Bernoulli Numbers and the Riemann Zeta Function

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B Sury is with the Indian Statistical Institute. He introduces this article by:

Bernoulli truly stunned us
with his numb-er;
woke us up from a deep
and ignorant slumber.
Its relation with Riemann
zeta
makes us think nothing
could be neater.
The connection is much
deeper
– ask any plumber!

Keywords
Sums of powers, Bernoulli numbers, Riemann zeta function, Abel summation, Riemann hypothesis.

Introduction

It is a beautiful discovery due to J Bernoulli that for any positive integer k, the sum $\sum_{i=1}^n i^k$ can be evaluated in terms of, what are now known as, Bernoulli numbers.

In this article, we shall discuss several methods of evaluating the above sum. For instance, Marikkannan and Ravichandran have written about a method of evaluation using integration. Apart from Bernoulli's method which we shall recall, we give a method akin to using integration, and one using differentiation. These methods are often useful in evaluating more general sums too as we shall indicate. Finally, we discuss the connections with the Riemann Zeta function.

Bernoulli Polynomials and Numbers

To motivate the introduction of the Bernoulli polynomials, let us start with the sum that we want to evaluate viz., $\sum_{i=1}^n i^k$. Evidently, $\frac{\sum_{i=1}^n i^k}{k!}$ is the coefficient of x^k in the power series expansion of $e^x + e^{2x} + \cdots + e^{nx}$. In other words,

$$\frac{e^{(n+1)x} - 1}{e^x - 1} = 1 + \sum_{k\geq 0} \frac{1^k + 2^k + \cdots + n^k}{k!} x^k.$$

Now, for x in a small interval around 0, the function $\frac{x}{e^x-1}$ can be represented by a power series $\frac{x}{e^x-1} = \sum_{r\geq 0} B_r \frac{x^r}{r!}$. The numbers B_r are known as Bernoulli numbers and it is easy to evaluate them as follows.

Since the power series x and $(e^x - 1)\sum_{r\geq 0} B_r \frac{x^r}{r!}$ agree in an interval around 0, the numbers are determined recursively as

$$B_0 = 1\ , \ \sum_{s<r} \binom{r}{s} B_s = 0 \ \forall\ r \geq 2.$$

The first few values are $B_0 = 1, B_1 = -1/2, B_2 = 1/6$ and $B_3 = B_5 = B_7 = \cdots = 0$.

Now, consider the function $F_t(x) = \frac{xe^{tx}}{e^x-1}$ for $x \neq 0$ and $F_t(0) = 1$. Once again, in an interval around 0, F_t has a power series expansion $F_t(x) = \sum_{k\geq 0} B_k(t)\frac{x^k}{k!}$.

The functions $B_k(t)$ are actually polynomials in t since

$$\sum_{k\geq 0} B_k(t)\frac{x^k}{k!} = F_t(x) = e^{tx}\frac{x}{e^x-1} = e^{tx}\sum_{k\geq 0} B_k\frac{x^k}{k!}$$

and thus

$$B_k(t) = \sum_{l=0}^{k}\binom{k}{l}B_l t^{k-l}.$$

$B_k(t)$ are called Bernoulli polynomials; note that $B_k(0) = B_k$.

Returning to our sum, we have that $\frac{1^k+2^k+\cdots+n^k}{k!}$ is the coefficient of x^k in $\frac{e^{(n+1)x}-1}{e^x-1}$ i.e., it is the coefficient of x^{k+1} in $\frac{x(e^{(n+1)x}-1)}{e^x-1} = F_{n+1}(x) - F_0(x)$.

Thus, $\frac{1^k+2^k+\cdots+n^k}{k!} = \frac{B_{k+1}(n+1)-B_{k+1}}{(k+1)!} = \frac{1}{(k+1)!}\sum_{l=0}^{k}\binom{k+1}{l}$ $B_l(n+1)^{k+1-l}$.

In other words,

$$1^k + 2^k + \cdots + n^k = \frac{1}{k+1}\sum_{l=0}^{k}\binom{k+1}{l}B_l(n+1)^{k+1-l}.$$

Note that it is evident from this formula that the sum of the k-th powers of the first n natural numbers is a polynomial function of n of degree $k+1$.

Method of 'Integration'

For convenience, let us denote $S_k(n) = 1^k + 2^k + \cdots + n^k$. This is a polynomial function of n i.e., there is a polynomial $S_k(x)$ of degree $k+1$ such that the above equality holds for all n.

The basic idea of the method we will discuss now is that (since $n^k = S_k(n) - S_k(n-1)$), x^k can be thought of as a 'derivative' of the function $S_k(x)$. In other words, $S_k(x)$ itself may be thought of as an 'integral' of the function x^k. Of course, this is only heuristic at the moment because x^k will be the derivative of S_k at some point between $x-1$ and x. The correct tool to make this precise is the 'method of differences' which is really a discrete analogue of differentiation. More precisely, let us recall that the 'backward difference' operator is defined on any function f by $(\nabla f)(x) = f(x) - f(x-1)$ for all x. It is trivial to see that if $P_r(x) = x(x+1)\cdots(x+r-1)$ for $r \geq 1$ and for all x, then $(\nabla P_r)(x) = rP_{r-1}(x)$ for all x.

Let us call g an anti-difference of f if $\Delta g = f$. Note that if f is a polynomial such that $(\nabla f)(n) = 0$ for infinitely many n, then f is a constant. So, if f_1, f_2 are polynomials with $\nabla f_1 = \nabla f_2$, then $f_1 - f_2$ is a constant.

Let us look at our sums $S_k(n)$ now. Let us keep in mind that the polynomial $S_k(x)$ has no constant term. Writing $f_k(x) = x^k$ and $g_k(x)$ for any anti-difference of f_k which is a polynomial function, then we have $(\nabla g_k)(n) = f_k(n) = n^k = S_k(n) - S_k(n-1) = (\nabla S_k)(n)$ for all $n \geq 2$.

Hence, $S_k(x) = g_k(x) + c$ for some constant c. Since $S_k(x)$ has no constant term, we have $c = -g_k(0)$.

In other words, $S_k(n) = g_k(n) - g_k(0)$ for any anti-difference (polynomial) function g_k of f_k.

Note the similarity with the fundamental theorem of calculus.

So, our problem reduces to finding an anti-difference of the function x^k. We observed earlier that the function $P_r(x) = x(x+1)\cdots(x+r-1)$ has an anti-difference $\frac{P_{r+1}(x)}{r+1}$. Therefore, it is just a matter of writing x^k in terms of the P_r's.

For instance, $k = 1$ gives $f_1(x) = x = P_1(x)$ so that

$g_1(x)$ can be taken to be $\frac{P_2(x)}{2} = \frac{x(x+1)}{2}$ so that $S_1(n) = g_1(n) - g_1(0) = \frac{n(n+1)}{2}$.

For $k = 2$, one has $f_2(x) = x^2 = x(x+1) - x = P_2(x) - P_1(x)$ so that g_2 can be taken as $g_2(x) = \frac{P_3(x)}{3} - \frac{P_2(x)}{2} = \frac{x(x+1)(x+2)}{3} - \frac{x(x+1)}{2} = \frac{x(x+1)(2x+1)}{6}$.

This gives $S_2(n) = \frac{n(n+1)(2n+1)}{6}$ for all n.

The fact that one can indeed write x^k as an integer linear combination of $P_k, P_{k-1}, \cdots, P_1$ can be seen as follows.

Now $P_r(x) = x(x+1)\cdots(x+r-1) = x^r + a_{r-1,r}x^{r-1} + \cdots + a_{0,r}$ for some integers $a_{i,r}$. Indeed, these integers are the symmetric polynomials in $1, 2, \cdots, r-1$.

Then, we have the matrix equation $AF = P$, where A is the upper triangular integer matrix

$$\begin{pmatrix} 1 & a_{k-1,k} & a_{k-2,k} & \cdots & a_{0,k} \\ 0 & 1 & a_{k-2,k-1} & \cdots & a_{0,k-1} \\ & & & \cdots & \\ & & & \cdots & \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

F is the column vector $(x^k, x^{k-1}, \cdots, x)$ and, P is the column vector $(P_k(x), P_{k-1}(x), \cdots, P_1(x))$.

The matrix A has an inverse which is also an upper triangular integer matrix B with 1's on the diagonal.

Thus, $F = BP$ gives the required expression.

Let us remark here that the above method is general enough to work atleast for any complex polynomial function f instead of f_k. Thus, to evaluate $f(1) + \cdots + f(n)$, one writes f as a linear combination of the polynomials P_r, say,

$$f(x) = a_0 + a_1P_1(x) + \cdots + a_dP_d(x),$$

where $d = \deg f$ and a_i are complex numbers. Then, one has

The reasons for not being able to evaluate ζ at odd values (or even say whether it is irrational in general) are deep.

$f(1)+\cdots+f(n)=a_0n+a_1\frac{n(n+1)}{2}+\cdots+a_d\frac{n(n+1)\cdots(n+d)}{d+1}$.

A Method Involving Differentiation

This is an elementary and pretty useful method involving the differential operator $x\frac{d}{dx}$.

Note that $(x\frac{d}{dx})x^n = nx^n$. Therefore, applying it repetitively, one obtains $(x\frac{d}{dx})^k x^n = n^k x^n$.

Hence $1^k+2^k+\cdots+n^k=(x\frac{d}{dx})^k(1+x+x^2+\cdots+x^n)$ at $x=1$.

This can be rewritten in a more convenient form as

$$\sum_{i=1}^{n} i^k = \lim_{x\to 1}(x\frac{d}{dx})^k\frac{x^{n+1}-1}{x-1}.$$

Riemann Zeta Function

In this final section we discuss the sums of the infinite series $\sum_{n\geq 1}\frac{1}{n^k}$ for integers $k\geq 2$. This is a special value of the so-called Riemann zeta function $\zeta(s)$ defined as the sum of the series $\sum_{n\geq 1}\frac{1}{n^s}$ for any real number $s>1$ (actually, it can be defined as a complex valued function for any complex number s with Re $s>1$ by the same series).

Some of the values are $\zeta(2)=\frac{\pi^2}{6},\zeta(4)=\frac{\pi^4}{90},\zeta(6)=\frac{\pi^6}{945}$.

The reader will notice that we have not written $\zeta(k)$ for any odd value of k and that, for even k, the value seems to be a rational multiple of π^k. In fact, the value $\zeta(3)$ is known to be irrational but it is still unkown if it can be expressed in terms of 'known' constants ! We shall show now that $\zeta(2k)$ is indeed a rational multiple of π^{2k} for any natural number k. In fact, the Bernoulli numbers will surface here again! The reasons for not being able to evaluate ζ at odd values (or even say whether it is irrational in general) are deep.

Now, for any complex number z, we have $\sin z = z\prod_{n\geq 1}(1-\frac{z^2}{n^2\pi^2})$.

Its logarithmic derivative gives us

$$z\cot z = 1 + 2\sum_{n\geq 1} \frac{z^2}{z^2 - n^2\pi^2} = 1 - 2\sum_{n\geq 1}\sum_{k\geq 1} \frac{1}{n^{2k}}\frac{z^{2k}}{\pi^{2k}}.$$

On the other hand, in the definition of the Bernoulli numbers as $\frac{x}{e^x-1} = \sum_{r\geq 0} B_r \frac{x^r}{r!}$, if we put $x = 2iz$, we obtain (recalling that $B_{2r+1} = 0$ for $r \geq 1$),

$$z\cot z = 1 - \sum_{k\geq 1}(-1)^{k-1} B_{2k} \frac{2^{2k} z^{2k}}{(2k)!}.$$

Comparing the two expressions, we obtain

$$\zeta(2k) = (-1)^{k-1} B_{2k} \frac{2^{2k-1}}{(2k)!}\pi^{2k}.$$

It turns out interestingly that for natural numbers k, the value $\zeta(-k)$ is related to the partial sums $\sum_{n\leq N} n^k$.

Here is a rather surprising observation. The Riemann zeta function $\zeta(s)$ is defined by the series $\sum_{n\geq 1} n^{-s}$ for any complex number with Re $(s) > 1$. The theory of the zeta function implies that its definition can be extended (not by the same series, of course) to all values of s other than $s = 1$. Moreover, the values at s and $1 - s$ are related by what is known as a functional equation (thus there is the mysterious half line Re $(s) = 1/2$ in the middle on which the Riemann hypothesis predicts all the nontrivial zeroes of $\zeta(s)$ ought to lie). Let us now think of the naive idea that since $\zeta(k)$ for any natural number $k > 1$ is given by the series $\sum_{n\geq 1} n^{-k}$, it is possible that the value $\zeta(-k)$ is related to the partial sums $\sum_{n\leq N} n^k$. That this is indeed so is a simple, beautiful observation due to J Minac. Recall from the previous discussion that there is a unique polynomial $S_k(x)$ which coincides with the sum $1^k + \cdots + n^k$ at $x = n$ for any natural number n and that S_k has degree $k + 1$. In fact, we saw that

$$S_k(x) = \frac{B_{k+1}(x+1) - B_{k+1}(1)}{k+1}.$$

As $B'_m(x) = mB_{m-1}(x)$ for all m, we see

$$\int_0^1 S_k(x-1)dx = \int_0^1 \frac{B_{k+1}(x+1) - B_{k+1}(1)}{k+1} =$$

Abel's partial summation formula is an elementary yet very powerful formula – the readers are well aware of its continuous analogue – integration by parts.

$$(-1)^k \frac{B_{k+1}}{k+1}.$$

We claim :

$$\zeta(-k) = \int_0^1 S_k(x-1)dx = (-1)^k \frac{B_{k+1}}{k+1}.$$

Actually, one can use the functional equation for the zeta function to conclude this but we follow a more elementary method of obtaining analytic continuation of the zeta function which will also prove this claim.

The analytic continuation of the zeta function to all $s \neq 1$ and the fact that $\lim_{s\to 1}(s-1)\zeta(s) = 1$ are obtainable as follows. Now, the zeta function $\zeta(s)$ is defined for a complex variable s by the series $\sum_{n=1}^{\infty} n^{-s}$ which converges for Re $(s) > 1$. We shall use Abel's partial summation formula which is an elementary yet very powerful formula – the readers are well aware of its continuous analogue – integration by parts.

If $\{a_n\}, \{b_n\}$ are two sequences of complex numbers, and if $A_n = a_1 + \cdots + a_n$, then we have the identity

$$a_1 b_1 + \cdots + a_n b_n = A_n b_{n+1} - \sum_{k=1}^{n} A_k (b_{k+1} - b_k).$$

Thus, $\sum_n a_n b_n$ converges if both the sequence $\{A_n b_{n+1}\}$ and the series $\sum_{k=1}^{\infty} A_k (b_{k+1} - b_k)$ converge.

The proof follows simply by observing that

$$\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n} (A_k - A_{k-1}) b_k - \sum_{k=1}^{n} A_k b_k - \sum_{k=1}^{n} A_k b_{k+1} + A_n b_{n+1}.$$

In our case, by using Abel's partial summation formula, one has

$$\zeta(s) = s \int_1^{\infty} \frac{[x]}{x^{s+1}} dx = \frac{s}{s-1} - s \int_1^{\infty} \frac{\{x\}}{x^{s+1}} dx.$$

Here $[x]$ and $\{x\}$ respectively denote the integral part and the fractional part of x. Note that the integral converges for Re $(s) > 0$ and thus the last expression gives the analytic continuation of the zeta function to the region Re $(s) > 0$. We shall proceed inductively now. On writing

$$\int_1^\infty \frac{\{x\}}{x^{s+1}}dx = \sum_{n=1}^\infty \int_n^{n+1} \frac{x-n}{x^{s+1}} = \sum_{n=1}^\infty \int_0^1 \frac{udu}{(u+n)^{s+1}}$$

and integrating the last integral by parts, we obtain

$$\zeta(s) = \frac{s}{s-1} - \frac{s}{2}(\zeta(s+1)-1) - \frac{s(s+1)}{2}\int_1^\infty \frac{\{x\}^2}{x^{s+2}}dx.$$

From this, we have analytic continuation of ζ for Re $(s) > -1$ and also that $\zeta(0) = -\frac{1}{2}$. Proceeding inductively, we get

$$\zeta(s) = 1 + \frac{1}{s-1} - \sum_{q=1}^m \frac{s(s+1)\cdots(s+q-1)}{(q+1)!}(\zeta(s+q)-1)$$

$$-\frac{s(s+1)\cdots(s+m)}{(m+1)!}\sum_{n=1}^\infty \int_0^1 \frac{u^{m+1}}{(u+n)^{s+m+1}}.$$

The infinite sum on the right hand side converges for Re $(s) > -m$ and thus we have an expression for $\zeta(s)$ for such s. At this point, we evaluate it at $s = 1-m$. Rather surprisingly, this pretty but simple idea does not seem to have been thought of until very recently when it was done so by Ram Murty and M Reece. We get

$$\zeta(1-m) = 1 - \frac{1}{m} + \frac{(-1)^m}{m(m+1)} -$$

$$\sum_{q=1}^{m-1}(-1)^q\binom{m-1}{q}\frac{1}{q+1}(\zeta(1-m+q)-1).$$

The first few values at nonpositive integers are

$$\zeta(0) = -\frac{1}{2}, \zeta(-1) = -\frac{1}{12}, \zeta(-2) = 0, \zeta(-3) = \frac{1}{120}.$$

On the other hand, for $M = 1, 2, 3, \cdots$, we have

$$M^{k+1} = \sum_{r=0}^k \binom{k+1}{r+1}(-1)^r S_{k-r}(M-1).$$

Therefore, we get

$$\sum_{r=0}^{k}\binom{k+1}{r+1}(-1)^r\int_0^1 S_{k-r}(x-1)dx=\frac{(-1)^{k+1}}{k+2}.$$

As $\zeta(0)=-\frac{1}{2}$, we arrive at the formula

$$\zeta(-k)=\int_0^1 S_k(x-1)dx=(-1)^k\frac{B_{k+1}}{k+1},$$

which was claimed.

Let us finally remark that the Riemann zeta function vanishes at the negative even integers $-2,-4,-6,\cdots$ and these are its so-called trivial zeroes. The Riemann hypothesis asserts that all other zeroes lie on the line $\mathrm{Re}\,(s)=1/2$.

Suggested Reading

[1] J Minac, *Expo. Math.*, Vol.12, pp.459-462, 1994.

[2] M Ram Murty and M Reece, *Funct. et Approx.*, Vol.28, pp.141-154, 2000.

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Errata

Resonance, Vol. 8, No.5, May 2003, page 10

Equations (1), (2) and (3) should read as follows:

$\nabla\cdot\mathbf{E}=\rho/\varepsilon$	Gauss law	(1)
$\nabla\times\vec{\mathbf{E}}=-\dfrac{\partial\mathbf{B}}{\partial t}$	Faraday law	(2)
$\nabla\cdot\mathbf{B}=0$	No magnetic charges	(3)

We regret the typographical error.

Editors

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