} \qquad (n = 0, 1, 2,\ldots). \tag{144}$$

Proof. If in (68) we set $n - \nu = q$, we obtain the formula

$$B_{q+\nu} = \sum_{r=0}^{\nu} (-1)^{\nu-r} r!\, S(\nu+1, r+1)\, T_q(r+1) \qquad (q, \nu \geqslant 0). \tag{145}$$

Multiplying (145) with $(-1)^\nu s(n+1, \nu+1)$ and summing from $\nu = 0$ to $\nu = n$, we obtain after replacing the order of summation,

$$\begin{aligned} &\sum_{\nu=0}^{n} (-1)^\nu s(n+1, \nu+1)\, B_{q+\nu} \\ &\quad = \sum_{\nu=0}^{n} s(n+1, \nu+1) \sum_{r=0}^{\nu} (-1)^r r!\, S(\nu+1, r+1)\, T_q(r+1) \\ &\quad = \sum_{r=0}^{n} (-1)^r r! T_q(r+1)\, \delta(n+1, r+1), \end{aligned} \tag{144$'$}$$

where

$$\delta(n+1, r+1) = \sum_{\nu=r+1}^{n+1} s(n+1, \nu)\, S(\nu, r+1) \tag{53$'$}$$

is the Kronecker symbol (see formula (53)), which is equal to zero if $r \neq n$ and is equal to unity if $r = n$. Hence, we have the formula

$$\sum_{\nu=0}^{n} (-1)^{\nu} s(n+1, \nu+1)\, B_{q+\nu} = (-1)^{n} n!\, T_q(n+1), \tag{144''}$$

which on the basis of (141) is reduced to (144).

This completes the proof of Theorem 16.

The following recurrence formula generates the equalities (144):

THEOREM 17. *The expression*

$$A_n(q) = n!\, T_q(n+1) \qquad (n, q = 0, 1, 2, \ldots) \tag{146}$$

is a solution of the recurrence formula

$$A_n(q) = nA_{n-1}(q) + A_{n-1}(q+1) \qquad (A_0(q) = B_q) \tag{147}$$

with respect to $n = 1, 2, \ldots$ *for an arbitrary* q $(q = 0, 1, 2, \ldots)$.

For $n = 1, 2, \ldots$ *the formula* (147) *generates successively the sum* (144).

Proof. If for $z = n$ $(n = 1, 2, \ldots)$ and $n - 1 = q$ $(q = 0, 1, 2, \ldots)$ we write the formula (62) in the form

$$nT_q(n+1) = nT_q(n) + T_{q+1}(n), \tag{148}$$

multiply the last formula with $(n-1)!$ and take into consideration (146), we obtain the formula (147), by which the first assertion is proved.

Now we shall show that the successive application of (147) generates the sum (144); i.e., it generates the numbers $\sigma(n+1, \nu+1)$ as well. Indeed, for $n = 1$ from (147) we have

$$A_1(q) = B_q + B_{q+1} \qquad (q \geqslant 0). \tag{147_1}$$

For $n = 2$ from (147) and (147_1) we obtain

$$\begin{aligned} A_2(q) &= 2A_1(q) + A_1(q+1) = 2(B_q + B_{q+1}) + (B_{q+1} + B_{q+2}) \\ &= 2B_q + 3B_{q+1} + B_{q+2} \qquad (q \geqslant 0). \end{aligned} \tag{147_2}$$

Continuing in such a way for $n = 3, 4, \ldots$ as well we obtain

$$A_3(q) = 6B_q + 11B_{q+1} + 6B_{q+2} + B_{q+3} \qquad (q \geqslant 0), \tag{147_3}$$

$$A_4(q) = 24B_q + 50B_{q+1} + 35B_{q+2} + 10B_{q+3} + B_{q+4} \qquad (q \geqslant 0), \tag{147_4}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots;$$

i.e., we obtain the sums in the right-hand side of (144) for $n = 1, 2, 3, 4, \ldots$.

In general, is we assume that to some natural $n \geqslant 1$ for an arbitrary integer $q \geqslant 0$ the magnitude $A_n(q)$ is represented by the sum on the right-hand side of (144), at the following step we shall obtain from (147) and (144)

$$\begin{aligned} A_{n+1}(q) &= (n+1)\,A_n(q) + A_n(q+1) \\ &= (n+1) \sum_{v=0}^{n} \sigma(n+1, v+1)\,B_{q+v} + \sum_{v=0}^{n} \sigma(n+1, v+1)\,B_{q+1+v} \\ &= \sum_{v=0}^{n+1} \left[(n+1)\,\sigma(n+1, v+1) + \sigma(n+1, v)\right] B_{q+v} \\ &= \sum_{v=0}^{n+1} \sigma(n+2, v+1)\,B_{q+v}, \end{aligned} \tag{147_{n+1}}$$

taking into consideration the equalities from (143); i.e., $A_{n+1}(q)$ is represented by the sum in (144) after replacing n and $n+1$ as well.

Theorem 17 is proved and, of course, a second proof (by induction by means of (146)–(147)) of the formula (144) is given as well.

In particular, on the basis of (146) from (147_3) for even and odd q we have the following formulas for the Bernoulli numbers,

$$B_{2m} = T_{2m}(2) \qquad (m = 1, 2, \ldots) \tag{149}$$

and

$$B_{2m} = T_{2m-1}(2) \qquad (m = 2, 3, \ldots), \tag{150}$$

respectively, and from (147_2) for odd q we have the formula

$$B_{2m} = \tfrac{2}{3} T_{2m-1}(3) \qquad (m = 2, 3, \ldots). \tag{151}$$

From these formulas if we express $T_{2m}(2)$. $T_{2m-1}(2)$ and $T_{2m-1}(3)$ by any of the representations of the functions (55) obtained we shall obtain other corresponding representations of the Bernoulli numbers. For example, from (149)–(150) by (67) with the help of (78) we can obtain two series of representations of the Bernoulli numbers with an even index—these series will be analogous to the series (68), which, however, has the most general and, simultaneously, the simplest character (therefore, we set the series (68) as the basic one in our investigation). Other simple representations are obtained if we express the right-hand sides of (149)–(151) by (55):

$$B_{2m} = \sum_{k=1}^{2m} (-1)^k \frac{k!\, S(2m, k)}{k+2} \qquad (m \geqslant 1), \tag{152}$$

$$B_{2m} = \sum_{k=1}^{2m-1} (-1)^k \frac{k!\, S(2m-1, k)}{k+2} \qquad (m \geqslant 2), \quad (153)$$

$$B_{2m} = \frac{2}{3} \sum_{k=1}^{2m-1} (-1)^k \frac{S(2m-1, k)}{k+3} \qquad (m \geqslant 2). \quad (154)$$

Further, from (149)–(151) by (66) we have

$$B_{2m} = 2 \sum_{k=1}^{2m} (-1)^{k+1} \frac{(k-1)!\, S(2m, k)}{k+2} \qquad (m \geqslant 1), \quad (155)$$

$$B_{2m} = 2 \sum_{k=1}^{2m-1} (-1)^{k+1} \frac{(k-1)!\, S(2m-1, k)}{k+2} \qquad (m \geqslant 2), \quad (156)$$

$$B_{2m} = 2 \sum_{k=1}^{2m-1} (-1)^{k+1} \frac{(k-1)!\, S(2m-1, k)}{k+3} \qquad (m \geqslant 2). \quad (157)$$

From (149)–(151) by (63) we have

$$B_{2m} = 2 \sum_{k=1}^{2m-1} (-1)^{k+1} \frac{k!\, S(2m-1, k)}{(k+2)(k+3)} \qquad (m \geqslant 1), \quad (158)$$

$$B_{2m} = 2 \sum_{k=1}^{2m-2} (-1)^{k+1} \frac{k!\, S(2m-2, k)}{(k+2)(k+3)} \qquad (m \geqslant 2), \quad (159)$$

$$B_{2m} = 2 \sum_{k=1}^{2m-2} (-1)^{k+1} \frac{k!\, S(2m-2, k)}{(k+3)(k+4)} \qquad (m \geqslant 2). \quad (160)$$

From (149)–(151) by (115) we have

$$B_{2m} = 4 \sum_{k=1}^{2m-1} (-1)^{k+1} \frac{k!\, S(2m-1, k)}{(k+1)(k+2)(k+3)} \qquad (m \geqslant 1), \quad (161)$$

$$B_{2m} = 4 \sum_{k=1}^{2m-2} (-1)^{k} \frac{k!\, S(2m-2, k)}{(k+1)(k+2)(k+3)} \qquad (m \geqslant 2), \quad (162)$$

$$B_{2m} = 12 \sum_{k=1}^{2m-2} (-1)^{k} \frac{k!\, S(2m-2, k)}{(k+1)(k+2)(k+3)(k+4)} \qquad (m \geqslant 2). \quad (163)$$

From (149)–(151) by (125) we have

$$B_{2m} = 4 \sum_{k=0}^{2m-2} (-1)^{k+1} \frac{k!\, S(2m-2, k)(2k-1)}{(k+1)(k+2)(k+3)(k+4)} \qquad (m \geqslant 1), \quad (164)$$

$$B_{2m} = 4 \sum_{k=1}^{2m-3} (-1)^{k} \frac{k!\, S(2m-3, k)(2k-1)}{(k+1)(k+2)(k+3)(k+4)} \qquad (m \geqslant 2), \quad (165)$$

$$B_{2m} = 12 \sum_{k=1}^{2m-3} (-1)^k \frac{k!\, S(2m-3, k)(3k-1)}{(k+1)(k+2)(k+3)(k+4)(k+5)} \quad (m \geqslant 2), \quad (166)$$

etc.

Another table of formulas is obtained if in (144) we set $q = 0, 1, 2, 3, 4, \ldots, n, n+1, \ldots$ $(n \geqslant 0)$ (for calculation of the left-hand side of (144) we use (55) for $q = 0, 1$ and (103) with the table (113) for $q \geqslant 2$):

$$\sum_{\nu=0}^{n} \sigma(n+1, \nu+1)\, B_\nu = \frac{n!}{n+1}, \qquad (144_0)$$

$$\sum_{\nu=0}^{n} \sigma(n+1, \nu+1)\, B_{\nu+1} = -\frac{n!}{n+2}, \qquad (144_1)$$

$$\sum_{\nu=0}^{n} \sigma(n+1, \nu+1)\, B_{\nu+2} = \frac{(n+1)!}{\langle n+2\rangle_2}, \qquad (144_2)$$

$$\sum_{\nu=0}^{n} \sigma(n+1, \nu+1)\, B_{\nu+3} = -\frac{(n+1)!\, n}{\langle n+2\rangle_3}, \qquad (144_3)$$

$$\sum_{\nu=0}^{n} \sigma(n+1, \nu+1)\, B_{\nu+4} = \frac{(n+1)!\, (n+1)(n-4)}{\langle n+2\rangle_4}, \qquad (144_4)$$

$$\sum_{\nu=0}^{n} \sigma(n+1, \nu+1)\, B_{\nu+5} = -\frac{(n+1)!\, n(n^2-13n-18)}{\langle n+2\rangle_5}, \qquad (144_5)$$

$$\sum_{\nu=0}^{n} \sigma(n+1, \nu+1)\, B_{\nu+6} = \frac{(n+1)!\, (n+1)(n^3-39n^2+38n+120)}{\langle n+2\rangle_6}, \qquad (144_6)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$\sum_{\nu=0}^{n} \sigma(n+1, \nu+1)\, B_{\nu+n} = n!\, T_n(n+1), \qquad (144_n)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Analogously, we can obtain a series of formulas for the values of q, multiple of n; i.e., for $q = \mu n$ $(\mu = 0, 1, 2, \ldots)$ from (144) we obtain

$$\sum_{\nu=0}^{n} \sigma(n+1, \nu+1)\, B_{\mu n+\nu} = n!\, T_{\mu n}(n+1) \qquad (n, \mu \geqslant 0). \qquad (167)$$

2.8. *Taylor Expansions of the Functions $G_k(t)$*

The coefficients of the power expansions of the functions (10) from Theorem 1 are also expressed with the help of the functions (55):

THEOREM 18. *In the disc $|t| < 2\pi$ we have the Taylor expansions*

$$e^t G_k(t) = \sum_{n=0}^{\infty} (-1)^n T_n(k+1) \frac{t^n}{n!} \qquad (k = 0, 1, 2, \ldots) \tag{168}$$

and

$$G_k(t) = \frac{1}{k+1} + \frac{1}{k} \sum_{m=1}^{\infty} (-1)^m T_{m+1}(k) \frac{t^m}{m!} \qquad (k = 1, 2, \ldots), \tag{169}$$

where $T_n(k+1)$ and $T_{m+1}(k)$ are the denoted values of the functions (55) (*in* (55) *the summing index k should be replaced by another suitable letter*).

For $k = 0$ and after replacing t with $-t$ the expansion (168) *is reduced to the classical expansion* (1)–(2).

Proof. If $t \in H_0(t)$ and moreover $|1 - e^{-t}| < 1$, the formula (23) yields

$$e^t G_k(t) = \int_0^1 dx \sum_{\nu=0}^{\infty} (1 - e^{-t})^\nu x^{\nu+k} = \sum_{\nu=0}^{\infty} \frac{(1 - e^{-t})^\nu}{\nu + k + 1} \qquad (k = 0, 1, 2, \ldots). \tag{170}$$

From here on the basis of (21) and (55) we obtain the expansion

$$\begin{aligned} e^t G_k(t) &= \sum_{\nu=0}^{\infty} \frac{\nu!}{\nu + k + 1} \sum_{n=\nu}^{\infty} (-1)^{n-\nu} S(n, \nu) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{t^n}{n!} \sum_{\nu=0}^{n} (-1)^\nu \frac{\nu!\, S(n, \nu)}{k + 1 + \nu} \\ &= \sum_{n=0}^{\infty} (-1)^n T_n(k+1) \frac{t^n}{n!} \qquad (k = 0, 1, 2, \ldots), \end{aligned} \tag{168'}$$

i.e., the expansion (168) which is valid in the disc $|t| < 2\pi$ since the nearest singular points of the functions (10) are the poles $\pm 2\pi i$. The corollary from (168) for $k = 0$ and after replacing t with $-t$, noted in the theorem, is evident.

Further, from (168) we obtain the expansion

$$\begin{aligned} G_k(t) &= \sum_{n=0}^{\infty} (-1)^n T_n(k+1) \frac{t^n}{n!} \sum_{m=n}^{\infty} (-1)^{m-n} \frac{t^{m-n}}{(m-n)!} \\ &= \sum_{m=0}^{\infty} (-1)^m \frac{t^m}{m!} \sum_{n=0}^{m} \binom{m}{n} T_n(k+1) \qquad (k = 0, 1, 2, \ldots). \end{aligned} \tag{171}$$

For each $m = 0, 1, 2,...$ we have, however,

$$\begin{aligned}
&\sum_{n=0}^{m} \binom{m}{n} T_n(k+1) \\
&\quad = \sum_{n=0}^{m} \binom{m}{n} \sum_{\nu=0}^{n} (-1)^\nu \frac{\nu!\, S(n,\nu)}{k+1+\nu} \\
&\quad = \sum_{\nu=0}^{m} (-1)^\nu \frac{\nu!}{k+\nu+1} \sum_{n=\nu}^{m} \binom{m}{n} S(n,\nu) \\
&\quad = \sum_{\nu=0}^{m} (-1)^\nu \frac{\nu!\, S(m+1,\nu+1)}{k+\nu+1} \qquad (k=0,1,2,...), \qquad (172)
\end{aligned}$$

where we have replaced the last internal sum with $S(m+1, \nu+1)$ according to the formula (79).

Let $k \geqslant 1$. Then for $m \geqslant 1$ if we replace ν with $\nu - 1$ in the last sum of (172) on the basis of (66) we obtain the formula (for $m = 0$ the value is evident)

$$\sum_{n=0}^{m} \binom{m}{n} T_n(k+1) = \begin{cases} (1/k)\, T_{m+1}(k), & m \geqslant 1 \\ 1/(k+1), & m = 0 \end{cases} \quad (k \geqslant 1), \qquad (173)$$

by the aid of which the expansion (171) takes the form (169).

This completes the proof of Theorem 18.

Remark 6. For $k = 0$ the formula (172) on the basis of (56) and (76) yields the classical recurrence formula (for $m = 1$ the value is evident) (see, for example, [4, p. 233, formula (7)])

$$\sum_{n=0}^{m} \binom{m}{n} B_n = \begin{cases} B_m, & m \geqslant 0,\ m \neq 1, \\ \frac{1}{2}, & m = 1. \end{cases} \qquad (174)$$

The formula (174) reduces the expansion (171) (for $k = 0$) to the classical expansion (1)–(2) (we can eliminate the multiplier $(-1)^m$ since $B_m = 0$ for odd $m \geqslant 3$, and for $m = 0$ and $m = 1$ the coefficients are exactly the Bernoulli numbers $B_0 = 1$ and $B_1 = -\frac{1}{2}$).

We shall note that with the help of the expansion (169) we can obtain the basic formula (68) for the Bernoulli numbers. This is realized if we equate the coefficients of the Taylor expansion of $g^{(n)}(t)$ obtained in two ways: firstly, by aid of (169) and (9) and secondly, by differentiating (1). However, this manner does not give the full series in (68)—the missing cases can be complemented by an immediate verification, taking into consideration the other formulas already established. Indeed, for $n \geqslant 1$, entering in (9) with (169), we obtain the Taylor expansion

$$\begin{aligned} g^{(n)}(t) &= \sum_{k=1}^{n} (-1)^k k!\, S(n,k) \left(\frac{1}{k+1} + \frac{1}{k} \sum_{m=1}^{\infty} (-1)^m T_{m+1}(k) \frac{t^m}{m!} \right) \\ &= B_n + \sum_{k=1}^{n} (-1)^k (k-1)!\, S(n,k) \sum_{m=1}^{\infty} (-1)^m T_{m+1}(k) \frac{t^m}{m!} \\ &= B_n + \sum_{m=1}^{\infty} (-1)^m \frac{t^m}{m!} \sum_{k=1}^{n} (-1)^k (k-1)!\, S(n,k)\, T_{m+1}(k). \end{aligned} \tag{175}$$

On the other hand, from (1), replacing the summing index n with m, differentiating n times, we have

$$g^{(n)}(t) = \sum_{m=n}^{\infty} \frac{B_m}{(m-n)!} t^{m-n} = \sum_{m=0}^{\infty} \frac{B_{m+n}}{m!} t^m. \tag{176}$$

Equating the coefficients of (175) and (176), we obtain the formula

$$B_{m+n} = (-1)^m \sum_{k=1}^{n} (-1)^k (k-1)!\, S(n,k)\, T_{m+1}(k) \qquad (m, n \geqslant 1), \tag{177}$$

in which we can replace the multiplier $(-1)^m$ with $(-1)^n$ (indeed, since $m+n \geqslant 2$, then for odd $m+n$ we have $B_{m+n}=0$; i.e., the sum on the right-hand side of (177) is equal to zero as well, and for even $m+n$ we have $(-1)^m = (-1)^n$). If in (177) we replace n with $\nu+1$ and m with $n-\nu-1$ $(0 \leqslant \nu \leqslant n-2,\ n \geqslant 2)$, we obtain the formula (68) for $n \geqslant 2$ and $0 \leqslant \nu \leqslant n-2$. For $n \geqslant 2$ the case $\nu = n-1$ is added according to (66′), and the case $\nu = n$ is verified with the help of (5), (64) and (55). For $n=1$ and $n=0$ the corresponding cases $\nu = 0, 1$ and $\nu = 0$ are verified immediately.

2.9. *Higher Differences of the Functions $T_n(z)$ for $z = 0, 1, 2, \ldots$*

According to (8) for the higher differences of the functions (55) when the values of the argument are nonnegative integers we have the formula

$$\Delta^\alpha T_n(l) = \sum_{k=0}^{\alpha} (-1)^{\alpha-k} \binom{\alpha}{k} T_n(l+k), \tag{178}$$

valid for $n = 1, 2, \ldots$ for all integers $\alpha, l \geqslant 0$, and for $n=0$ for all integers $\alpha \geqslant 0$ and $l \geqslant 1$.

We shall obtain other representations of the higher differences $\Delta^\alpha T_n(l)$ on the basis of their specific properties.

First, with the help of the formula (115) we find the following simpler representation of the higher difference $\Delta^\alpha T_n(0)$ in comparison with the representation (178) for $l=0$:

THEOREM 19. *For an arbitrary* $n = 1, 2, \ldots$ *we have the formula*

$$\Delta^\alpha T_n(0) = (-1)^{n+\alpha+1} T_n(\alpha) \qquad (\alpha = 0, 1, 2, \ldots). \tag{179}$$

Proof. For $n = 1$ the formula (179) is verified by the formula (83) applied to the function $T_1(z) = -1/(z+1)$. For $n \geqslant 2$ from (63), (122), (121) and (115) we obtain

$$\Delta^\alpha T_n(0) \equiv \Delta^\alpha_{z=0} T_n(z)$$

$$= (-1)^\alpha \alpha!\, \alpha \sum_{k=1}^{n-1} (-1)^k \frac{k!\, S(n-1, k)}{\langle k+1 \rangle_{\alpha+1}} = (-1)^{n+\alpha+1} T_n(\alpha). \tag{179'}$$

This completes the proof of Theorem 19.

Now we shall generalize the formula (179).

THEOREM 20. *For an arbitrary* $n = 1, 2, \ldots$ *for all integers* α, $l \geqslant 0$ *we have the formula*

$$\Delta^\alpha T_n(l) = (-1)^{n+\alpha} \sum_{k=0}^{l} (-1)^{k+1} \binom{l}{k} T_n(\alpha + k), \tag{180}$$

and for $n = 0$ for all integers $\alpha \geqslant 0$ and $l \geqslant 1$ the formula

$$\Delta^\alpha T_0(l) = \sum_{k=0}^{l-1} (-1)^{\alpha+k} \binom{l-1}{k} T_0(\alpha + k + 1). \tag{181}$$

Proof. We shall perform an induction with respect to l, regarding the natural n as fixed and the integer $\alpha \geqslant 0$ as arbitrary. For $l = 0$ the formula (180) coincides with (179). Assuming that the formula (180) is true to some $l \geqslant 0$ inclusively for each integer $\alpha \geqslant 0$, at the next step on the basis of (7) we shall have

$$\begin{aligned}
&(-1)^{n+\alpha} \Delta^\alpha T_n(l+1) \\
&\quad = (-1)^{n+\alpha} \Delta^\alpha T_n(l) - (-1)^{n+\alpha+1} \Delta^{\alpha+1} T_n(l) \\
&\quad = \sum_{k=0}^{l} (-1)^{k+1} \binom{l}{k} T_n(\alpha+k) - \sum_{k=0}^{l} (-1)^{k+1} \binom{l}{k} T_n(\alpha+k+1) \\
&\quad = \sum_{k=0}^{l+1} (-1)^{k+1} \left[\binom{l}{k} + \binom{l}{k-1} \right] T_n(\alpha+k) \\
&\quad = \sum_{k=0}^{l+1} (-1)^{k+1} \binom{l+1}{k} T_n(\alpha+k),
\end{aligned} \tag{180'}$$

by which the formula (180) is proved.

The formula (181) is obtained from (180) for $n = 1$ replacing l with $l-1$ and keeping in mind that $T_1(z) = -T_0(z+1)$.

This completes the proof of Theorem 20.

By the aid of Theorem 20 we can generalize the formula (82):

THEOREM 21. *For an arbitrary $n = 1, 2, \ldots$ for all natural α, $l \geqslant 1$ we have the formula (for $l = 1$ we have the formula* (82))

$$\Delta^\alpha T_n(l) = (-1)^{n+\alpha} \sum_{k=0}^{l-1} (-1)^k \binom{l-1}{k} \frac{T_{n+1}(\alpha+k)}{\alpha+k}. \tag{182}$$

For $n = 0$ for all integers $\alpha \geqslant 0$ and $l \geqslant 2$ we have the formula

$$\Delta^\alpha T_0(l) = \sum_{k=0}^{l-2} (-1)^{\alpha+k+1} \binom{l-2}{k} \frac{T_1(\alpha+k+1)}{\alpha+k+1}. \tag{183}$$

Proof. From the formula (180) taken for $l \geqslant 1$ in order to be able to decompose the binomial coefficients, and for $\alpha \geqslant 1$ in order to be able to apply the formula (62), as is indicated below, we have

$$\begin{aligned}
&(-1)^{n+\alpha} \Delta^\alpha T_n(l) \\
&\quad = \sum_{k=0}^{l-1} (-1)^{k+1} \binom{l-1}{k} T_n(\alpha+k) + \sum_{k=1}^{l} (-1)^{k+1} \binom{l-1}{k-1} T_n(\alpha+k) \\
&\quad = \sum_{k=0}^{l-1} (-1)^k \binom{l-1}{k} [T_n(\alpha+k+1) - T_n(\alpha+k)] \\
&\quad = \sum_{k=0}^{l-1} (-1)^k \binom{l-1}{k} \frac{T_{n+1}(\alpha+k)}{\alpha+k};
\end{aligned} \tag{182'}$$

i.e., we have obtained the formula (182).

Analogously, from (181) for $l \geqslant 2$ (and $\alpha \geqslant 0$) we have

$$\begin{aligned}
\Delta^\alpha T_0(l) &= \sum_{k=0}^{l-2} (-1)^{\alpha+k} \binom{l-2}{k} T_0(\alpha+k+1) \\
&\quad + \sum_{k=1}^{l-1} (-1)^{\alpha+k} \binom{l-2}{k-1} T_0(\alpha+k+1) \\
&= \sum_{k=0}^{l-2} (-1)^{\alpha+k+1} \binom{l-2}{k} [T_0(\alpha+k+2) - T_0(\alpha+k+1)] \\
&= \sum_{k=0}^{l-2} (-1)^{\alpha+k+1} \binom{l-2}{k} \frac{T_1(\alpha+k+1)}{\alpha+k+1};
\end{aligned} \tag{183'}$$

i.e., we have obtained the formula (183) as well, by which Theorem 21 is proved.

It is evident that the comparison of the formulas (178), (180), (181), (182) and (183) yields corresponding combinatorial identities.

We can formulate Theorem 20 in another way:

THEOREM 22. *For an arbitrary $n = 1, 2, \ldots$ for all integers α, $l \geqslant 0$ we have the formula*

$$\Delta^\alpha T_n(l) = (-1)^{n+\alpha+l+1} \Delta^l T_n(\alpha), \tag{184}$$

and for $n = 0$ for all integers $\alpha \geqslant 0$ and $l \geqslant 1$ the formula

$$\Delta^\alpha T_0(l) = (-1)^{\alpha+l+1} \Delta^{l-1} T_0(\alpha+1). \tag{185}$$

In particular, we have

$$\Delta^\alpha T_{2m}(\alpha) = 0 \qquad (m = 1, 2, \ldots;\ \alpha = 0, 1, 2, \ldots). \tag{186}$$

Proof. For $n = 1, 2, \ldots$ if in (178) we exchange the roles of α and l, we obtain

$$\Delta^l T_n(\alpha) = \sum_{k=0}^{l} (-1)^{l-k} \binom{l}{k} T_n(\alpha+k), \tag{187}$$

whence, multiplying by $(-1)^{l+1}$ and comparing with (180), we obtain the formula (184).

For $n = 0$, if in (178) we replace α with $l-1$ and l with $\alpha+1$, we obtain

$$\Delta^{l-1} T_0(\alpha+1) = \sum_{k=0}^{l-1} (-1)^{l-1-k} \binom{l-1}{k} T_0(\alpha+1+k), \tag{188}$$

whence, multiplying by $(-1)^{l+1+\alpha}$ and comparing with (181), we obtain the formula (185).

In particular, for $n = 2m$ $(m = 1, 2, \ldots)$ and $\alpha = l = 0, 1, 2, \ldots$ from (184) we have $\Delta^\alpha T_{2m}(\alpha) = -\Delta^\alpha T_{2m}(\alpha)$; i.e., we have the formula (186).

This completes the proof of Theorem 22.

An important application of the formula (186) is the following:

THEOREM 23. *For even $v \geqslant 2$ the polynomials* (87) *are identically equal to zero for $z = k$; i.e., for an arbitrary k we have*

$$P_{2m}(k, k) = 0 \qquad (m = 1, 2, \ldots). \tag{189}$$

Proof. On the basis of (186) from the first equality of (81) we obtain

$$(z\Delta)_{z=\alpha}^{2m} \frac{1}{z+\alpha} = 0 \qquad (m, \alpha = 1, 2, \ldots), \tag{190}$$

whence and (91) we obtain

$$P_{2m}(k, k) = 0 \qquad (m, k = 1, 2, \ldots). \tag{191}$$

Regarding m as fixed, the equality (191) indicates that the polynomial $P_{2m}(k, k)$ of degree $2m$ with respect to k vanishes for infinite values $k = 1, 2, \ldots$. Hence, the polynomial $P_{2m}(k, k)$ is identically equal to zero for an arbitrary (complex) k, by which the equality (189) is proved (in this way our conjecture at the end of Section 2.4, which has arisen from table ($93_{0,1,2,\ldots}$), is proved as well).

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