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# Congruences involving Bernoulli polynomials

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**Abstract**

Let $\{B_n(x)\}$ be the Bernoulli polynomials. In the paper we establish some congruences for $B_j(x) \pmod{p^n}$, where $p$ is an odd prime and $x$ is a rational $p$-integer. Such congruences are concerned with the properties of $p$-regular functions, the congruences for $h(-sp) \pmod p$ $(s = 3, 5, 8, 12)$ and the sum $\sum_{k\equiv r \pmod m}\binom{p}{k}$, where $h(d)$ is the class number of the quadratic field $\mathbb{Q}(\sqrt{d})$ of discriminant $d$ and $p$-regular functions are those functions $f$ such that $f(k)$ $(k = 0, 1, \ldots)$ are rational $p$-integers and $\sum_{k=0}^{n}\binom{n}{k}(-1)^k f(k) \equiv 0 \pmod{p^n}$ for $n = 1, 2, 3, \ldots$ . We also establish many congruences for Euler numbers.


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## 1. Introduction

The Bernoulli numbers $\{B_n\}$ and Bernoulli polynomials $\{B_n(x)\}$ are defined by

$$B_0 = 1, \quad \sum_{k=0}^{n-1}\binom{n}{k}B_k = 0 \ (n \geqslant 2) \quad \text{and} \quad B_n(x) = \sum_{k=0}^{n}\binom{n}{k}B_k x^{n-k} \ (n \geqslant 0).$$

The Euler numbers $\{E_n\}$ and Euler polynomials $\{E_n(x)\}$ are defined by

$$\frac{2e^t}{e^{2t}+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!} \ \left(|t| < \frac{\pi}{2}\right) \quad \text{and} \quad \frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} \ (|t| < \pi),$$

which are equivalent to (see [12])

$$E_0 = 1, \quad E_{2n-1} = 0, \quad \sum_{r=0}^{n}\binom{2n}{2r}E_{2r} = 0 \ (n \geqslant 1)$$

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and

$$E_n(x)+\sum_{r=0}^{n}\binom{n}{r}E_r(x)=2x^n \quad (n\geqslant 0). \tag{1.1}$$

It is well known that [12]

$$E_n(x)=\frac{1}{2^n}\sum_{r=0}^{n}\binom{n}{r}(2x-1)^{n-r}E_r=\frac{2}{n+1}\left(B_{n+1}(x)-2^{n+1}B_{n+1}\left(\frac{x}{2}\right)\right). \tag{1.2}$$

Let $\mathbb{Z}$ and $\mathbb{N}$ be the set of integers and the set of positive integers, respectively. Let $[x]$ be the integral part of $x$ and $\{x\}$ be the fractional part of $x$. If $m,s\in\mathbb{N}$ and $p$ is an odd prime not dividing $m$, in Section 2 we show that

$$(-1)^s\frac{m}{p}\sum_{\substack{k=1\\ k\equiv sp\ (\mathrm{mod}\ m)}}^{p-1}\binom{p}{k}$$

$$\equiv\begin{cases} B_{p-1}\left(\left\{\frac{(s-1)p}{m}\right\}\right)-B_{p-1}\left(\left\{\frac{sp}{m}\right\}\right)\ (\mathrm{mod}\ p) & \text{if } 2\mid m,\\ \frac{1}{2}\left((-1)^{[(s-1)p/m]}E_{p-2}\left(\left\{\frac{(s-1)p}{m}\right\}\right)-(-1)^{[sp/m]}E_{p-2}\left(\left\{\frac{sp}{m}\right\}\right)\right)\ (\mathrm{mod}\ p) & \text{if } 2\nmid m.\end{cases}$$

For a discriminant $d$ let $h(d)$ be the class number of the quadratic field $\mathbb{Q}(\sqrt{d})$ ($\mathbb{Q}$ is the set of rational numbers). If $p>3$ is a prime of the form $4m+3$, it is well known that (cf. [8])

$$h(-p)\equiv -2B_{(p+1)/2}\ (\mathrm{mod}\ p). \tag{1.3}$$

If $p$ is a prime of the form $4m+1$, according to [5] we have

$$2h(-4p)\equiv E_{(p-1)/2}\ (\mathrm{mod}\ p). \tag{1.4}$$

Let $(\frac{a}{n})$ be the Kronecker symbol. For odd primes $p$, in Section 3 we establish the following congruences:

$$h(-8p)\equiv E_{(p-1)/2}(\tfrac{1}{4})\ (\mathrm{mod}\ p);$$

$$h(-3p)\equiv -4\left(\frac{p}{3}\right)B_{(p+1)/2}\left(\frac{1}{3}\right)\ (\mathrm{mod}\ p)\quad \text{for } p\equiv 1\ (\mathrm{mod}\ 4);$$

$$h(-12p)\equiv 8\left(\frac{p}{3}\right)B_{(p+1)/2}\left(\frac{1}{12}\right)\ (\mathrm{mod}\ p)\quad \text{for } p\equiv 7,11,23\ (\mathrm{mod}\ 24);$$

$$h(-5p)\equiv -8B_{(p+1)/2}\left(\frac{1}{5}\right)\ (\mathrm{mod}\ p)\quad \text{for } p\equiv 11,19\ (\mathrm{mod}\ 20).$$

For $m\in\mathbb{N}$ let $\mathbb{Z}_m$ be the set of rational numbers whose denominator is coprime to $m$. For a prime $p$, in [18] the author introduced the notion of $p$-regular functions. If $f(k)\in\mathbb{Z}_p$ for any nonnegative integers $k$ and $\sum_{k=0}^{n}\binom{n}{k}(-1)^k f(k)\equiv 0\ (\mathrm{mod}\ p^n)$ for all $n\in\mathbb{N}$, then $f$ is called a $p$-regular function. If $f$ is a $p$-regular function and $k,m,n,t\in\mathbb{N}$, in Section 4 we show that

$$f(ktp^{m-1})\equiv\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}f(rtp^{m-1})\ (\mathrm{mod}\ p^{mn}), \tag{1.5}$$

which was announced by the author in [18, (2.4)]. We also show that

$$f(kp^{m-1})\equiv(1-kp^{m-1})f(0)+kp^{m-1}f(1)\ (\mathrm{mod}\ p^{m+1})\quad \text{for } p>2. \tag{1.6}$$

Let $p$ be a prime, $x\in\mathbb{Z}_p$ and let $b$ be a nonnegative integer. Let $\langle t\rangle_p$ be the least nonnegative residue of $t$ modulo $p$ and $x'=(x+\langle -x\rangle_p)/p$. From [17, Theorem 3.1] we know that $f(k)=p(pB_{k(p-1)+b}(x)-p^{k(p-1)+b}B_{k(p-1)+b}(x'))$

is a $p$-regular function. If $p-1\nmid b$, in [18] the author showed that $f(k)=(B_{k(p-1)+b}(x)-p^{k(p-1)+b-1}B_{k(p-1)+b}(x'))/(k(p-1)+b)$ is also a $p$-regular function. Using such results in [17,18] and (1.5), in Section 5 we obtain general congruences for $pB_{k\varphi(p^s)+b}(x)$, $pB_{k\varphi(p^s)+b,\chi} \pmod{p^{sn}}$, where $k,n,s\in\mathbb{N}$, $\varphi$ is Euler's totient function and $\chi$ is a Dirichlet character modulo a positive integer. As a consequence of (1.6), if $2|b$ and $p-1\nmid b$, we have

$$\frac{B_{k\varphi(p^s)+b}}{k\varphi(p^s)+b}\equiv(1-kp^{s-1})(1-p^{b-1})\frac{B_b}{b}+kp^{s-1}\frac{B_{p-1+b}}{p-1+b} \pmod{p^{s+1}}.$$

In Section 6 we establish some congruences for $\sum_{k=0}^{n}\binom{n}{k}(-1)^k pB_{k(p-1)+b}(x)$ modulo $p^{n+1}$, where $p$ is an odd prime, $n\in\mathbb{N}$, $x\in\mathbb{Z}_p$ and $b$ is a nonnegative integer.

Let $p$ be an odd prime and $b\in\{0,2,4,\ldots\}$. In Section 7 we show that $f(k)=(1-(-1)^{(p-1)/2}p^{k(p-1)+b})E_{k(p-1)+b}$ is a $p$-regular function. Using this and (1.5) we give congruences for $E_{k\varphi(p^m)+b} \pmod{p^{mn}}$, where $k,m\in\mathbb{N}$. By (1.6) we have

$$E_{k\varphi(p^m)+b}\equiv(1-kp^{m-1})(1-(-1)^{(p-1)/2}p^b)E_b+kp^{m-1}E_{p-1+b} \pmod{p^{m+1}}.$$

We also show that $f(k)=E_{2k+b}$ is a 2-regular function and

$$E_{2^mkt+b}\equiv\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}E_{2^mrt+b} \pmod{2^{mn+n-\alpha}},$$

where $k,m,n,t\in\mathbb{N}$ and $\alpha\in\mathbb{N}$ is given by $2^{\alpha-1}\leqslant n<2^{\alpha}$.

## 2. Congruences for $B_k(\{\frac{(s-1)p}{m}\})-B_k(\{\frac{sp}{m}\}) \pmod p$

We begin with two useful identities concerning Bernoulli and Euler polynomials. In the case $m=1$ the result is well known. See [12].

**Theorem 2.1.** *Let $p,m\in\mathbb{N}$ and $k,r\in\mathbb{Z}$ with $k\geqslant 0$. Then*

$$\sum_{\substack{x=0\\ x\equiv r \pmod m}}^{p-1} x^k=\frac{m^k}{k+1}\left(B_{k+1}\left(\frac{p}{m}+\left\{\frac{r-p}{m}\right\}\right)-B_{k+1}\left(\left\{\frac{r}{m}\right\}\right)\right)$$

*and*

$$\sum_{\substack{x=0\\ x\equiv r \pmod m}}^{p-1}(-1)^{(x-r)/m}x^k=-\frac{m^k}{2}\left((-1)^{[(r-p)/m]}E_k\left(\frac{p}{m}+\left\{\frac{r-p}{m}\right\}\right)-(-1)^{[r/m]}E_k\left(\left\{\frac{r}{m}\right\}\right)\right).$$

**Proof.** For any real number $t$ and nonnegative integer $n$ it is well known that (cf. [12])

$$B_n(t+1)-B_n(t)=nt^{n-1}(n\neq 0)\quad\text{and}\quad E_n(t+1)+E_n(t)=2t^n. \tag{2.1}$$

Hence, for $x\in\mathbb{Z}$ we have

$$B_{k+1}\left(\frac{x+1}{m}+\left\{\frac{r-x-1}{m}\right\}\right)-B_{k+1}\left(\frac{x}{m}+\left\{\frac{r-x}{m}\right\}\right)$$
$$=\begin{cases}B_{k+1}\left(\frac{x+1}{m}+\left\{\frac{r-x}{m}\right\}-\frac{1}{m}\right)-B_{k+1}\left(\frac{x}{m}+\left\{\frac{r-x}{m}\right\}\right)=0 & \text{if } m\nmid x-r,\\ B_{k+1}\left(\frac{x+1}{m}+\frac{m-1}{m}\right)-B_{k+1}\left(\frac{x}{m}\right)=(k+1)\left(\frac{x}{m}\right)^k & \text{if } m|x-r.\end{cases}$$

Thus

$$\begin{aligned}&B_{k+1}\left(\frac{p}{m}+\left\{\frac{r-p}{m}\right\}\right)-B_{k+1}\left(\left\{\frac{r}{m}\right\}\right)\\&=\sum_{x=0}^{p-1}\left(B_{k+1}\left(\frac{x+1}{m}+\left\{\frac{r-x-1}{m}\right\}\right)-B_{k+1}\left(\frac{x}{m}+\left\{\frac{r-x}{m}\right\}\right)\right)\\&=\frac{k+1}{m^k}\sum_{\substack{x=0\\x\equiv r\ (\mathrm{mod}\ m)}}^{p-1}x^k.\end{aligned}$$

Similarly, if $x\in\mathbb{Z}$, by (2.1) we have

$$\begin{aligned}&(-1)^{[(r-x-1)/m]}E_k\left(\frac{x+1}{m}+\left\{\frac{r-x-1}{m}\right\}\right)-(-1)^{[(r-x)/m]}E_k\left(\frac{x}{m}+\left\{\frac{r-x}{m}\right\}\right)\\&=\begin{cases}(-1)^{[(r-x)/m]}\left(E_k\left(\frac{x+1}{m}+\left\{\frac{r-x}{m}\right\}-\frac{1}{m}\right)-E_k\left(\frac{x}{m}+\left\{\frac{r-x}{m}\right\}\right)\right)=0 & \text{if } m\nmid x-r,\\(-1)^{(r-x)/m-1}E_k\left(\frac{x+1}{m}+\frac{m-1}{m}\right)-(-1)^{(r-x)/m}E_k\left(\frac{x}{m}\right)=-(-1)^{(r-x)/m}\cdot 2\left(\frac{x}{m}\right)^k & \text{if } m\mid x-r.\end{cases}\end{aligned}$$

Thus

$$\begin{aligned}&(-1)^{[(r-p)/m]}E_k\left(\frac{p}{m}+\left\{\frac{r-p}{m}\right\}\right)-(-1)^{[r/m]}E_k\left(\left\{\frac{r}{m}\right\}\right)\\&=\sum_{x=0}^{p-1}\left\{(-1)^{[(r-x-1)/m]}E_k\left(\frac{x+1}{m}+\left\{\frac{r-x-1}{m}\right\}\right)-(-1)^{[(r-x)/m]}E_k\left(\frac{x}{m}+\left\{\frac{r-x}{m}\right\}\right)\right\}\\&=-\frac{2}{m^k}\sum_{\substack{x=0\\x\equiv r\ (\mathrm{mod}\ m)}}^{p-1}(-1)^{(x-r)/m}x^k.\end{aligned}$$

This completes the proof. □

**Corollary 2.1.** *Let $p$ be an odd prime and $k\in\{0,1,\dots,p-2\}$. Let $r\in\mathbb{Z}$ and $m\in\mathbb{N}$ with $p\nmid m$. Then*

$$\sum_{\substack{x=0\\x\equiv r\ (\mathrm{mod}\ m)}}^{p-1}x^k\equiv\frac{m^k}{k+1}\left(B_{k+1}\left(\left\{\frac{r-p}{m}\right\}\right)-B_{k+1}\left(\left\{\frac{r}{m}\right\}\right)\right)\ (\mathrm{mod}\ p)$$

*and*

$$\sum_{\substack{x=0\\x\equiv r\ (\mathrm{mod}\ m)}}^{p-1}(-1)^{(x-r)/m}x^k\equiv-\frac{m^k}{2}\left((-1)^{[(r-p)/m]}E_k\left(\left\{\frac{r-p}{m}\right\}\right)-(-1)^{[r/m]}E_k\left(\left\{\frac{r}{m}\right\}\right)\right)\ (\mathrm{mod}\ p).$$

**Proof.** If $x_1,x_2\in\mathbb{Z}_p$ and $x_1\equiv x_2\ (\mathrm{mod}\ p)$, by [18, Lemma 3.1] and [16, Lemma 3.3] we have

$$\frac{B_{k+1}(x_1)-B_{k+1}(x_2)}{k+1}\equiv\frac{x_1-x_2}{p}\cdot pB_k\equiv 0\ (\mathrm{mod}\ p)\tag{2.2}$$

and

$$E_k(x_1)\equiv E_k(x_2)\ (\mathrm{mod}\ p).\tag{2.3}$$

Thus the result follows from Theorem 2.1. □

**Remark 2.1.** Putting $k = p-2$ in Corollary 2.1 and then applying Fermat's little theorem we see that if $p$ is an odd prime not dividing $m$, then

$$\sum_{\substack{x=1\\ x\equiv r \pmod m}}^{p-1} \frac{1}{x} \equiv -\frac{1}{m}\left(B_{p-1}\left(\left\{\frac{r-p}{m}\right\}\right) - B_{p-1}\left(\left\{\frac{r}{m}\right\}\right)\right) \pmod p \tag{2.4}$$

and

$$\sum_{\substack{x=1\\ x\equiv r \pmod m}}^{p-1} (-1)^{(x-r)/m}\frac{1}{x} \equiv -\frac{1}{2m}\left((-1)^{[(r-p)/m]}E_{p-2}\left(\left\{\frac{r-p}{m}\right\}\right) - (-1)^{[r/m]}E_{p-2}\left(\left\{\frac{r}{m}\right\}\right)\right) \pmod p. \tag{2.5}$$

Here (2.4) and (2.5) are due to the author's brother Z.W. Sun. See [20, Theorem 2.1]. Inspired by his work, the author established Theorem 2.1 and Corollary 2.1 in 1991.

**Corollary 2.2.** *Let $p$ be an odd prime. Let $k \in \{0, 1, \ldots, p-2\}$ and $m, s \in \mathbb{N}$ with $p \nmid m$. Then*

$$\frac{(-1)^k}{k+1}\left(B_{k+1}\left(\left\{\frac{(s-1)p}{m}\right\}\right) - B_{k+1}\left(\left\{\frac{sp}{m}\right\}\right)\right) \equiv \sum_{(s-1)p/m<r\leqslant sp/m} r^k \pmod p$$

*and*

$$(-1)^{[(s-1)p/m]}E_k\left(\left\{\frac{(s-1)p}{m}\right\}\right) - (-1)^{[sp/m]}E_k\left(\left\{\frac{sp}{m}\right\}\right) \equiv 2(-1)^{k-1}\sum_{(s-1)p/m<r\leqslant sp/m} (-1)^r r^k \pmod p.$$

**Proof.** It is clear that (see [16, Lemma 3.1, Corollaries 3.1 and 3.3])

$$\sum_{\substack{x=0\\ x\equiv sp \pmod m}}^{p-1} x^k = \sum_{\substack{r\in\mathbb{Z}\\ 0\leqslant sp-rm<p}} (sp-rm)^k = \sum_{(s-1)p/m<r\leqslant sp/m} (sp-rm)^k \equiv (-m)^k \sum_{(s-1)p/m<r\leqslant sp/m} r^k \pmod p \tag{2.6}$$

and

$$\sum_{\substack{x=0\\ x\equiv sp \pmod m}}^{p-1} (-1)^{(x-sp)/m}x^k = \sum_{\substack{r\in\mathbb{Z}\\ 0\leqslant sp-rm<p}} (-1)^r(sp-rm)^k = \sum_{(s-1)p/m<r\leqslant sp/m} (-1)^r(sp-rm)^k \equiv (-m)^k \sum_{(s-1)p/m<r\leqslant sp/m} (-1)^r r^k \pmod p. \tag{2.7}$$

Thus applying Corollary 2.1 we obtain the result. □

**Remark 2.2.** In the case $s=1$, the first part of Corollary 2.2 is due to Lehmer [10, p. 351]. In the case $k = p-2$, the first part of Corollary 2.2 can be deduced from [7, p. 126].

**Corollary 2.3.** *Let p be a prime*.

(i) (Karpinski[9,22]) *If* $p \equiv 3 \pmod 8$, *then* $\sum_{x=1}^{(p-3)/4}(\frac{x}{p})=0$.
(ii) (Karpinski[9,22]) *If* $p \equiv 5 \pmod 8$, *then* $\sum_{x=1}^{[p/6]}(\frac{x}{p})=0$.
(iii) (Berndt[1,22]) *If* $p \equiv 5 \pmod{24}$, *then* $\sum_{x=1}^{(p-5)/12}(\frac{x}{p})=0$.

**Proof.** By Corollary 2.2 and the known fact $B_{2n+1}=0$, for $m \in \mathbb{N}$ with $p \nmid m$ we have

$$\sum_{x=1}^{[p/m]}\left(\frac{x}{p}\right) \equiv \sum_{x=1}^{[p/m]} x^{(p-1)/2} \equiv \frac{(-1)^{(p-1)/2}}{(p+1)/2}\left(B_{(p+1)/2}-B_{(p+1)/2}\left(\left\{\frac{p}{m}\right\}\right)\right)$$
$$\equiv \begin{cases} -2B_{(p+1)/2}\left(\left\{\frac{p}{m}\right\}\right) \pmod p & \text{if } p\equiv 1 \pmod 4,\\ -2B_{(p+1)/2}+2B_{(p+1)/2}\left(\left\{\frac{p}{m}\right\}\right) \pmod p & \text{if } p\equiv 3 \pmod 4.\end{cases} \tag{2.8}$$

It is well known that $B_{2n}(\frac{3}{4})=B_{2n}(\frac{1}{4})=(1-2^{2n-1})B_{2n}/2^{4n-1}$. Thus, if $p\equiv 3 \pmod 8$, by (2.8) we see that

$$\sum_{x=1}^{\frac{p-3}{4}}\left(\frac{x}{p}\right) \equiv -2B_{(p+1)/2}+2B_{(p+1)/2}\left(\frac{3}{4}\right)=\frac{1}{2^{p-1}}(1-2^{(p-1)/2})B_{(p+1)/2}-2B_{(p+1)/2}$$
$$\equiv \left(1-\left(\frac{2}{p}\right)-2\right)B_{(p+1)/2}=0 \pmod p.$$

As $-\frac{p-3}{4} \leqslant \sum_{x=1}^{(p-3)/4}(\frac{x}{p}) \leqslant \frac{p-3}{4}$, we must have $\sum_{x=1}^{(p-3)/4}(\frac{x}{p})=0$. This proves (i).

Now we consider (ii). For $n\in\{0,1,2,\ldots\}$ and $m\in\mathbb{N}$ it is well known that (cf. [8,12])

$$B_n(1-x)=(-1)^nB_n(x) \quad \text{and} \quad \sum_{k=0}^{m-1}B_n\left(x+\frac{k}{m}\right)=m^{1-n}B_n(mx). \tag{2.9}$$

Thus

$$B_{(p+1)/2}\left(\frac{1}{2n}\right)+B_{(p+1)/2}\left(\frac{1}{2n}+\frac{1}{2}\right)=2^{-(p-1)/2}B_{(p+1)/2}\left(\frac{1}{n}\right)$$

and so

$$B_{(p+1)/2}\left(\frac{1}{2n}\right)\equiv\left(\frac{2}{p}\right)B_{(p+1)/2}\left(\frac{1}{n}\right)-(-1)^{(p+1)/2}B_{(p+1)/2}\left(\frac{n-1}{2n}\right) \pmod p. \tag{2.10}$$

Since $p\equiv 5 \pmod 8$, taking $n=3$ in (2.10) we find

$$B_{(p+1)/2}(\tfrac{1}{6})\equiv -B_{(p+1)/2}(\tfrac{1}{3})+B_{(p+1)/2}(\tfrac{1}{3})=0 \pmod p. \tag{2.11}$$

This together with (2.8) and (2.9) yields

$$\sum_{x=1}^{[p/6]}\left(\frac{x}{p}\right)\equiv -2B_{(p+1)/2}\left(\left\{\frac{p}{6}\right\}\right)=-2\left(\frac{p}{3}\right)B_{(p+1)/2}\left(\frac{1}{6}\right)\equiv 0 \pmod p.$$

As $|\sum_{x=1}^{[p/6]}(\frac{x}{p})|\leqslant[\frac{p}{6}]$ we have $\sum_{x=1}^{[p/6]}(\frac{x}{p})=0$. This proves (ii).

Finally we consider (iii). Assume $p \equiv 5 \pmod{24}$. By (2.10) and (2.11) we have

$$B_{(p+1)/2}(\tfrac{1}{12}) \equiv (\tfrac{2}{p})B_{(p+1)/2}(\tfrac{1}{6}) + B_{(p+1)/2}(\tfrac{5}{12}) \equiv B_{(p+1)/2}(\tfrac{5}{12}) \pmod{p}.$$

On the other hand, by (2.9) we have

$$\begin{aligned} B_{(p+1)/2}\left(\frac{1}{12}\right) + B_{(p+1)/2}\left(\frac{5}{12}\right) &= 3^{-(p-1)/2}B_{(p+1)/2}\left(\frac{1}{4}\right) - B_{(p+1)/2}\left(\frac{9}{12}\right) \\ &\equiv \left(\frac{3}{p}\right)B_{(p+1)/2}\left(\frac{1}{4}\right) - (-1)^{(p+1)/2}B_{(p+1)/2}\left(\frac{1}{4}\right) \\ &= 0 \pmod{p}. \end{aligned}$$

Thus $B_{(p+1)/2}(\frac{1}{12}) \equiv B_{(p+1)/2}(\frac{5}{12}) \equiv 0 \pmod{p}$. Now applying (2.8) we see that

$$\sum_{x=1}^{[p/12]}\left(\frac{x}{p}\right) \equiv -2B_{(p+1)/2}\left(\left\{\frac{p}{12}\right\}\right) = -2B_{(p+1)/2}\left(\frac{5}{12}\right) \equiv 0 \pmod{p}.$$

This yields (iii) and so the corollary is proved. □

**Corollary 2.4.** *Suppose* $p, q, m \in \mathbb{N}, n \in \mathbb{Z}, \gcd(p, m) = 1$ *and* $q \leqslant m$. *For* $r \in \mathbb{Z}$ *let* $A_r(m, p)$ *be the least positive solution of the congruence* $px \equiv r \pmod{m}$. *Then*

$$|\{r : A_r(m, p) \leqslant q,\ \ r \in \mathbb{Z}, -n \leqslant r \leqslant p-1-n\}| = \left[\frac{pq+n}{m}\right] - \left[\frac{n}{m}\right].$$

**Proof.** Using Theorem 2.1 we see that

$$\begin{aligned} &|\{r : A_r(m, p) \leqslant q, r \in \mathbb{Z}, -n \leqslant r \leqslant p-1-n\}| \\ &= \sum_{x=1}^{q} \sum_{\substack{r=-n \\ r \equiv px \pmod{m}}}^{p-1-n} 1 = \sum_{x=1}^{q} \sum_{\substack{s=0 \\ s \equiv px+n \pmod{m}}}^{p-1} 1 \\ &= \sum_{x=1}^{q}\left(B_1\left(\frac{p}{m} + \left\{\frac{px+n-p}{m}\right\}\right) - B_1\left(\left\{\frac{px+n}{m}\right\}\right)\right) \\ &= \sum_{x=1}^{q}\left(\frac{p}{m} + \left\{\frac{p(x-1)+n}{m}\right\} - \left\{\frac{px+n}{m}\right\}\right) \\ &= \frac{pq}{m} + \left\{\frac{n}{m}\right\} - \left\{\frac{pq+n}{m}\right\} = \frac{pq+n}{m} - \left\{\frac{pq+n}{m}\right\} - \left(\frac{n}{m} - \left\{\frac{n}{m}\right\}\right) \\ &= \left[\frac{pq+n}{m}\right] - \left[\frac{n}{m}\right]. \end{aligned}$$

This proves the corollary. □

**Theorem 2.2.** *Let $m, s \in \mathbb{N}$ and let $p$ be an odd prime not dividing $m$. Then*

$$(-1)^s \frac{m}{p} \sum_{\substack{k=1\\ k\equiv sp \pmod m}}^{p-1} \binom{p}{k} \equiv \sum_{(s-1)p/m<k<sp/m} \frac{(-1)^{km}}{k}$$
$$\equiv \begin{cases} B_{p-1}\left(\left\{\frac{(s-1)p}{m}\right\}\right) - B_{p-1}\left(\left\{\frac{sp}{m}\right\}\right) \pmod p & \text{if } 2\mid m,\\ \frac{1}{2}\left((-1)^{[(s-1)p/m]}E_{p-2}\left(\left\{\frac{(s-1)p}{m}\right\}\right) - (-1)^{[sp/m]}E_{p-2}\left(\left\{\frac{sp}{m}\right\}\right)\right) \pmod p & \text{if } 2\nmid m.\end{cases}$$

**Proof.** Let $r \in \mathbb{Z}$. Since $\binom{p-1}{j} \equiv (-1)^j \pmod p$ for $j \in \{0, 1, \dots, p-1\}$ we see that

$$\frac{1}{p} \sum_{\substack{k=1\\ k\equiv r \pmod m}}^{p-1} \binom{p}{k} = \sum_{\substack{k=1\\ k\equiv r \pmod m}}^{p-1} \frac{1}{k}\binom{p-1}{k-1} \equiv \sum_{\substack{k=1\\ k\equiv r \pmod m}}^{p-1} \frac{(-1)^{k-1}}{k}$$
$$= \begin{cases} (-1)^{r-1} \sum_{\substack{k=1\\ k\equiv r \pmod m}}^{p-1} \frac{1}{k} \pmod p & \text{if } 2\mid m,\\ (-1)^{r-1} \sum_{\substack{k=1\\ k\equiv r \pmod m}}^{p-1} (-1)^{(k-r)/m}\frac{1}{k} \pmod p & \text{if } 2\nmid m.\end{cases}$$

Putting this together with (2.4) and (2.5) we see that

$$\frac{1}{p} \sum_{\substack{k=1\\ k\equiv r \pmod m}}^{p-1} \binom{p}{k}$$
$$\equiv \begin{cases} \frac{(-1)^r}{m}\left(B_{p-1}\left(\left\{\frac{r-p}{m}\right\}\right) - B_{p-1}\left(\left\{\frac{r}{m}\right\}\right)\right) \pmod p & \text{if } 2\mid m,\\ \frac{(-1)^r}{2m}\left((-1)^{[(r-p)/m]}E_{p-2}\left(\left\{\frac{r-p}{m}\right\}\right) - (-1)^{[r/m]}E_{p-2}\left(\left\{\frac{r}{m}\right\}\right)\right) \pmod p & \text{if } 2\nmid m.\end{cases}$$

Taking $r = sp$ we obtain

$$(-1)^s \frac{m}{p} \sum_{\substack{k=1\\ k\equiv sp \pmod m}}^{p-1} \binom{p}{k}$$
$$\equiv \begin{cases} B_{p-1}\left(\left\{\frac{(s-1)p}{m}\right\}\right) - B_{p-1}\left(\left\{\frac{sp}{m}\right\}\right) \pmod p & \text{if } 2\mid m,\\ \frac{1}{2}\left((-1)^{[(s-1)p/m]}E_{p-2}\left(\left\{\frac{(s-1)p}{m}\right\}\right) - (-1)^{[sp/m]}E_{p-2}\left(\left\{\frac{sp}{m}\right\}\right)\right) \pmod p & \text{if } 2\nmid m.\end{cases}$$

On the other hand, putting $k = p-2$ in Corollary 2.2 we see that

$$B_{p-1}\left(\left\{\frac{(s-1)p}{m}\right\}\right) - B_{p-1}\left(\left\{\frac{sp}{m}\right\}\right) \equiv \sum_{(s-1)p/m<r<sp/m} \frac{1}{r} \pmod p$$

and

$$(-1)^{[(s-1)p/m]}E_{p-2}\left(\left\{\frac{(s-1)p}{m}\right\}\right)-(-1)^{[sp/m]}E_{p-2}\left(\left\{\frac{sp}{m}\right\}\right)$$
$$\equiv 2\sum_{(s-1)p/m<r<sp/m}\frac{(-1)^r}{r} \pmod p.$$

Now combining the above we prove the theorem. □

**Corollary 2.5.** *Let $m,n\in\mathbb{N}$ and let $p$ be an odd prime not dividing $m$.*

(i) *If $2|m$, then*

$$B_{p-1}\left(\left\{\frac{np}{m}\right\}\right)-B_{p-1}\equiv\frac{m}{p}\sum_{s=1}^{n}(-1)^{s-1}\sum_{\substack{k=1\\ k\equiv sp \ (\mathrm{mod}\ m)}}^{p-1}\binom{p}{k} \pmod p.$$

(ii) *If $2\nmid m$, then*

$$(-1)^{[np/m]}E_{p-2}\left(\left\{\frac{np}{m}\right\}\right)+\frac{2^p-2}{p}\equiv\frac{2m}{p}\sum_{s=1}^{n}(-1)^{s-1}\sum_{\substack{k=1\\ k\equiv sp \ (\mathrm{mod}\ m)}}^{p-1}\binom{p}{k} \pmod p.$$

**Proof.** It is well known that $pB_{p-1}\equiv p-1 \pmod p$. Thus, by (1.2) we have $E_{p-2}(0)=2(1-2^{p-1})B_{p-1}/(p-1)\equiv -(2^p-2)/p \pmod p$. Note that $\sum_{s=1}^{n}(f(s)-f(s-1))=f(n)-f(0)$. Then the result follows from Theorem 2.2 and the above immediately. □

Combining Theorem 2.2, Corollary 2.5 with the formulae for $\sum_{k\equiv r \ (\mathrm{mod}\ m)}\binom{p}{k}$ in the cases $m=3,4,5,6,8,9,10,12$ (see [14–16,21,19]) we may deduce many useful results, which have been given in [7] and [16].

## 3. Some congruences for $h(-3p), h(-5p), h(-8p), h(-12p) \pmod p$

Let $\{S_n\}$ be defined by

$$S_0=1 \quad\text{and}\quad S_n=1-\sum_{k=0}^{n-1}\binom{n}{k}2^{2n-2k-1}S_k \quad (n\geqslant 1). \tag{3.1}$$

Then clearly $S_n\in\mathbb{Z}$. The first few $S_n$ are shown below:

$$S_1=-1,\quad S_2=-3,\quad S_3=11,\quad S_4=57,\quad S_5=-361,\quad S_6=-2763.$$

**Theorem 3.1.** *Let $p$ be an odd prime. Then*

$$h(-8p)\equiv E_{(p-1)/2}(\tfrac14)\equiv S_{(p-1)/2} \pmod p.$$

**Proof.** From [22, p. 58] we know that

$$h(-8p)=2\sum_{\substack{a=1\\ a\equiv 1 \ (\mathrm{mod}\ 4)}}^{p-1}\left(\frac{8p}{a}\right). \tag{3.2}$$

Thus applying Corollary 2.1 in the case $r=1$, $m=4$ and $k=\frac{p-1}{2}$ we see that

$$h(-8p)=2\sum_{\substack{a=0\\ a\equiv 1\ (\mathrm{mod}\ 4)}}^{p-1}\left(\frac{2}{a}\right)\left(\frac{a}{p}\right)\equiv 2\sum_{\substack{a=0\\ a\equiv 1\ (\mathrm{mod}\ 4)}}^{p-1}(-1)^{(a-1)/4}a^{(p-1)/2}$$

$$\equiv -4^{(p-1)/2}\left((-1)^{[(1-p)/4]}E_{(p-1)/2}\left(\left\{\frac{1-p}{4}\right\}\right)-E_{(p-1)/2}\left(\frac{1}{4}\right)\right)\ (\mathrm{mod}\ p).$$

Since $E_{2n}(0)=\frac{2}{2n+1}(B_{2n+1}-2^{2n+1}B_{2n+1})=0$ by (1.2), we see that

$$E_{(p-1)/2}\left(\left\{\frac{1-p}{4}\right\}\right)=\begin{cases}E_{2n}(0)=0 & \text{if } p=4n+1,\\ E_{2n-1}(\frac{1}{2})=2^{1-2n}E_{2n-1}=0 & \text{if } p=4n-1.\end{cases}$$

Thus

$$h(-8p)\equiv 4^{(p-1)/2}E_{(p-1)/2}(\tfrac{1}{4})\equiv E_{(p-1)/2}(\tfrac{1}{4})\ (\mathrm{mod}\ p).$$

Let $S_n'=4^nE_n(\frac{1}{4})$. Now we show that $S_n=S_n'$ for $n\geqslant 0$. By (1.1) we have

$$4^{-n}S_n'+\sum_{k=0}^{n}\binom{n}{k}4^{-k}S_k'=2\cdot 4^{-n}\quad\text{and so}\quad S_n'+\sum_{k=0}^{n}\binom{n}{k}4^{n-k}S_k'=2.$$

That is, $S_n'=1-\sum_{k=0}^{n-1}\binom{n}{k}2^{2n-2k-1}S_k'$. Since $S_0'=S_0=1$ we see that $S_n'=S_n$. That is,

$$S_n=4^nE_n(\tfrac{1}{4}). \tag{3.3}$$

Hence $S_{(p-1)/2}=4^{(p-1)/2}E_{(p-1)/2}(\frac{1}{4})\equiv h(-8p)\ (\mathrm{mod}\ p)$. This proves the theorem. □

**Corollary 3.1.** *Let p be an odd prime. Then* $p\nmid S_{(p-1)/2}$.

**Proof.** From (3.2) we have $1<h(-8p)<p$. Thus the result follows from Theorem 3.1. □

**Remark 3.1.** Since $S_n=4^nE_n(\frac{1}{4})$, by (1.2) and the binomial inversion formula we have

$$S_n=\sum_{r=0}^{n}\binom{n}{r}(-1)^{n-r}2^rE_r\quad\text{and}\quad\sum_{r=0}^{n}\binom{n}{r}S_r=2^nE_n. \tag{3.4}$$

**Theorem 3.2.** *Let p be a prime greater than* 3.

(i) *If* $p\equiv 1\ (\mathrm{mod}\ 4)$, *then*

$$h(-3p)\equiv\begin{cases}-4B_{(p+1)/2}(\frac{1}{3})\ (\mathrm{mod}\ p) & \text{if } p\equiv 1\ (\mathrm{mod}\ 12),\\ 4B_{(p+1)/2}(\frac{1}{3})\ (\mathrm{mod}\ p) & \text{if } p\equiv 5\ (\mathrm{mod}\ 12).\end{cases}$$

(ii) *If* $p\equiv 3\ (\mathrm{mod}\ 4)$, *then*

$$h(-12p)\equiv\begin{cases}8B_{(p+1)/2}(\frac{1}{12})\ (\mathrm{mod}\ p) & \text{if } p\equiv 7\ (\mathrm{mod}\ 24),\\ -8B_{(p+1)/2}(\frac{1}{12})\ (\mathrm{mod}\ p) & \text{if } p\equiv 11\ (\mathrm{mod}\ 12),\\ 8B_{(p+1)/2}(\frac{1}{12})+8B_{(p+1)/2}\ (\mathrm{mod}\ p) & \text{if } p\equiv 19\ (\mathrm{mod}\ 24)\end{cases}$$

*and*

$$h(-5p)\equiv\begin{cases}-8B_{(p+1)/2}(\frac{1}{5})\ (\mathrm{mod}\ p) & \text{if } p\equiv 11, 19\ (\mathrm{mod}\ 20)\\ 8B_{(p+1)/2}(\frac{1}{5})+4B_{(p+1)/2}\ (\mathrm{mod}\ p) & \text{if } p\equiv 3, 7\ (\mathrm{mod}\ 20).\end{cases}$$

**Proof.** We first assume $p \equiv 1 \pmod 4$. From [22, p. 40] or [1] we have

$$h(-3p) = 2\sum_{x=1}^{[p/3]} \left(\frac{p}{x}\right).$$

Thus applying (2.8), (2.9), and the quadratic reciprocity law we see that

$$h(-3p) = 2\sum_{x=1}^{[p/3]} \left(\frac{x}{p}\right) \equiv -4B_{(p+1)/2}\left(\left\{\frac{p}{3}\right\}\right) = -4\left(\frac{p}{3}\right)B_{(p+1)/2}\left(\frac{1}{3}\right) \pmod p.$$

This proves (i).

Now let us consider (ii). Assume $p \equiv 3 \pmod 4$. From [22, pp. 3–5] we have

$$h(-12p) = \begin{cases} 4\sum\limits_{p/12<x<2p/12} \left(\frac{x}{p}\right) & \text{if } p \equiv 7, 11, 23 \pmod{24}, \\ 4\sum\limits_{4p/12<x<5p/12} \left(\frac{x}{p}\right) & \text{if } p \equiv 19 \pmod{24}. \end{cases}$$

By Corollary 2.2 and the fact $B_{2n}(x) = B_{2n}(1-x)$ we find

$$\sum_{p/12<x<2p/12} \left(\frac{x}{p}\right) \equiv \sum_{p/12<x\leqslant 2p/12} x^{(p-1)/2} \equiv -2\left(B_{(p+1)/2}\left(\left\{\frac{p}{12}\right\}\right) - B_{(p+1)/2}\left(\frac{1}{6}\right)\right) \pmod p$$

and

$$\sum_{4p/12<x<5p/12} \left(\frac{x}{p}\right) \equiv \sum_{4p/12<x\leqslant 5p/12} x^{(p-1)/2} \equiv -2\left(B_{(p+1)/2}\left(\frac{1}{3}\right) - B_{(p+1)/2}\left(\left\{\frac{5p}{12}\right\}\right)\right) \pmod p.$$

Thus

$$h(-12p) \equiv \begin{cases} -8\left(B_{(p+1)/2}\left(\frac{5}{12}\right) - B_{(p+1)/2}\left(\frac{1}{6}\right)\right) \pmod p & \text{if } p \equiv 7 \pmod{24}, \\ -8\left(B_{(p+1)/2}\left(\frac{1}{12}\right) - B_{(p+1)/2}\left(\frac{1}{6}\right)\right) \pmod p & \text{if } p \equiv 11 \pmod{12}, \\ -8\left(B_{(p+1)/2}\left(\frac{1}{3}\right) - B_{(p+1)/2}\left(\frac{1}{12}\right)\right) \pmod p & \text{if } p \equiv 19 \pmod{24}. \end{cases}$$

By (2.10) we have

$$B_{(p+1)/2}\left(\frac{1}{12}\right) \equiv \left(\frac{2}{p}\right)B_{(p+1)/2}\left(\frac{1}{6}\right) - B_{(p+1)/2}\left(\frac{5}{12}\right) \pmod p.$$

Thus, if $p \equiv 7 \pmod{24}$, then $h(-12p) \equiv 8(B_{(p+1)/2}(\frac{1}{6}) - B_{(p+1)/2}(\frac{5}{12})) \equiv 8B_{(p+1)/2}(\frac{1}{12}) \pmod p$. It is well known that [7]

$$B_{2n}\left(\frac{1}{3}\right) = \frac{3^{1-2n}-1}{2}B_{2n} \quad \text{and} \quad B_{2n}\left(\frac{1}{6}\right) = \frac{(2^{1-2n}-1)(3^{1-2n}-1)}{2}B_{2n}.$$

Thus

$$B_{(p+1)/2}\left(\frac{1}{3}\right) = \frac{1}{2}(3^{-(p-1)/2}-1)B_{(p+1)/2} \equiv \frac{1}{2}\left(\left(\frac{3}{p}\right)-1\right)B_{(p+1)/2} \pmod p$$

and

$$B_{(p+1)/2}\left(\frac{1}{6}\right)=\frac{(2^{-(p-1)/2}-1)(3^{-(p-1)/2}-1)}{2}B_{(p+1)/2}\equiv\frac{1}{2}\left(\left(\frac{2}{p}\right)-1\right)\left(\left(\frac{3}{p}\right)-1\right)B_{(p+1)/2}\ (\mathrm{mod}\ p).$$

If $p\equiv 11\ (\mathrm{mod}\ 12)$, then $(\frac{3}{p})=1$ and so $B_{(p+1)/2}(\frac{1}{6})\equiv 0\ (\mathrm{mod}\ p)$. Hence $h(-12p)\equiv -8B_{(p+1)/2}(\frac{1}{12})\ (\mathrm{mod}\ p)$. If $p\equiv 19\ (\mathrm{mod}\ 24)$, then $(\frac{3}{p})=-1$ and so $B_{(p+1)/2}(\frac{1}{3})\equiv -B_{(p+1)/2}\ (\mathrm{mod}\ p)$. Thus $h(-12p)\equiv 8(B_{(p+1)/2}(\frac{1}{12})+B_{(p+1)/2})\ (\mathrm{mod}\ p)$.

Finally we consider $h(-5p)\ (\mathrm{mod}\ p)$. From [22, p. 40] or [1] we have

$$h(-5p)=2\sum_{p/5<a<2p/5}\left(\frac{-p}{a}\right).$$

Observe that $(\frac{-p}{a})=(\frac{a}{p})$ by the quadratic reciprocity law. Thus applying Corollary 2.2 and (2.9) we obtain

$$\begin{aligned}h(-5p)&=2\sum_{p/5<a<2p/5}\left(\frac{a}{p}\right)\equiv 2\sum_{p/5<a<2p/5}a^{(p-1)/2}\\&\equiv 2\cdot\frac{(-1)^{(p-1)/2}}{(p+1)/2}\left(B_{(p+1)/2}\left(\left\{\frac{p}{5}\right\}\right)-B_{(p+1)/2}\left(\left\{\frac{2p}{5}\right\}\right)\right)\\&\equiv -4\left(\frac{p}{5}\right)\left(B_{(p+1)/2}\left(\frac{1}{5}\right)-B_{(p+1)/2}\left(\frac{2}{5}\right)\right)\ (\mathrm{mod}\ p).\end{aligned}$$

From (2.9) we see that

$$B_{(p+1)/2}+2B_{(p+1)/2}\left(\frac{1}{5}\right)+2B_{(p+1)/2}\left(\frac{2}{5}\right)=\sum_{k=0}^{4}B_{(p+1)/2}\left(\frac{k}{5}\right)=5^{-(p-1)/2}B_{(p+1)/2}$$

and so

$$B_{(p+1)/2}\left(\frac{1}{5}\right)+B_{(p+1)/2}\left(\frac{2}{5}\right)\equiv\frac{1}{2}\left(\left(\frac{p}{5}\right)-1\right)B_{(p+1)/2}\ (\mathrm{mod}\ p).$$

Thus

$$\begin{aligned}h(-5p)&\equiv -4\left(\frac{p}{5}\right)\left(2B_{(p+1)/2}\left(\frac{1}{5}\right)+\frac{1}{2}\left(1-\left(\frac{p}{5}\right)\right)B_{(p+1)/2}\right)\\&=\begin{cases}-8B_{(p+1)/2}\left(\frac{1}{5}\right)\ (\mathrm{mod}\ p) & \text{if } p\equiv 11, 19\ (\mathrm{mod}\ 20),\\ 8B_{(p+1)/2}\left(\frac{1}{5}\right)+4B_{(p+1)/2}\ (\mathrm{mod}\ p) & \text{if } p\equiv 3, 7\ (\mathrm{mod}\ 20).\end{cases}\end{aligned}$$

The proof is now complete. □

When $d$ is a negative discriminant, it is known that $1\leqslant h(d)<p$. Thus, from Theorem 3.2 we deduce the following result.

**Corollary 3.2.** *Let p be a prime.*

(i) *If* $p\equiv 1\ (\mathrm{mod}\ 4)$, *then* $B_{(p+1)/2}(\frac{1}{3})\not\equiv 0\ (\mathrm{mod}\ p)$.
(ii) *If* $p\equiv 7, 11, 23\ (\mathrm{mod}\ 24)$, *then* $B_{(p+1)/2}(\frac{1}{12})\not\equiv 0\ (\mathrm{mod}\ p)$.
(iii) *If* $p\equiv 11, 19\ (\mathrm{mod}\ 20)$, *then* $B_{(p+1)/2}(\frac{1}{5})\not\equiv 0\ (\mathrm{mod}\ p)$.

**Remark 3.2.** For $n = 0, 1, \ldots$ it is well known that $\sum_{k=0}^{n}\binom{n}{k}\frac{1}{n-k+1}B_k(x) = x^n$. From this we deduce that if $m \in \mathbb{N}$ and $a_n = m^n B_n(\frac{1}{m})$, then $\sum_{k=0}^{n}\binom{n+1}{k}m^{n-k}a_k = n+1$.

## 4. $p$-Regular functions

For a prime $p$, in [18] the author introduced the notion of $p$-regular functions. If $f(k)$ is a complex number congruent to an algebraic integer modulo $p$ for any given nonnegative interger $k$ and $\sum_{k=0}^{n}\binom{n}{k}(-1)^k f(k) \equiv 0 \pmod{p^n}$ for all $n \in \mathbb{N}$, then $f$ is called a $p$-regular function. If $f$ and $g$ are $p$-regular functions, in [18] the author showed that $f \cdot g$ is also a $p$-regular function. Thus we see that $p$-regular functions form a ring. In the section we discuss further properties of $p$-regular functions.

Suppose $n \in \mathbb{N}$ and $k \in \{0, 1, \ldots, n\}$. Let $s(n, k)$ be the unsigned Stirling number of the first kind and $S(n, k)$ be the Stirling number of the second kind defined by

$$x(x-1)\cdots(x-n+1) = \sum_{k=0}^{n}(-1)^{n-k}s(n,k)x^k$$

and

$$x^n = \sum_{k=0}^{n} S(n,k)x(x-1)\cdots(x-k+1).$$

For our convenience we also define $s(n, k) = S(n, k) = 0$ for $k > n$. For $m \in \mathbb{N}$ it is well known that

$$\sum_{r=0}^{n}\binom{n}{r}(-1)^{n-r}r^m = n!S(m,n). \tag{4.1}$$

In particular, taking $m = n$ we have the following Euler's identity:

$$\sum_{r=0}^{n}\binom{n}{r}(-1)^{n-r}r^n = n!. \tag{4.2}$$

**Lemma 4.1.** *Let $x, d$ be variables, $m, n \in \mathbb{N}$ and $i \in \mathbb{Z}$ with $i \geqslant 0$. Then*

$$\sum_{r=0}^{n}\binom{n}{r}(-1)^{n-r}\binom{rx+d}{m}r^i = \frac{n!}{m!}\sum_{j=n-i}^{m}\left(\sum_{k=j}^{m}\binom{k}{j}(-1)^{m-k}s(m,k)d^{k-j}\right)S(i+j,n)x^j.$$

*In particular we have*

$$\sum_{r=0}^{n}\binom{n}{r}(-1)^{n-r}\binom{rx}{m}r^i = \frac{n!}{m!}\sum_{j=n-i}^{m}(-1)^{m-j}s(m,j)S(i+j,n)x^j.$$

**Proof.** Since

$$\begin{aligned} m!\binom{rx+d}{m} &= (rx+d)(rx+d-1)\cdots(rx+d-m+1) \\ &= \sum_{k=0}^{m}(-1)^{m-k}s(m,k)(rx+d)^k \\ &= \sum_{k=0}^{m}(-1)^{m-k}s(m,k)\sum_{j=0}^{k}\binom{k}{j}(rx)^j d^{k-j} \\ &= \sum_{j=0}^{m}\left(\sum_{k=j}^{m}\binom{k}{j}(-1)^{m-k}s(m,k)d^{k-j}\right)r^j x^j, \end{aligned}$$

we have

$$\sum_{r=0}^{n}\binom{n}{r}(-1)^{n-r}\binom{rx+d}{m}r^i=\frac{1}{m!}\sum_{j=0}^{m}\left(\sum_{k=j}^{m}\binom{k}{j}(-1)^{m-k}s(m,k)d^{k-j}\right)x^j\cdot\sum_{r=0}^{n}\binom{n}{r}(-1)^{n-r}r^{i+j}.$$

Now applying (4.1) we obtain the result. □

**Lemma 4.2.** *Let p be a prime and* $m,n\in\mathbb{N}$. *Then*

$$\frac{m!s(n,m)}{n!}p^{n-m}\in\mathbb{Z}_p\quad\text{and}\quad\frac{m!S(n,m)}{n!}p^{n-m}\in\mathbb{Z}_p.$$

*Moreover, if* $m<n$, *we have*

$$\frac{m!s(n,m)}{n!}p^{n-m}\equiv\frac{m!S(n,m)}{n!}p^{n-m}\equiv 0\pmod p\quad\text{for } p>2$$

*and*

$$\frac{m!s(n,m)}{n!}2^{n-m}\equiv\binom{m}{n-m}\pmod 2.$$

**Proof.** It is well known that

$$\frac{(e^x-1)^m}{m!}=\sum_{n=m}^{\infty}S(n,m)\frac{x^n}{n!}.$$

Thus, applying the multinomial theorem we see that

$$(e^x-1)^m=\left(\sum_{k=1}^{\infty}\frac{x^k}{k!}\right)^m=\sum_{n=m}^{\infty}\left(\sum_{\substack{k_1+k_2+\cdots+k_n=m\\ k_1+2k_2+\cdots+nk_n=n}}\frac{m!}{k_1!k_2!\cdots k_n!}\prod_{r=1}^{n}\frac{1}{r!^{k_r}}\right)x^n$$

and so

$$S(n,m)=\sum_{\substack{k_1+k_2+\cdots+k_n=m\\ k_1+2k_2+\cdots+nk_n=n}}\frac{n!}{1!^{k_1}k_1!2!^{k_2}k_2!\cdots n!^{k_n}k_n!}. \tag{4.3}$$

Hence

$$\frac{m!S(n,m)}{n!}p^{n-m}=\sum_{\substack{k_1+k_2+\cdots+k_n=m\\ k_1+2k_2+\cdots+nk_n=n}}\frac{(k_1+k_2+\cdots+k_n)!}{k_1!k_2!\cdots k_n!}\prod_{r=1}^{n}\left(\frac{p^{r-1}}{r!}\right)^{k_r}.$$

From [18, pp. 196,197] we also have

$$s(n,m)=\sum_{\substack{k_1+k_2+\cdots+k_n=m\\ k_1+2k_2+\cdots+nk_n=n}}\frac{n!}{1^{k_1}k_1!2^{k_2}k_2!\cdots n^{k_n}k_n!} \tag{4.4}$$

and

$$\frac{m!s(n,m)}{n!}p^{n-m}=\sum_{\substack{k_1+k_2+\cdots+k_n=m\\ k_1+2k_2+\cdots+nk_n=n}}\frac{(k_1+k_2+\cdots+k_n)!}{k_1!k_2!\cdots k_n!}\prod_{r=1}^{n}\left(\frac{p^{r-1}}{r}\right)^{k_r}.$$

It is known that $(k_1+\cdots+k_n)!/(k_1!\cdots k_n!)\in\mathbb{Z}$. For $r\in\mathbb{N}$ we know that if $p^\alpha\|r!$ (that is $p^\alpha|r!$ but $p^{\alpha+1}\nmid r!$), then $\alpha=\sum_{i=1}^{\infty}[\frac{r}{p^i}]\leqslant[\frac{r}{p}]$. Thus $p^{r-1}/r,\ p^{r-1}/r!\in\mathbb{Z}_p$. For $p>2$ we see that $p^{r-1}/r\equiv p^{r-1}/r!\equiv 0\ (\mathrm{mod}\ p)$ for $r>1$. Hence the result follows from the above. For $p=2$ we see that $2^{r-1}/r\equiv 0\ (\mathrm{mod}\ 2)$ for $r>2$. Thus

$$\frac{m!s(n,m)}{n!}2^{n-m}\equiv\sum_{\substack{k_1+k_2=m\\k_1+2k_2=n}}\frac{(k_1+k_2)!}{k_1!k_2!}=\binom{m}{n-m}\ (\mathrm{mod}\ 2).$$

Summarizing the above we prove the lemma. □

From Lemma 4.1 we have the following identities, which are generalizations of Euler's identity.

**Theorem 4.1.** *Let $x,d$ be variables and $m,n\in\mathbb{N}$.*

(i) *If $m\leqslant n$, then*

$$\sum_{r=0}^{n}\binom{n}{r}(-1)^{n-r}\binom{rx+d}{m}r^{n-m}=\frac{n!}{m!}x^m.$$

*In particular, when $m=n$ we have*

$$\sum_{r=0}^{n}\binom{n}{r}(-1)^{n-r}\binom{rx+d}{n}=x^n.$$

(ii) *If $m\leqslant n+1$, then*

$$\sum_{r=0}^{n}\binom{n}{r}(-1)^{n-r}\binom{rx+d}{m}r^{n+1-m}=\frac{n!}{m!}\left(\frac{n(n+1)}{2}x^m-\frac{m(m-1-2d)}{2}x^{m-1}\right).$$

*In particular, when $m=n+1$ we have*

$$\sum_{r=0}^{n}\binom{n}{r}(-1)^{n-r}\binom{rx+d}{n+1}=\left(d+\frac{n(x-1)}{2}\right)x^n.$$

**Proof.** Observe that $s(m,m)=1$ and $S(n,n)=1$. Putting $i=n-m$ in Lemma 4.1 we obtain (i). By (4.3) and (4.4) we have

$$s(n,n-1)=S(n,n-1)=n(n-1)/2\quad\text{for } n=2,3,4,\ldots.$$

Thus applying Lemma 4.1 we see that if $m\leqslant n+1$, then

$$\begin{aligned}&\sum_{r=0}^{n}\binom{n}{r}(-1)^{n-r}\binom{rx+d}{m}r^{n+1-m}\\&\quad=\frac{n!}{m!}\sum_{j=m-1}^{m}\left(\sum_{k=j}^{m}\binom{k}{j}(-1)^{m-k}s(m,k)d^{k-j}\right)S(n+1-m+j,n)x^j\\&\quad=\frac{n!}{m!}\left(S(n+1,n)x^m+\sum_{k=m-1}^{m}\binom{k}{m-1}(-1)^{m-k}s(m,k)d^{k-(m-1)}x^{m-1}\right)\\&\quad=\frac{n!}{m!}\left(\frac{n(n+1)}{2}x^m+\left(dm-\frac{m(m-1)}{2}\right)x^{m-1}\right).\end{aligned}$$

This yields (ii) and so the theorem is proved. □

**Corollary 4.1.** *Let $p$ be an odd prime, $m \in \mathbb{Z}$ and $d \in \{0, 1, \ldots, p-1\}$. Then $m^p \equiv m \pmod p$ and*

$$\frac{m^p - m}{p} \equiv \sum_{k=1}^{p-1} \frac{1}{k}\left[\frac{km+d}{p}\right] + m\sum_{k=1}^{d}\frac{1}{k} \pmod p.$$

**Proof.** From Theorem 4.1(i) we have

$$\begin{aligned} m^p &= \sum_{k=0}^{p}\binom{p}{k}(-1)^{p-k}\binom{km+d}{p} \\ &= \binom{mp+d}{p} + \sum_{k=1}^{p-1}\binom{p}{k}(-1)^{p-k}\binom{km+d}{p}. \end{aligned}$$

As $\sum_{k=1}^{p-1}\frac{1}{k} \equiv 0 \pmod p$, we see that

$$\begin{aligned} \binom{mp+d}{p} &= \frac{(mp+d)(mp+d-1)\cdots(mp+d-p+1)}{p!} \\ &= \frac{mp}{p}\cdot\frac{(mp+1)\cdots(mp+d)((m-1)p+d+1)\cdots((m-1)p+p-1)}{(p-1)!} \\ &\equiv m\left(1 + mp\sum_{k=1}^{d}\frac{1}{k} + (m-1)p\sum_{k=d+1}^{p-1}\frac{1}{k}\right) \\ &\equiv m\left(1 + mp\sum_{k=1}^{d}\frac{1}{k} - (m-1)p\sum_{k=1}^{d}\frac{1}{k}\right) \\ &= m\left(1 + p\sum_{k=1}^{d}\frac{1}{k}\right) \pmod{p^2}. \end{aligned}$$

Let $r_k$ be the least nonnegative residue of $km+d$ modulo $p$. For $k \in \{1, 2, \ldots, p-1\}$ we see that

$$\binom{p}{k} = \frac{p(p-1)\cdots(p-k+1)}{k!} \equiv \frac{(-1)^{k-1}}{k}p \pmod{p^2}.$$

Thus,

$$\begin{aligned} &\sum_{k=1}^{p-1}\binom{p}{k}(-1)^{p-k}\binom{km+d}{p} \\ &\quad\equiv \sum_{k=1}^{p-1}\frac{p}{k}\cdot\frac{(km+d)(km+d-1)\cdots(km+d-p+1)}{p!} \\ &\quad= p\sum_{k=1}^{p-1}\frac{1}{k}\cdot\frac{km+d-r_k}{p}\cdot\frac{1}{(p-1)!}\prod_{\substack{i=0\\ i\neq r_k}}^{p-1}(km+d-i) \\ &\quad\equiv p\sum_{k=1}^{p-1}\frac{1}{k}\cdot\frac{km+d-r_k}{p} = p\sum_{k=1}^{p-1}\frac{1}{k}\left[\frac{km+d}{p}\right] \pmod{p^2}. \end{aligned}$$

Now putting all the above together we obtain the result. □

**Remark 4.1.** In the case $d = 0$, Corollary 4.1 was first found by Lerch [11]. For a different proof of Lerch's result, see [18].

**Theorem 4.2.** *Let $p$ be a prime. Let $f$ be a $p$-regular function. Suppose $m, n \in \mathbb{N}$ and $d, t \in \mathbb{Z}$ with $d, t \geqslant 0$. Then*

$$\sum_{r=0}^{n} \binom{n}{r} (-1)^r f(p^{m-1}rt + d) \equiv 0 \pmod{p^{mn}}.$$

*Moreover, if $A_k = p^{-k}\sum_{r=0}^{k}\binom{k}{r}(-1)^r f(r)$, then*

$$\sum_{r=0}^{n} \binom{n}{r} (-1)^r f(p^{m-1}rt + d)$$
$$\equiv \begin{cases} p^{mn}t^n A_n \pmod{p^{mn+1}} & \text{if } p > 2 \text{ or } m = 1, \\ 2^{mn}t^n \sum_{r=0}^{n} \binom{n}{r} A_{r+n} \pmod{2^{mn+1}} & \text{if } p = 2 \text{ and } m \geqslant 2. \end{cases}$$

**Proof.** Since $f$ is a $p$-regular function, we have $A_k \in \mathbb{Z}_p$ for $k \geqslant 0$. Set

$$a_0 = A_0 \quad \text{and} \quad a_i = (-1)^i \sum_{r=i}^{n} s(r, i)\frac{p^r}{r!}A_r \;\; \text{for } i = 1, 2, \ldots, n.$$

As $p^r/r! \in \mathbb{Z}_p$ and $A_r \in \mathbb{Z}_p$ we have $a_0, \ldots, a_n \in \mathbb{Z}_p$. From [18, p. 197] we have

$$f(k) \equiv \sum_{i=0}^{n} a_i k^i \pmod{p^{n+1}} \quad \text{for } k = 0, 1, 2, \ldots .$$

Thus applying (4.1) and (4.2) we see that

$$\begin{aligned} \sum_{r=0}^{n} \binom{n}{r} (-1)^r f(rt + d) &\equiv \sum_{r=0}^{n} \binom{n}{r} (-1)^r \sum_{i=0}^{n} a_i (rt + d)^i \\ &= \sum_{r=0}^{n} \binom{n}{r} (-1)^r (a_n t^n r^n + b_{n-1}r^{n-1} + \cdots + b_1 r + b_0) \\ &= a_n(-t)^n n! = (-1)^n s(n, n)\frac{p^n}{n!}A_n \cdot (-t)^n n! \\ &= p^n t^n A_n \pmod{p^{n+1}}, \end{aligned}$$

where $b_0, b_1, \ldots, b_{n-1} \in \mathbb{Z}_p$. Thus the result is true for $m = 1$.

Now assume $m\geqslant 2$. By the binomial inversion formula we have $f(k)=\sum_{s=0}^{k}\binom{k}{s}(-p)^s A_s$. Thus

$$\begin{aligned}
&\sum_{r=0}^{n}\binom{n}{r}(-1)^r f(p^{m-1}rt)\\
&=\sum_{r=0}^{n}\binom{n}{r}(-1)^r\sum_{k=0}^{p^{m-1}rt}\binom{p^{m-1}rt}{k}(-p)^k A_k\\
&=\sum_{k=0}^{p^{m-1}nt}(-p)^k A_k\sum_{r=0}^{n}\binom{n}{r}(-1)^r\binom{p^{m-1}rt}{k}\\
&=\sum_{k=n}^{p^{m-1}nt}(-p)^k A_k\cdot(-1)^n\frac{n!}{k!}\sum_{j=n}^{k}(-1)^{k-j}s(k,j)S(j,n)(p^{m-1}t)^j \quad \text{(by Lemma 4.1)}\\
&=\sum_{k=n}^{p^{m-1}nt}(-p)^n(-1)^k A_k\sum_{j=n}^{k}(-1)^{k-j}\frac{s(k,j)j!}{k!}p^{k-j}\cdot\frac{S(j,n)n!}{j!}p^{j-n}\cdot(p^{m-1}t)^j\\
&=A_n t^n p^{mn}+\sum_{k=n+1}^{p^{m-1}nt}(-p)^n(-1)^k A_k\left(\frac{(-1)^{k-n}s(k,n)n!}{k!}p^{k-n}\cdot p^{(m-1)n}t^n\right.\\
&\quad\left.+\sum_{j=n+1}^{k}\frac{(-1)^{k-j}s(k,j)j!}{k!}p^{k-j}\cdot\frac{S(j,n)n!}{j!}p^{j-n}\cdot(p^{m-1}t)^j\right).
\end{aligned}$$

By Lemma 4.2, for $j,k,n\in\mathbb{N}$ we have

$$\frac{s(k,j)j!}{k!}p^{k-j}\in\mathbb{Z}_p \quad\text{and}\quad \frac{S(j,n)n!}{j!}p^{j-n}\in\mathbb{Z}_p.$$

Hence, by the above, Lemma 4.2 and the fact $(m-1)(n+1)+n\geqslant mn+1$ we obtain

$$\begin{aligned}
&\sum_{r=0}^{n}\binom{n}{r}(-1)^r f(p^{m-1}rt)\\
&\equiv p^{mn}t^n\left(A_n+\sum_{k=n+1}^{p^{m-1}nt}\frac{s(k,n)n!}{k!}p^{k-n}A_k\right)\\
&\equiv\begin{cases} p^{mn}t^n A_n \pmod{p^{mn+1}} & \text{if } p>2,\\ 2^{mn}t^n\sum_{k=n}^{2^{m-1}nt}\binom{n}{k-n}A_k=2^{mn}t^n\sum_{r=0}^{n}\binom{n}{r}A_{r+n} \pmod{2^{mn+1}} & \text{if } p=2.\end{cases}
\end{aligned}$$

Thus the result holds for $d=0$.

Now suppose $g(r)=f(r+d)$. By the previous argument,

$$\sum_{r=0}^{n}\binom{n}{r}(-1)^r g(r)\equiv p^n A_n \pmod{p^{n+1}}.$$

Thus $g$ is also a $p$-regular function. Note that

$$\sum_{r=0}^{n}\binom{n}{r}(-1)^r f(p^{m-1}rt+d)=\sum_{r=0}^{n}\binom{n}{r}(-1)^r g(p^{m-1}rt).$$

By the above we see that the result is also true for $d>0$. The proof is now complete. □

**Theorem 4.3.** *Let $p$ be a prime, $k,m,n,t\in\mathbb{N}$ and $d\in\{0,1,2,\ldots\}$. Let $f$ be a $p$-regular function. Then*

$$f(ktp^{m-1}+d)\equiv\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}f(rtp^{m-1}+d)\ (\mathrm{mod}\ p^{mn}).$$

*Moreover, setting $A_s=p^{-s}\sum_{r=0}^{s}\binom{s}{r}(-1)^r f(r)$ we then have*

$$f(ktp^{m-1}+d)-\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}f(rtp^{m-1}+d)$$
$$\equiv\begin{cases}p^{mn}\binom{k}{n}(-t)^nA_n\ (\mathrm{mod}\ p^{mn+1}) & \text{if } p>2 \text{ or } m=1,\\ 2^{mn}\binom{k}{n}(-t)^n\sum_{r=0}^{n}\binom{n}{r}A_{r+n}\ (\mathrm{mod}\ 2^{mn+1}) & \text{if } p=2 \text{ and } m\geqslant 2.\end{cases}$$

**Proof.** From [17, Lemma 2.1] we know that for any function $F$,

$$F(k)=\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}F(r)+\sum_{r=n}^{k}\binom{k}{r}(-1)^r\sum_{s=0}^{r}\binom{r}{s}(-1)^sF(s),\tag{4.5}$$

where the second sum vanishes when $k<n$.

Now taking $F(k)=f(ktp^{m-1}+d)$ we obtain

$$f(ktp^{m-1}+d)=\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}f(rtp^{m-1}+d)$$
$$+\sum_{r=n}^{k}\binom{k}{r}(-1)^r\sum_{s=0}^{r}\binom{r}{s}(-1)^sf(stp^{m-1}+d).$$

By Theorem 4.2 we have

$$\sum_{r=n}^{k}\binom{k}{r}(-1)^r\sum_{s=0}^{r}\binom{r}{s}(-1)^sf(stp^{m-1}+d)$$
$$\equiv(-1)^n\binom{k}{n}\sum_{s=0}^{n}\binom{n}{s}(-1)^sf(stp^{m-1}+d)$$
$$\equiv\begin{cases}\binom{k}{n}p^{mn}(-t)^nA_n\ (\mathrm{mod}\ p^{mn+1}) & \text{if } p>2 \text{ or } m=1,\\ \binom{k}{n}2^{mn}(-t)^n\sum_{r=0}^{n}\binom{n}{r}A_{r+n}\ (\mathrm{mod}\ 2^{mn+1}) & \text{if } p=2 \text{ and } m\geqslant 2.\end{cases}$$

Now combining the above we prove the theorem. □

Putting $n=1,2,3$ and $d=0$ in Theorem 4.3 we deduce the following result.

**Corollary 4.2.** *Let $p$ be a prime, $k, m, t \in \mathbb{N}$. Let $f$ be a $p$-regular function. Then*

(i) [18, Corollary 2.1] $f(kp^{m-1}) \equiv f(0) \pmod{p^m}$.
(ii) $f(ktp^{m-1}) \equiv kf(tp^{m-1}) - (k-1)f(0) \pmod{p^{2m}}$.
(iii) *We have*

$$f(ktp^{m-1}) \equiv \frac{k(k-1)}{2} f(2tp^{m-1}) - k(k-2)f(tp^{m-1}) + \frac{(k-1)(k-2)}{2} f(0) \pmod{p^{3m}}.$$

(iv) *We have*

$$f(kp^{m-1}) \equiv \begin{cases} f(0) - k(f(0) - f(1))p^{m-1} \pmod{p^{m+1}} & \text{if } p > 2 \text{ or } m = 1, \\ f(0) - 2^{m-2}k(f(2) - 4f(1) + 3f(0)) \pmod{2^{m+1}} & \text{if } p = 2 \text{ and } m \geqslant 2. \end{cases}$$

**Theorem 4.4.** *Let $p$ be a prime and let $f$ be a $p$-regular function. Let $n \in \mathbb{N}$.*

(i) *For $d, x \in \mathbb{Z}_p$ and $m \in \{0, 1, \ldots, n-1\}$ we have*

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k \binom{kx+d}{m} f(k) \equiv 0 \pmod{p^{n-m}}.$$

(ii) *We have*

$$\sum_{k=1}^{n} \binom{n}{k} (-1)^k f(k-1) \equiv -f(p^{n-1} - 1) \pmod{p^n}.$$

**Proof.** From [18, Theorem 2.1] we know that there are $a_0, a_1, \ldots, a_{n-m-1} \in \mathbb{Z}$ such that

$$f(k) \equiv a_{n-m-1}k^{n-m-1} + \cdots + a_1 k + a_0 \pmod{p^{n-m}} \quad \text{for } k = 0, 1, 2, \ldots .$$

Thus applying Lemma 4.1 and (4.1) we have

$$\begin{aligned} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \binom{kx+d}{m} f(k) &\equiv \sum_{k=0}^{n} \binom{n}{k} (-1)^k \binom{kx+d}{m} \sum_{i=0}^{n-m-1} a_i k^i \\ &= \sum_{i=0}^{n-m-1} a_i \sum_{k=0}^{n} \binom{n}{k} (-1)^k \binom{kx+d}{m} k^i = 0 \pmod{p^{n-m}}. \end{aligned}$$

This proves (i).

Now we consider (ii). By [18, Theorem 2.1] there are $a_0, a_1, \ldots, a_{n-1} \in \mathbb{Z}_p$ such that $s!a_s/p^s \in \mathbb{Z}_p$ $(s = 0, 1, \ldots, n-1)$ and

$$f(k) \equiv a_{n-1}k^{n-1} + \cdots + a_1 k + a_0 \pmod{p^n} \quad \text{for } k = 0, 1, 2, \ldots .$$

Note that $p^{s-1}/s! \in \mathbb{Z}_p$ for $s \in \mathbb{N}$. We then have $a_1 \equiv \cdots \equiv a_{n-1} \equiv 0 \pmod p$. Let

$$a_{n-1}(k-1)^{n-1} + \cdots + a_1(k-1) + a_0 = b_{n-1}k^{n-1} + \cdots + b_1 k + b_0.$$

Then clearly $b_1 \equiv \cdots \equiv b_{n-1} \equiv 0 \pmod p$ and

$$f(k-1) \equiv b_{n-1}k^{n-1} + \cdots + b_1 k + b_0 \pmod{p^n} \quad \text{for } k = 1, 2, 3, \ldots .$$

Thus

$$f(p^{n-1} - 1) \equiv b_{n-1}(p^{n-1})^{n-1} + \cdots + b_1 p^{n-1} + b_0 \equiv b_0 \pmod{p^n}.$$

Hence, applying (4.1) we have

$$\begin{aligned}\sum_{k=1}^{n}\binom{n}{k}(-1)^k f(k-1) &\equiv \sum_{k=1}^{n}\binom{n}{k}(-1)^k(b_{n-1}k^{n-1}+\cdots+b_1k+b_0)\\ &=\sum_{i=1}^{n-1}b_i\sum_{k=0}^{n}\binom{n}{k}(-1)^k k^i + b_0\sum_{k=1}^{n}\binom{n}{k}(-1)^k\\ &= -b_0 \equiv -f(p^{n-1}-1)\pmod{p^n}.\end{aligned}$$

So the theorem is proved. □

## 5. Congruences for $pB_{k\varphi(p^m)+b}(x)$ and $pB_{k\varphi(p^m)+b,\chi}\ (\mathrm{mod}\ p^{mn})$

For given prime $p$ and $t \in \mathbb{Z}_p$ we recall that $\langle t\rangle_p$ denotes the least nonnegative residue of $t$ modulo $p$.

**Theorem 5.1.** *Let $p$ be a prime, and $k, m, n, t, b \in \mathbb{Z}$ with $m, n \geqslant 1$ and $k, b, t \geqslant 0$. Let $x \in \mathbb{Z}_p$ and $x' = (x + \langle -x\rangle_p)/p$. Then*

$$\begin{aligned}&pB_{kt\varphi(p^m)+b}(x) - p^{kt\varphi(p^m)+b}B_{kt\varphi(p^m)+b}(x')\\ &\quad -\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}(pB_{rt\varphi(p^m)+b}(x) - p^{rt\varphi(p^m)+b}B_{rt\varphi(p^m)+b}(x'))\\ &\quad\equiv\begin{cases}\delta(b,n,p)\binom{k}{n}(-t)^n p^{mn-1}\pmod{p^{mn}} & \text{if } p>2 \text{ or } m=1,\\ 0\pmod{2^{mn}} & \text{if } p=2 \text{ and } m\geqslant 2,\end{cases}\end{aligned}$$

*where*

$$\delta(b,n,p)=\begin{cases}1 & \text{if } p=2 \text{ and } n\in\{1,2,4,6,\ldots\}\\ & \text{or if } p>2,\ p-1\mid b \text{ and } p-1\mid n,\\ 0 & \text{otherwise}.\end{cases}$$

**Proof.** From [17, Theorem 3.1] we know that

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k(pB_{k(p-1)+b}(x) - p^{k(p-1)+b}B_{k(p-1)+b}(x')) \equiv p^{n-1}\delta(b,n,p)\pmod{p^n}.$$

Set $f(k) = p(pB_{k(p-1)+b}(x) - p^{k(p-1)+b}B_{k(p-1)+b}(x'))$. Then $\sum_{k=0}^{n}\binom{n}{k}(-1)^k f(k) \equiv \delta(b,n,p)p^n \pmod{p^{n+1}}$. Thus $f$ is a $p$-regular function. Hence appealing to Theorem 4.3 we have

$$\begin{aligned}&f(ktp^{m-1}) - \sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}f(rtp^{m-1})\\ &\quad\equiv\begin{cases}p^{mn}\binom{k}{n}(-t)^n\delta(b,n,p)\pmod{p^{mn+1}} & \text{if } p>2 \text{ or } m=1,\\ 2^{mn}\binom{k}{n}(-t)^n\sum_{r=0}^{n}\binom{n}{r}\delta(b,n+r,2)\pmod{2^{mn+1}} & \text{if } p=2 \text{ and } m\geqslant 2.\end{cases}\end{aligned}$$

Note that

$$\delta(b, n+r, 2) = \begin{cases} 1 & \text{if } n+r \in \{1, 2, 4, 6, \ldots\}, \\ 0 & \text{if } n+r \in \{3, 5, 7, \ldots\}. \end{cases}$$

We then have

$$\sum_{r=0}^{n} \binom{n}{r} \delta(b, n+r, 2)$$
$$= \begin{cases} \delta(b,1,2)+\delta(b,2,2) = 1+1 \equiv 0 \pmod 2 & \text{if } n=1, \\ \sum\limits_{\substack{r=0 \\ 2\mid n+r}}^{n} \binom{n}{r} = 2^{n-1} \equiv 0 \pmod 2 & \text{if } n>1. \end{cases}$$

Thus

$$\frac{f(ktp^{m-1})}{p} - \sum_{r=0}^{n-1} (-1)^{n-1-r} \binom{k-1-r}{n-1-r} \binom{k}{r} \frac{f(rtp^{m-1})}{p}$$
$$\equiv \begin{cases} p^{mn-1} \binom{k}{n} (-t)^n \delta(b, n, p) \pmod{p^{mn}} & \text{if } p>2 \text{ or } m=1, \\ 0 \pmod{2^{mn}} & \text{if } p=2 \text{ and } m \geqslant 2. \end{cases}$$

This is the result. □

**Corollary 5.1.** *Let $p$ be a prime, and $k, m, b \in \mathbb{Z}$ with $k, m \geqslant 1$ and $b \geqslant 0$. Let $x \in \mathbb{Z}_p$ and $x' = (x + \langle -x \rangle_p)/p$. Suppose $p>2$ or $m>1$. Then*

$$pB_{k\varphi(p^m)+b}(x) \equiv \begin{cases} 3 \pmod 4 & \text{if } p=m=2,\ k=1 \text{ and } b=0, \\ pB_b(x) - p^b B_b(x') \pmod{p^m} & \text{otherwise}. \end{cases}$$

**Proof.** Putting $n=t=1$ in Theorem 5.1 we see that

$$pB_{k\varphi(p^m)+b}(x) - p^{k\varphi(p^m)+b} B_{k\varphi(p^m)+b}(x') \equiv pB_b(x) - p^b B_b(x') \pmod{p^m}.$$

If $p=m=2$, $k=1$ and $b=0$, then $pB_{k\varphi(p^m)+b}(x) = 2B_2(x) = 2(x^2 - x + \frac{1}{6}) \equiv 3 \pmod 4$. Otherwise, we have $k\varphi(p^m)+b \geqslant m+1$ and so $p^{k\varphi(p^m)+b} B_{k\varphi(p^m)+b}(x') \equiv 0 \pmod{p^m}$. Thus the result follows from the above.

In the case $p>2$, Corollary 5.1 has been proved by the author in [17].

Let $\chi$ be a primitive Dirichlet character of conductor $m$. The generalized Bernoulli number $B_{n,\chi}$ is defined by

$$\sum_{r=1}^{m} \frac{\chi(r) t e^{rt}}{e^{mt}-1} = \sum_{n=0}^{\infty} B_{n,\chi} \frac{t^n}{n!}.$$

Let $\chi_0$ be the trivial character. It is well known that (see [23])

$$B_{1,\chi_0} = \frac{1}{2}, \quad B_{n,\chi_0} = B_n \ (n \neq 1) \quad \text{and} \quad B_{n,\chi} = m^{n-1} \sum_{r=1}^{m} \chi(r) B_n\left(\frac{r}{m}\right).$$

If $\chi$ is nontrivial and $n \in \mathbb{N}$, then clearly $\sum_{r=1}^{m} \chi(r) = 0$ and so

$$\frac{B_{n,\chi}}{n} = m^{n-1} \sum_{r=1}^{m} \chi(r) \left( \frac{B_n\left(\frac{r}{m}\right) - B_n}{n} + \frac{B_n}{n} \right) = m^{n-1} \sum_{r=1}^{m} \chi(r) \frac{B_n\left(\frac{r}{m}\right) - B_n}{n}.$$

When $p$ is a prime with $p \nmid m$, by [17, Lemma 2.3] we have $(B_n(\frac{r}{m}) - B_n)/n \in \mathbb{Z}_p$. Thus $B_{n,\chi}/n$ is congruent to an algebraic integer modulo $p$.

**Lemma 5.1.** *Let $p$ be a prime and let $b$ be a nonnegative integer.*

(i) [18, Theorem 3.2, 25] *If $p-1\nmid b$, $x\in\mathbb{Z}_p$ and $x'=(x+\langle -x\rangle_p)/p$, then $f(k)=(B_{k(p-1)+b}(x)-p^{k(p-1)+b-1}B_{k(p-1)+b}(x'))/(k(p-1)+b)$ is a $p$-regular function.*

(ii) [18, (3.1), Theorem 3.1 and Remark 3.1] *If $a,b\in\mathbb{N}$ and $p\nmid a$, then $f(k)=(1-p^{k(p-1)+b-1})(a^{k(p-1)+b}-1)B_{k(p-1)+b}/(k(p-1)+b)$ is a $p$-regular function.*

(iii) [26, Theorem 4.2, 24, p. 216, 6, 18, Lemma 8.1(a)] *If $b,m\in\mathbb{N}$, $p\nmid m$ and $\chi$ is a nontrivial primitive Dirichlet character of conductor $m$, then $f(k)=(1-\chi(p)\,p^{k(p-1)+b-1})B_{k(p-1)+b,\chi}/(k(p-1)+b)$ is a $p$-regular function.*

(iv) [18, Lemma 8.1(b)] *If $m\in\mathbb{N}$, $p\nmid m$ and $\chi$ is a nontrivial Dirichlet character of conductor $m$, then $f(k)=(1-\chi(p)p^{k(p-1)+b-1})pB_{k(p-1)+b,\chi}$ is a $p$-regular function.*

From Lemma 5.1 and Theorem 4.3 we deduce the following theorem.

**Theorem 5.2.** *Let $p$ be a prime, $k,n,s,t\in\mathbb{N}$ and $b\in\{0,1,2,\ldots\}$.*

(i) *If $p-1\nmid b$, $x\in\mathbb{Z}_p$ and $x'=(x+\langle -x\rangle_p)/p$, then*

$$\begin{aligned}&\frac{B_{ktp^{s-1}(p-1)+b}(x)-p^{ktp^{s-1}(p-1)+b-1}B_{ktp^{s-1}(p-1)+b}(x')}{ktp^{s-1}(p-1)+b}\\&\quad\equiv\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}\\&\quad\times\frac{B_{rtp^{s-1}(p-1)+b}(x)-p^{rtp^{s-1}(p-1)+b-1}B_{rtp^{s-1}(p-1)+b}(x')}{rtp^{s-1}(p-1)+b}\ (\mathrm{mod}\ p^{sn}).\end{aligned}$$

(ii) *If $a,b\in\mathbb{N}$ and $p\nmid a$, then*

$$\begin{aligned}&(1-p^{ktp^{s-1}(p-1)+b-1})(a^{ktp^{s-1}(p-1)+b}-1)\frac{B_{ktp^{s-1}(p-1)+b}}{ktp^{s-1}(p-1)+b}\\&\quad\equiv\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}(1-p^{rtp^{s-1}(p-1)+b-1})\\&\quad\times\left(a^{rtp^{s-1}(p-1)+b}-1\right)\frac{B_{rtp^{s-1}(p-1)+b}}{rtp^{s-1}(p-1)+b}\ (\mathrm{mod}\ p^{sn}).\end{aligned}$$

(iii) *If $b,m\in\mathbb{N}$, $p\nmid m$ and $\chi$ is a nontrivial primitive Dirichlet character of conductor $m$, then*

$$\begin{aligned}&\frac{(1-\chi(p)p^{ktp^{s-1}(p-1)+b-1})B_{ktp^{s-1}(p-1)+b,\chi}}{ktp^{s-1}(p-1)+b}\\&\quad\equiv\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}\frac{(1-\chi(p)p^{rtp^{s-1}(p-1)+b-1})B_{rtp^{s-1}(p-1)+b,\chi}}{rtp^{s-1}(p-1)+b}\ (\mathrm{mod}\ p^{sn}).\end{aligned}$$

(iv) *If $m\in\mathbb{N}$, $p\nmid m$ and $\chi$ is a nontrivial Dirichlet character of conductor $m$, then*

$$\begin{aligned}&(1-\chi(p)p^{ktp^{s-1}(p-1)+b-1})pB_{ktp^{s-1}(p-1)+b,\chi}\\&\quad\equiv\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}(1-\chi(p)p^{rtp^{s-1}(p-1)+b-1})pB_{rtp^{s-1}(p-1)+b,\chi}\ (\mathrm{mod}\ p^{sn}).\end{aligned}$$

**Remark 5.1.** Theorem 5.2 can be viewed as generalizations of some congruences in [18]. In the case $n=1$, Theorem 5.2(i) was given by Eie and Ong [4], and independently by the author in [18, p. 204]. In the case $s=t=1$, Theorem 5.2(i) was announced by the author in [17] and proved in [18], and Theorem 5.2(iii) (in the case $p-1\nmid b$) and Theorem 5.2(iv) were also given in [18]. When $n=1$, Theorem 5.2(iii) was given in [23, p. 141].

Combining Lemma 5.1 and Corollary 4.2(iv) we obtain the following result.

**Theorem 5.3.** *Let $p$ be an odd prime, $k,s\in\mathbb{N}$ and $b\in\{0,1,2,\ldots\}$.*

(i) *If $p-1\nmid b$, $x\in\mathbb{Z}_p$ and $x'=(x+\langle -x\rangle_p)/p$, then*

$$\frac{B_{k\varphi(p^s)+b}(x)}{k\varphi(p^s)+b}\equiv(1-kp^{s-1})\frac{B_b(x)-p^{b-1}B_b(x')}{b}+kp^{s-1}\frac{B_{p-1+b}(x)}{p-1+b}\pmod{p^{s+1}}.$$

(ii) *If $b,m\in\mathbb{N}$, $p\nmid m$ and $\chi$ is a nontrivial primitive Dirichlet character of conductor $m$, then*

$$\frac{B_{k\varphi(p^s)+b,\chi}}{k\varphi(p^s)+b}\equiv(1-kp^{s-1})(1-\chi(p)p^{b-1})\frac{B_{b,\chi}}{b}+kp^{s-1}\frac{B_{p-1+b,\,\chi}}{p-1+b}\pmod{p^{s+1}}.$$

(iii) *If $m\in\mathbb{N}$, $p\nmid m$ and $\chi$ is a nontrivial Dirichlet character of conductor $m$, then*

$$\begin{aligned}&(1-\chi(p)p^{k\varphi(p^s)+b-1})pB_{k\varphi(p^s)+b,\chi}\\&\quad\equiv(1-kp^{s-1})(1-\chi(p)p^{b-1})pB_{b,\chi}+kp^{s-1}(1-\chi(p)p^{p-2+b})pB_{p-1+b,\chi}\pmod{p^{s+1}}.\end{aligned}$$

**Corollary 5.2.** *Let $p$ be an odd prime and $k,s,b\in\mathbb{N}$ with $2\mid b$ and $p-1\nmid b$. Then*

$$\frac{B_{k\varphi(p^s)+b}}{k\varphi(p^s)+b}\equiv(1-kp^{s-1})(1-p^{b-1})\frac{B_b}{b}+kp^{s-1}\frac{B_{p-1+b}}{p-1+b}\pmod{p^{s+1}}.$$

**Theorem 5.4.** *Let $p$ be a prime, $a,n\in\mathbb{N}$ and $p\nmid a$.*

(i) *There are integers $b_0,b_1,\ldots,b_{n-1}$ such that*

$$(1-p^{k(p-1)-1})(a^{k(p-1)}-1)\frac{B_{k(p-1)}}{k(p-1)}\equiv b_{n-1}k^{n-1}+\cdots+b_1k+b_0\pmod{p^n}\quad\text{for }k=1,2,3,\ldots.$$

(ii) *If $p>2$ or $n>2$, then*

$$\sum_{k=1}^{n}\binom{n}{k}(-1)^k(1-p^{k(p-1)-1})(a^{k(p-1)}-1)\frac{B_{k(p-1)}}{k(p-1)}\equiv\frac{1-a^{\varphi(p^n)}}{p^n}\pmod{p^n}.$$

**Proof.** Suppose $b\in\mathbb{N}$. From Lemma 5.1(ii) we know that

$$f(k)=(1-p^{k(p-1)+b-1})(a^{k(p-1)+b}-1)\frac{B_{k(p-1)+b}}{k(p-1)+b}$$

is a $p$-regular function. Hence taking $b=p-1$ and applying [18, Theorem 2.1] we know that there exist integers $a_0,a_1,\ldots,a_{n-1}$ such that

$$\begin{aligned}&(1-p^{(k+1)(p-1)-1})(a^{(k+1)(p-1)}-1)\frac{B_{(k+1)(p-1)}}{(k+1)(p-1)}\\&\quad\equiv a_{n-1}k^{n-1}+\cdots+a_1k+a_0\pmod{p^n}\quad\text{for }k=0,1,2,\ldots\end{aligned}$$

That is,

$$(1-p^{k(p-1)-1})(a^{k(p-1)}-1)\frac{B_{k(p-1)}}{k(p-1)}$$
$$\equiv a_{n-1}(k-1)^{n-1}+\cdots+a_1(k-1)+a_0 \pmod{p^n} \quad \text{for } k=1,2,3,\ldots$$

On setting

$$a_{n-1}(k-1)^{n-1}+\cdots+a_1(k-1)+a_0=b_{n-1}k^{n-1}+\cdots+b_1k+b_0$$

we obtain (i).

Now we consider (ii). Suppose $p>2$ or $n>2$. Since $f(k)$ is a $p$-regular function, by Theorem 4.4(ii) we have

$$\sum_{k=1}^{n}\binom{n}{k}(-1)^k(1-p^{(k-1)(p-1)+b-1})(a^{(k-1)(p-1)+b}-1)\frac{B_{(k-1)(p-1)+b}}{(k-1)(p-1)+b}$$
$$\equiv -(1-p^{(p^{n-1}-1)(p-1)+b-1})(a^{(p^{n-1}-1)(p-1)+b}-1)\frac{B_{(p^{n-1}-1)(p-1)+b}}{(p^{n-1}-1)(p-1)+b} \pmod{p^n}.$$

Substituting $b$ by $p-1+b$ we see that for $b\geqslant 0$,

$$\sum_{k=1}^{n}\binom{n}{k}(-1)^k(1-p^{k(p-1)+b-1})(a^{k(p-1)+b}-1)\frac{B_{k(p-1)+b}}{k(p-1)+b}$$
$$\equiv -(1-p^{\varphi(p^n)+b-1})(a^{\varphi(p^n)+b}-1)\frac{B_{\varphi(p^n)+b}}{\varphi(p^n)+b} \pmod{p^n}. \tag{5.1}$$

By Corollary 5.1 we have $pB_{\varphi(p^n)}\equiv p-1 \pmod{p^n}$. Thus taking $b=0$ in (5.1) and noting that $\varphi(p^n)\geqslant n+1$ we obtain

$$\sum_{k=1}^{n}\binom{n}{k}(-1)^k(1-p^{k(p-1)-1})(a^{k(p-1)}-1)\frac{B_{k(p-1)}}{k(p-1)}$$
$$\equiv -(1-p^{\varphi(p^n)-1})(a^{\varphi(p^n)}-1)\frac{B_{\varphi(p^n)}}{\varphi(p^n)}$$
$$= -(1-p^{\varphi(p^n)-1})\frac{a^{\varphi(p^n)}-1}{p^n}\cdot\frac{pB_{\varphi(p^n)}}{p-1}\equiv -\frac{a^{\varphi(p^n)}-1}{p^n} \pmod{p^n}.$$

This completes the proof of the theorem. □

## 6. Congruences for $\sum_{k=0}^{n}\binom{n}{k}(-1)^k pB_{k(p-1)+b}(x) \pmod{p^{n+1}}$

For $a\in\mathbb{N}$ and $b\in\mathbb{Z}$ we define $\chi(a|b)=1$ or $0$ according as $a|b$ or $a\nmid b$.

**Lemma 6.1.** *Let $p$ be an odd prime and $n\in\mathbb{N}$. Then*

$$\sum_{\substack{s=1\\ s\equiv n+1 \pmod{p-1}}}^{n}\binom{n}{s}\equiv -\chi(p-1|n) \pmod p.$$

**Proof.** Let $n_0\in\{1,2,\ldots,p-1\}$ be such that $n\equiv n_0 \pmod{p-1}$. Since Glaisher (see [3]) it is well known that

$$\sum_{\substack{s=0\\ s\equiv r \pmod{p-1}}}^{n}\binom{n}{s}\equiv\sum_{\substack{s=0\\ s\equiv r \pmod{p-1}}}^{n_0}\binom{n_0}{s} \pmod p \quad \text{for } r\in\mathbb{Z}.$$

From [14] we know that

$$\sum_{\substack{s=0\\ s\equiv r\ (\mathrm{mod}\ p-1)}}^{n} \binom{n}{s} = \sum_{\substack{s=0\\ s\equiv n-r\ (\mathrm{mod}\ p-1)}}^{n} \binom{n}{s}.$$

Thus

$$\sum_{\substack{s=0\\ s\equiv n+1\ (\mathrm{mod}\ p-1)}}^{n} \binom{n}{s} = \sum_{\substack{s=0\\ s\equiv -1\ (\mathrm{mod}\ p-1)}}^{n} \binom{n}{s} \equiv \sum_{\substack{s=0\\ s\equiv p-2\ (\mathrm{mod}\ p-1)}}^{n_0} \binom{n_0}{s}$$

$$= \begin{cases} p-1 \equiv -1\ (\mathrm{mod}\ p) & \text{if } n_0 = p-1,\\ 1\ (\mathrm{mod}\ p) & \text{if } n_0 = p-2,\\ 0\ (\mathrm{mod}\ p) & \text{if } n_0 < p-2. \end{cases}$$

Hence

$$\sum_{\substack{s=1\\ s\equiv n+1\ (\mathrm{mod}\ p-1)}}^{n} \binom{n}{s} = \sum_{\substack{s=0\\ s\equiv n+1\ (\mathrm{mod}\ p-1)}}^{n} \binom{n}{s} - \chi(p-1|n+1) \equiv -\chi(p-1|n)\ (\mathrm{mod}\ p).$$

This proves the lemma. □

**Proposition 6.1.** *Let $p$ be an odd prime, $n \in \mathbb{N}$ and $x \in \mathbb{Z}_p$. Let $b$ be a nonnegative integer. Then*

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k \left( pB_{k(p-1)+b}(x) - p^{k(p-1)+b} B_{k(p-1)+b}\left(\frac{x+\langle -x\rangle_p}{p}\right)\right)$$

$$\equiv \sum_{\substack{j=0\\ j\neq\langle -x\rangle_p}}^{p-1} (x+j)^{b-n} p^n B_n \left(\frac{(x+j)^p-(x+j)}{p(p-1)}\right) + p^n \Delta(b,n,p)\ (\mathrm{mod}\ p^{n+1}),$$

*where*

$$\Delta(b,n,p) = \begin{cases} (n-b)T - n & \text{if } p-1|b \text{ and } p-1|n,\\ (n-b)T & \text{if } p-1\nmid b \text{ and } p-1|n,\\ b-n & \text{if } p-1|b \text{ and } p-1|n+1,\\ 0 & \text{otherwise} \end{cases}$$

*and*

$$T = \sum_{\substack{j=0\\ j\neq\langle -x\rangle_p}}^{p-1} \frac{(x+j)^{p-1+b}-(x+j)^b}{p}.$$

**Proof.** Let

$$S_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \left( pB_{k(p-1)+b}(x) - p^{k(p-1)+b} B_{k(p-1)+b}\left(\frac{x+\langle -x\rangle_p}{p}\right)\right).$$

From [17, p.157] we know that

$$S_n = \sum_{r=0}^{n(p-1)+b} p^r B_r \sum_{\substack{j=0\\ j\neq\langle -x\rangle_p}}^{p-1} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \binom{k(p-1)+b}{r} (x+j)^{k(p-1)+b-r}.$$

By [18, p.199] we know that for any functions $f$ and $g$ we have

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k f(k)g(k) = \sum_{s=0}^{n}\binom{n}{s}\left(\sum_{i=0}^{n-s}\binom{n-s}{i}(-1)^i f(i+s)\right)\sum_{j=0}^{s}\binom{s}{j}(-1)^j g(j). \tag{6.1}$$

Now taking $f(k)=\binom{k(p-1)+b}{r}$ and $g(k)=a^{k(p-1)+b-r}$ $(a\neq 0)$ in (6.1) we obtain

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\binom{k(p-1)+b}{r}a^{k(p-1)+b-r} &= \sum_{s=0}^{n}\binom{n}{s}\left(\sum_{i=0}^{n-s}\binom{n-s}{i}(-1)^i\binom{(i+s)(p-1)+b}{r}\right)\sum_{j=0}^{s}\binom{s}{j}(-1)^j a^{j(p-1)+b-r}\\ &= \sum_{s=0}^{n}\binom{n}{s}a^{b-r}(1-a^{p-1})^s\sum_{i=0}^{n-s}\binom{n-s}{i}(-1)^i\binom{i(p-1)+s(p-1)+b}{r}.\end{aligned}$$

Thus applying the above and Lemma 4.1 we have

$$\begin{aligned}S_n &= \sum_{r=0}^{n(p-1)+b}p^rB_r\sum_{\substack{j=0\\ j\neq\langle -x\rangle_p}}^{p-1}\sum_{s=0}^{n}\binom{n}{s}(x+j)^{b-r}(1-(x+j)^{p-1})^s\\ &\quad\times\sum_{i=0}^{n-s}\binom{n-s}{i}(-1)^i\binom{i(p-1)+s(p-1)+b}{r}\\ &= \sum_{\substack{j=0\\ j\neq\langle -x\rangle_p}}^{p-1}\sum_{s=0}^{n}\binom{n}{s}\left(\frac{1-(x+j)^{p-1}}{p}\right)^s\sum_{r=n-s}^{n(p-1)+b}p^{r+s}B_r\cdot(x+j)^{b-r}\\ &\quad\times\sum_{i=0}^{n-s}\binom{n-s}{i}(-1)^i\binom{i(p-1)+s(p-1)+b}{r}.\end{aligned}$$

Since $pB_r\in\mathbb{Z}_p$ and so $p^{r+s}B_r\equiv 0 \pmod{p^{n+1}}$ for $r\geqslant n-s+2$, by Theorem 4.1 we have

$$\begin{aligned}&\sum_{r=n-s}^{n(p-1)+b}(x+j)^{b-r}p^{r+s}B_r\sum_{i=0}^{n-s}\binom{n-s}{i}(-1)^i\binom{i(p-1)+s(p-1)+b}{r}\\ &\quad\equiv (x+j)^{b-(n-s)}p^nB_{n-s}\sum_{i=0}^{n-s}\binom{n-s}{i}(-1)^i\binom{i(p-1)+s(p-1)+b}{n-s}\\ &\qquad+(x+j)^{b-(n-s+1)}p^{n+1}B_{n-s+1}\sum_{i=0}^{n-s}\binom{n-s}{i}(-1)^i\binom{i(p-1)+s(p-1)+b}{n-s+1}\\ &\quad= (x+j)^{b-(n-s)}p^nB_{n-s}\cdot(1-p)^{n-s}+(x+j)^{b-(n-s+1)}p^{n+1}B_{n-s+1}\\ &\qquad\times(s(p-1)+b+(n-s)(p-2)/2)(1-p)^{n-s}\\ &\quad\equiv (x+j)^{b-(n-s)}(1-p)^{n-s}p^nB_{n-s}+(x+j)^{b-(n-s+1)}(b-n)p^{n+1}B_{n-s+1} \pmod{p^{n+1}}.\end{aligned}$$

Thus,

$$\begin{aligned}
S_n &\equiv \sum_{\substack{j=0\\ j\neq\langle -x\rangle_p}}^{p-1}\sum_{s=0}^{n}\binom{n}{s}\left(\frac{1-(x+j)^{p-1}}{p}\right)^s((x+j)^{b-n+s}(1-p)^{n-s}p^nB_{n-s}\\
&\quad+(x+j)^{b-n+s-1}(b-n)p^{n+1}B_{n-s+1})\\
&=\sum_{\substack{j=0\\ j\neq\langle -x\rangle_p}}^{p-1}(x+j)^{b-n}(1-p)^np^n\sum_{s=0}^{n}\binom{n}{s}\left(\frac{1-(x+j)^{p-1}}{p}\cdot\frac{x+j}{1-p}\right)^sB_{n-s}\\
&\quad+\sum_{\substack{j=0\\ j\neq\langle -x\rangle_p}}^{p-1}\sum_{s=0}^{n}\binom{n}{s}\left(\frac{1-(x+j)^{p-1}}{p}\right)^s(x+j)^{b-n+s-1}(b-n)p^{n+1}B_{n-s+1}\\
&\equiv\sum_{\substack{j=0\\ j\neq\langle -x\rangle_p}}^{p-1}(x+j)^{b-n}(1-p)^np^nB_n(x_j)+\sum_{\substack{j=0\\ j\neq\langle -x\rangle_p}}^{p-1}\sum_{\substack{s=0\\ p-1\mid n-s+1}}^{n}\binom{n}{s}\\
&\quad\times\left(\frac{1-(x+j)^{p-1}}{p}\right)^s(x+j)^{b-n+s-1}(n-b)p^n\ (\mathrm{mod}\ p^{n+1}),
\end{aligned}$$

where

$$x_j=\frac{(x+j)^p-(x+j)}{p(p-1)}.$$

In the last step we use the facts

$$B_n(t)=\sum_{s=0}^{n}\binom{n}{s}t^sB_{n-s}\quad\text{and}\quad pB_k\equiv-\chi(p-1|k)\ (\mathrm{mod}\ p)\ \ (k\geqslant 1).$$

For $a\in\mathbb{Z}$, using Lemma 6.1 and Fermat's little theorem we see that

$$\begin{aligned}
\sum_{\substack{s=0\\ s\equiv n+1\ (\mathrm{mod}\ p-1)}}^{n}\binom{n}{s}a^s &= \sum_{\substack{s=1\\ s\equiv n+1\ (\mathrm{mod}\ p-1)}}^{n}\binom{n}{s}a^s+\chi(p-1|n+1)\\
&\equiv a^{n+1}\sum_{\substack{s=1\\ s\equiv n+1\ (\mathrm{mod}\ p-1)}}^{n}\binom{n}{s}+\chi(p-1|n+1)\\
&\equiv -\chi(p-1|n)a^{n+1}+\chi(p-1|n+1)\\
&=\begin{cases}-a^{n+1}\equiv -a\ (\mathrm{mod}\ p) & \text{if } p-1|n,\\ 1\ (\mathrm{mod}\ p) & \text{if } p-1|n+1,\\ 0\ (\mathrm{mod}\ p) & \text{if } p-1\nmid n \text{ and } p-1\nmid n+1.\end{cases}
\end{aligned}$$

We also note that (see [18, (5.1)])

$$\sum_{\substack{j=0\\ j\neq\langle -x\rangle_p}}^{p-1}(x+j)^b\equiv\sum_{r=1}^{p-1}r^b\equiv-\chi(p-1|b)\ (\mathrm{mod}\ p). \tag{6.2}$$

Thus

$$\sum_{\substack{j=0\\ j\neq\langle -x\rangle_p}}^{p-1}\ \sum_{\substack{s=0\\ p-1\mid n-s+1}}^{n}\binom{n}{s}\left(\frac{1-(x+j)^{p-1}}{p}\right)^{s}(x+j)^{b-n+s-1}(n-b)p^n$$

$$\equiv p^n(n-b)\sum_{\substack{j=0\\ j\neq\langle -x\rangle_p}}^{p-1}(x+j)^b\sum_{\substack{s=0\\ s\equiv n+1\ (\mathrm{mod}\ p-1)}}^{n}\binom{n}{s}\left(\frac{1-(x+j)^{p-1}}{p}\right)^{s}$$

$$\equiv\begin{cases} p^n(n-b)\displaystyle\sum_{\substack{j=0\\ j\neq\langle -x\rangle_p}}^{p-1}(x+j)^b((x+j)^{p-1}-1)/p \pmod{p^{n+1}} & \text{if } p-1\mid n,\\ p^n(n-b)\displaystyle\sum_{\substack{j=0\\ j\neq\langle -x\rangle_p}}^{p-1}(x+j)^b\equiv -\chi(p-1\mid b)(n-b)p^n \pmod{p^{n+1}} & \text{if } p-1\mid n+1,\\ 0 \pmod{p^{n+1}} & \text{if } p-1\nmid n \text{ and } p-1\nmid n+1.\end{cases}$$

On the other hand, for $t\in\mathbb{Z}_p$ we have $B_n(t)-B_n\in\mathbb{Z}_p$ (cf. [17, Lemma 2.3]) and so

$$(-np)p^nB_n(x_j)\equiv -np^{n+1}B_n\equiv\begin{cases} np^n \pmod{p^{n+1}} & \text{if } p-1\mid n,\\ 0\pmod{p^{n+1}} & \text{if } p-1\nmid n.\end{cases}$$

Thus applying (6.2) we get

$$\sum_{\substack{j=0\\ j\neq\langle -x\rangle_p}}^{p-1}(x+j)^{b-n}\cdot(-np)p^nB_n(x_j)$$

$$\equiv\begin{cases} \displaystyle\sum_{\substack{j=0\\ j\neq\langle -x\rangle_p}}^{p-1}(x+j)^b\cdot np^n\equiv -np^n\chi(p-1\mid b)\pmod{p^{n+1}} & \text{if } p-1\mid n,\\ 0\pmod{p^{n+1}} & \text{if } p-1\nmid n.\end{cases}$$

Hence, by the above and the fact $(1-p)^n\equiv 1-np \pmod{p^2}$ we obtain

$$\sum_{\substack{j=0\\ j\neq\langle -x\rangle_p}}^{p-1}(x+j)^{b-n}(1-p)^np^nB_n(x_j)-\sum_{\substack{j=0\\ j\neq\langle -x\rangle_p}}^{p-1}(x+j)^{b-n}p^nB_n(x_j)$$

$$\equiv\sum_{\substack{j=0\\ j\neq\langle -x\rangle_p}}^{p-1}(x+j)^{b-n}\cdot(-np)p^nB_n(x_j)$$

$$\equiv\begin{cases} -np^n \pmod{p^{n+1}} & \text{if } p-1\mid b \text{ and } p-1\mid n,\\ 0\pmod{p^{n+1}} & \text{if } p-1\nmid b \text{ or } p-1\nmid n.\end{cases}$$

Now combining the above we see that

$$S_n - \sum_{\substack{j=0\\ j\neq\langle -x\rangle_p}}^{p-1} (x+j)^{b-n} p^n B_n(x_j)$$

$$\equiv \begin{cases} -np^n + (n-b)p^n T \pmod{p^{n+1}} & \text{if } p-1\mid b \text{ and } p-1\mid n,\\ p^n(n-b)T \pmod{p^{n+1}} & \text{if } p-1\nmid b \text{ and } p-1\mid n,\\ p^n(b-n) \pmod{p^{n+1}} & \text{if } p-1\mid b \text{ and } p-1\mid n+1,\\ 0 \pmod{p^{n+1}} & \text{otherwise.}\end{cases}$$

This is the result. □

**Remark 6.1.** When $p=2$, $b\geqslant 1$ and $n\geqslant 2$, setting $\Delta(b,n,p)=b-n$ we can show that the result of Proposition 6.1 is also true.

**Theorem 6.1.** *Let $p$ be a prime greater than 3, $x\in\mathbb{Z}_p$, $n\in\mathbb{N}$, $n\not\equiv 0,1 \pmod{p-1}$ and $b\in\{0,1,2,\dots\}$. Let $n_0$ be given by $n\equiv n_0 \pmod{p-1}$ and $n_0\in\{2,3,\dots,p-2\}$. Set*

$$S_n=\sum_{k=0}^{n}\binom{n}{k}(-1)^k\left(pB_{k(p-1)+b}(x)-p^{k(p-1)+b}B_{k(p-1)+b}\left(\frac{x+\langle -x\rangle_p}{p}\right)\right).$$

*Then*

$$S_n\equiv\begin{cases}\left(\dfrac{n}{n_0}\cdot\dfrac{S_{n_0}}{p^{n_0}}+\dfrac{(n+2)b}{2}\right)p^n \pmod{p^{n+1}} & \text{if } p-1\mid b \text{ and } p-1\mid n+1,\\ \dfrac{n}{n_0}\cdot\dfrac{S_{n_0}}{p^{n_0}}\cdot p^n \pmod{p^{n+1}} & \text{if } p-1\nmid b \text{ or } p-1\nmid n+1.\end{cases}$$

**Proof.** Since $p-1\nmid n$ we know that $B_n/n\in\mathbb{Z}_p$. For $t\in\mathbb{Z}_p$, by [17, Lemma 2.3] we have $(B_n(t)-B_n)/n\in\mathbb{Z}_p$. Thus

$$\frac{B_n(t)}{n}=\frac{B_n(t)-B_n}{n}+\frac{B_n}{n}\in\mathbb{Z}_p.$$

As $n\not\equiv 0,1 \pmod{p-1}$, by [18, Corollary 3.1] we have

$$\frac{B_n(t)}{n}\equiv\frac{B_{n_0}(t)-p^{n_0-1}B_{n_0}((t+\langle -t\rangle_p)/p)}{n_0}\equiv\frac{B_{n_0}(t)}{n_0} \pmod p.$$

Set $x_j=((x+j)^p-(x+j))/(p(p-1))$. Then $x_j\in\mathbb{Z}_p$. Thus $B_n(x_j)/n\in\mathbb{Z}_p$ and $B_n(x_j)/n\equiv B_{n_0}(x_j)/n_0 \pmod p$. From Proposition 6.1 and the above we see that

$$\frac{S_n}{p^n}\equiv\sum_{\substack{j=0\\ j\neq\langle -x\rangle_p}}^{p-1}(x+j)^{b-n}B_n(x_j)+(b-n)\chi(p-1\mid b)\chi(p-1\mid n+1)$$

$$\equiv n\sum_{\substack{j=0\\ j\neq\langle -x\rangle_p}}^{p-1}(x+j)^{b-n_0}\frac{B_{n_0}(x_j)}{n_0}+(b-n)\chi(p-1\mid b)\chi(p-1\mid n+1) \pmod p$$

and so

$$\frac{S_{n_0}}{p^{n_0}}\equiv n_0\sum_{\substack{j=0\\ j\neq\langle -x\rangle_p}}^{p-1}(x+j)^{b-n_0}\frac{B_{n_0}(x_j)}{n_0}+(b-n_0)\chi(p-1\mid b)\chi(p-1\mid n+1) \pmod p.$$

Thus

$$\begin{aligned}\frac{S_n}{p^n} &\equiv \frac{n}{n_0}\left(\frac{S_{n_0}}{p^{n_0}} - (b-n_0)\chi(p-1|b)\chi(p-1|n+1)\right) + (b-n)\chi(p-1|b)\chi(p-1|n+1)\\ &= \frac{n}{n_0}\cdot\frac{S_{n_0}}{p^{n_0}} + b\left(1-\frac{n}{n_0}\right)\chi(p-1|b)\chi(p-1|n+1)\\ &\equiv \frac{n}{n_0}\cdot\frac{S_{n_0}}{p^{n_0}} + b\left(1+\frac{n}{2}\right)\chi(p-1|b)\chi(p-1|n+1) \pmod p.\end{aligned}$$

This proves the theorem. □

**Theorem 6.2.** *Let $p$ be an odd prime, $x \in \mathbb{Z}_p$, $b, n \in \mathbb{Z}$ with $n \geqslant 1$ and $b \geqslant 0$. If $p|n$ and $p-1 \nmid n$, then*

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}(-1)^k\left(pB_{k(p-1)+b}(x) - p^{k(p-1)+b}B_{k(p-1)+b}\left(\frac{x+\langle -x\rangle_p}{p}\right)\right)\\ &\equiv \begin{cases} bp^n \pmod{p^{n+1}} & \text{if } p-1|b \text{ and } p-1|n+1,\\ 0 \pmod{p^{n+1}} & \text{if } p-1\nmid b \text{ or } p-1\nmid n+1.\end{cases}\end{aligned}$$

**Proof.** As $p-1 \nmid n$ and $p|n$, for $t \in \mathbb{Z}_p$ we see that $B_n(t)/n \in \mathbb{Z}_p$ and so $B_n(t) = nB_n(t)/n \equiv 0 \pmod p$. Thus the result follows from Proposition 6.1. □

**Theorem 6.3.** *Let $p$ be an odd prime, $n \in \mathbb{N}$ and $b \in \{0, 2, 4, \ldots\}$. If $p(p-1)|n$, then*

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k(1-p^{k(p-1)+b-1})pB_{k(p-1)+b} \equiv \begin{cases} p^{n-1}-2p^n \pmod{p^{n+1}} & \text{if } p-1|b,\\ 0 \pmod{p^{n+1}} & \text{if } p-1\nmid b.\end{cases}$$

**Proof.** From Proposition 6.1 we see that

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k(1-p^{k(p-1)+b-1})pB_{k(p-1)+b} \equiv \sum_{j=1}^{p-1} j^{b-n}p^nB_n\left(\frac{j^p-j}{p(p-1)}\right) - bTp^n \pmod{p^{n+1}},$$

where

$$T = \sum_{j=1}^{p-1}\frac{j^{p-1+b}-j^b}{p}.$$

For $p > 3$ and $m \in \mathbb{N}$, from [18, (5.1)] we have

$$\sum_{j=1}^{p-1} j^m \equiv pB_m + \frac{p^2}{2}mB_{m-1} + \frac{p^3}{6}m(m-1)B_{m-2} \pmod{p^3}.$$

If $m \geqslant 4$ is even, then $B_{m-1} = 0$ and $pB_{m-2} \in \mathbb{Z}_p$. Thus

$$\sum_{j=1}^{p-1} j^m \equiv pB_m \pmod{p^2} \quad \text{for } m = 2, 4, 6, \ldots . \tag{6.3}$$

Hence

$$T \equiv \begin{cases} \dfrac{pB_{p-1+b}-pB_b}{p} \pmod p & \text{if } p>3 \text{ and } b>0,\\[2mm] \dfrac{pB_{p-1}-(p-1)}{p} \pmod p & \text{if } p>3 \text{ and } b=0,\\[2mm] \dfrac{2^{2+b}-2^b}{3} = 2^b \equiv (-1)^b = 1 \equiv \dfrac{3B_2-2}{3} \pmod 3 & \text{if } p=3. \end{cases}$$

If $p>3$ and $b=k(p-1)$ for some $k\in\mathbb{N}$, by [17, Corollary 4.2] we have

$$pB_b = pB_{k(p-1)} \equiv kpB_{p-1} - (k-1)(p-1) \pmod{p^2} \tag{6.4}$$

and

$$pB_{p-1+b} = pB_{(k+1)(p-1)} \equiv (k+1)pB_{p-1} - k(p-1) \pmod{p^2}.$$

Thus

$$T \equiv \frac{pB_{p-1+b}-pB_b}{p} \equiv \frac{pB_{p-1}-(p-1)}{p} \pmod p.$$

If $p>3$ and $p-1\nmid b$, by Kummer's congruences we have

$$\frac{B_{p-1+b}}{p-1+b} \equiv \frac{B_b}{b} \pmod p \quad \text{and so } B_{p-1+b} \equiv (b-1)\frac{B_b}{b} \pmod p.$$

Thus

$$T \equiv \frac{pB_{p-1+b}-pB_b}{p} \equiv \frac{b-1}{b}B_b - B_b = -\frac{B_b}{b} \pmod p.$$

Summarizing the above we have

$$T \equiv \begin{cases} \dfrac{pB_{p-1}-(p-1)}{p} \pmod p & \text{if } p-1\mid b,\\[2mm] -\dfrac{B_b}{b} \pmod p & \text{if } p-1\nmid b. \end{cases} \tag{6.5}$$

As $p(p-1)\mid n$, from Corollary 5.1 we have $pB_n(x) \equiv p-1 \pmod{p^2}$ for $x\in\mathbb{Z}_p$. Note that $j^n \equiv 1 \pmod{p^2}$ for $j=1,2,\ldots,p-1$. Combining the above we obtain

$$\begin{aligned} &\sum_{k=0}^{n}\binom{n}{k}(-1)^k(1-p^{k(p-1)+b-1})pB_{k(p-1)+b}\\ &\quad\equiv \sum_{j=1}^{p-1} j^{b-n}p^{n-1}\cdot pB_n\left(\frac{j^p-j}{p(p-1)}\right) - bTp^n\\ &\quad\equiv \sum_{j=1}^{p-1} j^b p^{n-1}(p-1) - bTp^n \pmod{p^{n+1}}. \end{aligned}$$

From (6.3) and (6.4) we see that

$$\sum_{j=1}^{p-1} j^b \equiv \begin{cases} pB_b \equiv \dfrac{b}{p-1}\cdot pB_{p-1} - \left(\dfrac{b}{p-1}-1\right)(p-1) \pmod{p^2} & \text{if } p>3,\ b>0 \text{ and } p-1\mid b,\\ pB_b \pmod{p^2} & \text{if } p>3 \text{ and } p-1\nmid b,\\ p-1 \pmod{p^2} & \text{if } p>3 \text{ and } b=0,\\ 1+(1+3)^{b/2}\equiv 2+\dfrac{3b}{2}\equiv 2+6b \pmod 9 & \text{if } p=3. \end{cases}$$

That is,

$$\sum_{j=1}^{p-1} j^b \equiv \begin{cases} \dfrac{b}{p-1}(pB_{p-1}-(p-1))+p-1 \pmod{p^2} & \text{if } p-1\mid b,\\ pB_b \pmod{p^2} & \text{if } p-1\nmid b. \end{cases}$$

Hence

$$\begin{aligned} &\sum_{k=0}^{n}\binom{n}{k}(-1)^k(1-p^{k(p-1)+b-1})pB_{k(p-1)+b}\\ &\quad\equiv p^{n-1}(p-1)\sum_{j=1}^{p-1} j^b - bTp^n\\ &\quad\equiv \begin{cases} p^{n-1}(b(pB_{p-1}-(p-1))+(p-1)^2) - p^{n-1}b(pB_{p-1}-(p-1)) & \\ \quad = p^{n-1}(p-1)^2 \equiv p^{n-1}-2p^n \pmod{p^{n+1}} & \text{if } p-1\mid b,\\ p^{n-1}(p-1)\cdot pB_b - bp^n\cdot\left(-\dfrac{B_b}{b}\right) = p^{n+1}B_b \equiv 0 \pmod{p^{n+1}} & \text{if } p-1\nmid b. \end{cases} \end{aligned}$$

This completes the proof. □

**Theorem 6.4.** *Let $p$ be a prime greater than 3, $x\in\mathbb{Z}_p$, $n\in\mathbb{N}$, $n\not\equiv 0,1 \pmod{p-1}$ and $b\in\{0,1,2,\ldots\}$. Let $n_0$ be given by $n\equiv n_0 \pmod{p-1}$ and $n_0\in\{2,3,\ldots,p-2\}$. Let*

$$f(k) = pB_{k(p-1)+b}(x) - p^{k(p-1)+b}B_{k(p-1)+b}\left(\frac{x+\langle -x\rangle_p}{p}\right).$$

*Then for $k=0,1,2,\ldots$ we have*

$$\begin{aligned} f(k) \equiv{}& \sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}f(r) + \frac{n}{n_0}\cdot\frac{\sum_{s=0}^{n_0}\binom{n_0}{s}(-1)^s f(s)}{p^{n_0}}\binom{k}{n}(-p)^n\\ &+\chi(p-1\mid n+1)\chi(p-1\mid b)\left(\frac{(n+2)b}{2}\binom{k}{n}-\binom{k}{n+1}\right)(-p)^n \pmod{p^{n+1}}. \end{aligned}$$

**Proof.** From [17, Theorem 3.1] we have

$$\sum_{k=0}^{m}\binom{m}{k}(-1)^k f(k) \equiv p^{m-1}\chi(p-1\mid m)\chi(p-1\mid b) \pmod{p^m} \quad \text{for } m\in\mathbb{N}.$$

Thus applying [17, Lemma 2.1], Theorem 6.1, and the above we see that

$$\begin{aligned}
&f(k)-\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}f(r)\\
&\quad=\sum_{r=n}^{k}\binom{k}{r}(-1)^r\sum_{s=0}^{r}\binom{r}{s}(-1)^s f(s)\\
&\quad\equiv\binom{k}{n}(-1)^n\sum_{s=0}^{n}\binom{n}{s}(-1)^s f(s)+\binom{k}{n+1}(-1)^{n+1}\sum_{s=0}^{n+1}\binom{n+1}{s}(-1)^s f(s)\\
&\quad\equiv\binom{k}{n}(-1)^n p^n\left(\frac{n}{n_0}\cdot\frac{\sum_{s=0}^{n_0}\binom{n_0}{s}(-1)^s f(s)}{p^{n_0}}+\frac{(n+2)b}{2}\chi(p-1|n+1)\chi(p-1|b)\right)\\
&\qquad+\binom{k}{n+1}(-1)^{n+1}p^n\chi(p-1|n+1)\chi(p-1|b)\ (\mathrm{mod}\ p^{n+1}).
\end{aligned}$$

This yields the result. □

**Corollary 6.1.** *Let $k, n \in \mathbb{N}$.*

(i) *If $n \equiv 2\ (\mathrm{mod}\ 4)$, then*

$$(5-5^{4k})B_{4k}\equiv\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}(5-5^{4r})B_{4r}+3n\binom{k}{n}5^n\ (\mathrm{mod}\ 5^{n+1})$$

*and*

$$(5-5^{4k+2})B_{4k+2}\equiv\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}(5-5^{4r+2})B_{4r+2}-n\binom{k}{n}5^n\ (\mathrm{mod}\ 5^{n+1}).$$

(ii) *If $n \equiv 3\ (\mathrm{mod}\ 4)$, then*

$$(5-5^{4k})B_{4k}\equiv\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}(5-5^{4r})B_{4r}+\binom{k}{n+1}5^n\ (\mathrm{mod}\ 5^{n+1})$$

*and*

$$(5-5^{4k+2})B_{4k+2}\equiv\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}(5-5^{4r+2})B_{4r+2}+n\binom{k}{n}5^n\ (\mathrm{mod}\ 5^{n+1}).$$

## 7. Congruences for Euler numbers

We recall that the Euler numbers $\{E_n\}$ are given by

$$E_0=1,\quad E_{2n-1}=0\quad\text{and}\quad\sum_{r=0}^{n}\binom{2n}{2r}E_{2r}=0\ (n\geqslant 1).$$

The first few Euler numbers are shown below:

$$E_0=1,\quad E_2=-1,\quad E_4=5,\quad E_6=-61,\quad E_8=1385,\quad E_{10}=-50521,$$
$$E_{12}=2\,702\,765,\quad E_{14}=-199\,360\,981,\quad E_{16}=19\,391\,512\,145.$$

By (1.2) and (2.9) we have

$$\begin{aligned} E_{2n} &= 2^{2n} E_{2n}\left(\frac{1}{2}\right) = 2^{2n} \cdot \frac{2^{2n+1}}{2n+1}\left(B_{2n+1}\left(\frac{3}{4}\right) - B_{2n+1}\left(\frac{1}{4}\right)\right) \\ &= \frac{2^{4n+1}}{2n+1}\left(-B_{2n+1}\left(\frac{1}{4}\right) - B_{2n+1}\left(\frac{1}{4}\right)\right). \end{aligned}$$

That is,

$$E_{2n} = -4^{2n+1}\frac{B_{2n+1}\left(\frac{1}{4}\right)}{2n+1}. \tag{7.1}$$

**Lemma 7.1.** *Let $p$ be an odd prime and $b \in \{0, 2, 4, \ldots\}$. Then $f(k) = (1 - (-1)^{(p-1)/2} p^{k(p-1)+b}) E_{k(p-1)+b}$ is a $p$-regular function.*

**Proof.** As $p > 2$ and $2|b$ we see that $p-1\nmid b+1$. For $x \in \mathbb{Z}_p$, from Lemma 5.1(i) we know that $F(k)=(B_{k(p-1)+b+1}(x)-p^{k(p-1)+b}B_{k(p-1)+b+1}(x'))/(k(p-1)+b+1)$ is a $p$-regular function, where $x' = (x + \langle -x\rangle_p)/p$. It is clear that

$$\frac{\frac{1}{4} + \left\langle -\frac{1}{4}\right\rangle_p}{p} = \begin{cases} \frac{1}{p}\left(\frac{1}{4} + \frac{p-1}{4}\right) = \frac{1}{4} & \text{if } p \equiv 1 \pmod 4, \\ \frac{1}{p}\left(\frac{1}{4} + \frac{3p-1}{4}\right) = \frac{3}{4} & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Thus, using (2.9) we see that

$$B_{k(p-1)+b+1}\left(\frac{\frac{1}{4} + \left\langle -\frac{1}{4}\right\rangle_p}{p}\right) = B_{k(p-1)+b+1}\left(\left\{\frac{p}{4}\right\}\right) = (-1)^{(p-1)/2} B_{k(p-1)+b+1}\left(\frac{1}{4}\right).$$

Hence

$$\begin{aligned} g(k) &= (1 - (-1)^{(p-1)/2} p^{k(p-1)+b})\frac{B_{k(p-1)+b+1}\left(\frac{1}{4}\right)}{k(p-1)+b+1} \\ &= -4^{-(k(p-1)+b+1)}(1 - (-1)^{(p-1)/2} p^{k(p-1)+b}) E_{k(p-1)+b} \end{aligned}$$

is a $p$-regular function. For $n \in \mathbb{N}$ we see that

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k(-4^{k(p-1)+b+1}) = -4^{b+1}(1 - 4^{p-1})^n \equiv 0 \pmod{p^n}.$$

Namely, $-4^{k(p-1)+b+1}$ is a $p$-regular function. Hence, using [18, Theorem 2.3] we see that $f(k) = -4^{k(p-1)+b+1} g(k)$ is also a $p$-regular function. This proves the lemma.

From Lemma 7.1 and Theorem 4.3 we have:

**Theorem 7.1.** *Let $p$ be an odd prime, $k, m, n, t \in \mathbb{N}$ and $b \in \{0, 2, 4, \ldots\}$. Then*

$$\begin{aligned} &(1 - (-1)^{(p-1)/2} p^{ktp^{m-1}(p-1)+b}) E_{ktp^{m-1}(p-1)+b} \\ &\equiv \sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}(1 - (-1)^{(p-1)/2} p^{rtp^{m-1}(p-1)+b}) E_{rtp^{m-1}(p-1)+b} \pmod{p^{mn}}. \end{aligned}$$

Putting $n = 1, 2, 3$ and $t = 1$ in Theorem 7.1 we obtain the following result.

**Corollary 7.1.** *Let p be an odd prime*, $k, m \in \mathbb{N}$ *and* $b \in \{0, 2, 4, \ldots\}$. *Then*

(i) [2, p. 131] $E_{k\varphi(p^m)+b} \equiv (1-(-1)^{(p-1)/2}p^b)E_b \pmod{p^m}$.
(ii) $E_{k\varphi(p^m)+b} \equiv kE_{\varphi(p^m)+b} - (k-1)(1-(-1)^{(p-1)/2}p^b)E_b \pmod{p^{2m}}$.
(iii) *We have*

$$E_{k\varphi(p^m)+b} \equiv \frac{k(k-1)}{2}E_{2\varphi(p^m)+b} - k(k-2)(1-(-1)^{(p-1)/2}p^{\varphi(p^m)+b})E_{\varphi(p^m)+b} + \frac{(k-1)(k-2)}{2}(1-(-1)^{(p-1)/2}p^b)E_b \pmod{p^{3m}}.$$

From Lemma 7.1 and Corollary 4.2(iv) we have:

**Theorem 7.2.** *Let p be an odd prime*, $k, m \in \mathbb{N}$ *and* $b \in \{0, 2, 4, \ldots\}$. *Then*

$$E_{k\varphi(p^m)+b} \equiv (1-kp^{m-1})(1-(-1)^{(p-1)/2}p^b)E_b + kp^{m-1}E_{p-1+b} \pmod{p^{m+1}}.$$

**Corollary 7.2.** *Let p be an odd prime and* $k, m \in \mathbb{N}$. *Then*

$$E_{k\varphi(p^m)} \equiv \begin{cases} kp^{m-1}E_{p-1} \pmod{p^{m+1}} & \text{if } p \equiv 1 \pmod 4, \\ 2 + kp^{m-1}(E_{p-1}-2) \pmod{p^{m+1}} & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

From [18, Theorem 2.1] and Lemma 7.1 we have:

**Theorem 7.3.** *Let p be an odd prime*, $n \in \mathbb{N}$ *and* $b \in \{0, 2, 4, \ldots\}$. *Then there are integers* $a_0, a_1, \ldots, a_{n-1}$ *such that*

$$(1-(-1)^{(p-1)/2}p^{k(p-1)+b})E_{k(p-1)+b} \equiv a_{n-1}k^{n-1} + \cdots + a_1k + a_0 \pmod{p^n}$$

*for every* $k = 0, 1, 2, \ldots$. *Moreover, if* $p \geqslant n$, *then* $a_0, a_1, \ldots, a_{n-1} \pmod{p^n}$ *are uniquely determined.*

As examples, we have

$$(1+3^{2k})E_{2k} \equiv -12k + 2 \pmod{3^3}, \tag{7.2}$$

$$(1-5^{4k})E_{4k} \equiv -750k^3 + 1375k^2 - 620k \pmod{5^5}, \tag{7.3}$$

$$(1-5^{4k+2})E_{4k+2} \equiv 1000k^3 + 1500k^2 + 540k + 24 \pmod{5^5}. \tag{7.4}$$

**Theorem 7.4.** *Let* $n \in \mathbb{N}$ *and* $b \in \{0, 2, 4, \ldots\}$. *Suppose* $\alpha_n \in \mathbb{N}$ *and* $2^{\alpha_n - 1} \leqslant n < 2^{\alpha_n}$. *Then*

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k E_{2k+b} \equiv 0 \pmod{2^{2n-\alpha_n}}.$$

**Proof.** We first prove the result in the case $b = 0$. Taking $x = 0$ in (1.2) we find

$$\sum_{r=0}^{n} \binom{n}{r} (-1)^{n-r} E_r = \frac{2^{n+1}}{n+1}(B_{n+1} - 2^{n+1}B_{n+1}).$$

Thus applying the binomial inversion formula we have

$$E_n = \sum_{m=0}^{n} \binom{n}{m} \frac{2^{m+1}(1-2^{m+1})}{m+1} B_{m+1}.$$

Using this we see that

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}E_{2k} &= \sum_{k=0}^{n}\sum_{m=0}^{2k}\binom{n}{k}(-1)^{n-k}\binom{2k}{m}\frac{2^{m+1}(1-2^{m+1})}{m+1}B_{m+1}\\ &= \sum_{m=0}^{2n}\frac{2^{m+1}(1-2^{m+1})}{m+1}B_{m+1}\sum_{m/2\leqslant k\leqslant n}\binom{n}{k}(-1)^{n-k}\binom{2k}{m}\\ &= \sum_{m=1}^{2n}\frac{2^{m+1}(1-2^{m+1})}{m+1}B_{m+1}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\binom{2k}{m}.\end{aligned}$$

By Lemma 4.1 we have

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\binom{2k}{m} &= \frac{n!}{m!}\sum_{j=n}^{m}(-1)^{m-j}s(m,j)S(j,n)\cdot 2^{j}\\ &= \sum_{j=n}^{m}(-1)^{m-j}\frac{j!s(m,j)}{m!}2^{m-j}\cdot\frac{n!S(j,n)}{j!}2^{j-n}\cdot 2^{j+n-m}.\end{aligned}$$

Thus,

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}E_{2k}\\ &\quad= \sum_{m=1}^{2n}\frac{2^{m+1}(1-2^{m+1})}{m+1}B_{m+1}\sum_{j=n}^{m}(-1)^{m-j}\frac{j!s(m,j)}{m!}2^{m-j}\cdot\frac{n!S(j,n)}{j!}2^{j-n}\cdot 2^{j+n-m}\\ &\quad= \sum_{m=n}^{2n}\frac{2^{m+1}(1-2^{m+1})}{m+1}B_{m+1}\sum_{j=n}^{m}(-1)^{m-j}\frac{j!s(m,j)}{m!}2^{m-j}\cdot\frac{n!S(j,n)}{j!}2^{j-n}\cdot 2^{j+n-m}.\end{aligned}$$

It is well known that $2B_k\in\mathbb{Z}_2$. Suppose $2^{\mathrm{ord}_2(m+1)}\|m+1$. We then have

$$\frac{1}{2^{m-\mathrm{ord}_2(m+1)}}\cdot\frac{2^{m+1}B_{m+1}}{m+1}=\frac{2B_{m+1}}{2^{-\mathrm{ord}_2(m+1)}(m+1)}\in\mathbb{Z}_2.$$

On the other hand, by Lemma 4.2 we have $\frac{j!s(m,j)}{m!}2^{m-j}\in\mathbb{Z}_2$ and $\frac{n!S(j,n)}{j!}2^{j-n}\in\mathbb{Z}_2$. Hence, if $n\leqslant j\leqslant m\leqslant 2n$, then

$$\begin{aligned}&\frac{2^{m+1}(1-2^{m+1})}{m+1}B_{m+1}\cdot(-1)^{m-j}\frac{j!s(m,j)}{m!}2^{m-j}\cdot\frac{n!S(j,n)}{j!}2^{j-n}\cdot 2^{j+n-m}\\ &\quad\equiv 0\ (\mathrm{mod}\ 2^{j+n-\mathrm{ord}_2(m+1)}).\end{aligned}$$

When $n\leqslant j\leqslant m\leqslant 2n$, we also have $m+1<2(n+1)\leqslant 2^{\alpha_n+1}$ and so $\mathrm{ord}_2(m+1)\leqslant\alpha_n$, thus $j+n-\mathrm{ord}_2(m+1)\geqslant j+n-\alpha_n\geqslant 2n-\alpha_n$. Therefore, by the above we obtain $\sum_{k=0}^{n}\binom{n}{k}(-1)^k E_{2k}\equiv 0\ (\mathrm{mod}\ 2^{2n-\alpha_n})$. So the result holds for $b=0$.

From [18, (2.5)] we know that for any function $f$,

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k f(k+m)=\sum_{k=0}^{m}\binom{m}{k}(-1)^k\sum_{r=0}^{k+n}\binom{k+n}{r}(-1)^r f(r). \tag{7.5}$$

Thus,

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k E_{2k+b}=\sum_{k=0}^{b/2}\binom{\frac{b}{2}}{k}(-1)^k\sum_{r=0}^{k+n}\binom{k+n}{r}(-1)^r E_{2r}. \tag{7.6}$$

As $\alpha_{s+1}=\alpha_s$ or $\alpha_s+1$, we see that $2(s+1)-\alpha_{s+1}\geqslant 2s-\alpha_s$ and hence $2r-\alpha_r\geqslant 2s-\alpha_s$ for $r\geqslant s$. As the result holds for $b=0$ we have

$$\sum_{r=0}^{k+n}\binom{k+n}{r}(-1)^r E_{2r}\equiv 0\ (\mathrm{mod}\ 2^{2(k+n)-\alpha_{k+n}}).$$

Since $2(k+n)-\alpha_{k+n}\geqslant 2n-\alpha_n$, we must have $\sum_{r=0}^{k+n}\binom{k+n}{r}(-1)^r E_{2r}\equiv 0\ (\mathrm{mod}\ 2^{2n-\alpha_n})$. Hence applying (7.6) we obtain

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k E_{2k+b}\equiv 0\ (\mathrm{mod}\ 2^{2n-\alpha_n}).$$

This proves the theorem. □

**Corollary 7.3.** *Let $n\in\mathbb{N}$ and $b\in\{0,2,4,\ldots\}$. Then*

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k E_{2k+b}\equiv\begin{cases}2\ (\mathrm{mod}\ 4) & \text{if } n=1,\\ 0\ (\mathrm{mod}\ 2^{n+1}) & \text{if } n>1\end{cases}$$

*and thus $f(k)=E_{2k+b}$ is a 2-regular function.*

**Proof.** Suppose $\alpha_n\in\mathbb{N}$ and $2^{\alpha_n-1}\leqslant n<2^{\alpha_n}$. By Theorem 7.4 we have

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k E_{2k+b}\equiv 0\ (\mathrm{mod}\ 2^{2n-\alpha_n}).$$

If $\alpha_n\geqslant n$, then $2^{n-1}\leqslant 2^{\alpha_n-1}\leqslant n$. For $n\geqslant 3$ we have $2^{n-1}>n$, thus $\alpha_n<n$ and hence $2n-\alpha_n\geqslant n+1$. Therefore, for $n\geqslant 3$ we have $\sum_{k=0}^{n}\binom{n}{k}(-1)^k E_{2k+b}\equiv 0\ (\mathrm{mod}\ 2^{n+1})$. As $E_0-E_2=1-(-1)=2$ and $E_0-2E_2+E_4=1-2(-1)+5=8$, applying (7.6) and the above we see that $E_b-E_{b+2}\equiv E_0-E_2=2\ (\mathrm{mod}\ 8)$ and $E_b-2E_{b+2}+E_{b+4}\equiv 0\ (\mathrm{mod}\ 8)$. So the result follows. □

**Theorem 7.5.** *Suppose $k,m,n,t\in\mathbb{N}$ and $b\in\{0,2,4,\ldots\}$. For $s\in\mathbb{N}$ let $\alpha_s\in\mathbb{N}$ be given by $2^{\alpha_s-1}\leqslant s<2^{\alpha_s}$ and let $e_s=2^{-s}\sum_{r=0}^{s}\binom{s}{r}(-1)^r E_{2r}$. Then*

$$E_{2^m kt+b}\equiv\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}E_{2^m rt+b}+2^{mn}\binom{k}{n}(-t)^n e_n\ (\mathrm{mod}\ 2^{mn+n+1-\alpha_{n+1}}).$$

*Moreover, for $m\geqslant 2$ we have*

$$E_{2^m kt+b}\equiv\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}E_{2^m rt+b}$$
$$+2^{mn}\binom{k}{n}(-t)^n\left(e_n+ne_{n+1}+\frac{n(n-1)}{2}e_{n+2}\right)\ (\mathrm{mod}\ 2^{mn+n+2-\alpha_{n+1}}).$$

**Proof.** For $s\in\mathbb{N}$ set $A_s=2^{-s}\sum_{r=0}^{s}\binom{s}{r}(-1)^r E_{2r+b}$. Since $\alpha_s\leqslant s$, by Theorem 7.4 we have $A_s\in\mathbb{Z}_2$ and $2^{s-\alpha_s}\mid A_s$. As $\alpha_{s+1}\leqslant\alpha_s+1$ we have $s+1-\alpha_{s+1}\geqslant s-\alpha_s$ and hence $r-\alpha_r\geqslant s-\alpha_s$ for $r\geqslant s$. Therefore $2^s-\alpha_s\mid A_r$ for $r\geqslant s$.

As $1+\alpha_{n+1}\geqslant\alpha_{n+3}$ we see that $n+3-\alpha_{n+3}\geqslant n+2-\alpha_{n+1}$ and thus $2^{n+2-\alpha_{n+1}}\mid A_r$ for $r\geqslant n+3$. By (7.6) we have

$$A_n=\sum_{k=0}^{b/2}\binom{\frac{b}{2}}{k}(-1)^k2^ke_{k+n}.$$

Since $2^{n+2-\alpha_{n+1}}\mid e_r$ for $r\geqslant n+3$, $2^{n+2-\alpha_{n+1}}\mid 2e_{n+1}$ and $2^{n+2-\alpha_{n+1}}\mid 2^2e_{n+2}$, we see that $A_n\equiv e_n \pmod{2^{n+2-\alpha_{n+1}}}$.

From Corollary 7.3 and the proof of Theorem 4.2 we know that

$$\begin{aligned}&\sum_{r=0}^{n}\binom{n}{r}(-1)^rE_{2\cdot 2^{m-1}rt+b}\\ &\quad=A_nt^n\cdot 2^{mn}+\sum_{r=n+1}^{2^{m-1}nt}(-2)^n(-1)^rA_r\left(\frac{(-1)^{r-n}s(r,n)n!}{r!}2^{r-n}\cdot 2^{(m-1)n}t^n\right.\\ &\qquad\left.+\sum_{j=n+1}^{r}\frac{(-1)^{r-j}s(r,j)j!}{r!}2^{r-j}\cdot\frac{S(j,n)n!}{j!}2^{j-n}\cdot(2^{m-1}t)^j\right).\end{aligned}$$

By Lemma 4.2, for $n+1\leqslant j\leqslant r$ we have

$$\frac{s(r,j)j!}{r!}2^{r-j},\ \frac{S(j,n)n!}{j!}2^{j-n}\in\mathbb{Z}_2\quad\text{and}\quad\frac{s(r,n)n!}{r!}2^{r-n}\equiv\binom{n}{r-n}\pmod 2.$$

As $2^{n+1-\alpha_{n+1}}\mid A_r$ for $r\geqslant n+1$, by the above we obtain

$$\sum_{r=0}^{n}\binom{n}{r}(-1)^rE_{2^mrt+b}\equiv 2^{mn}A_nt^n\equiv 2^{mn}t^ne_n \pmod{2^{mn+n+1-\alpha_{n+1}}}\tag{7.7}$$

and so

$$\sum_{r=0}^{n}\binom{n}{r}(-1)^rE_{2^mrt+b}\equiv 0 \pmod{2^{mn+n-\alpha_n}}.\tag{7.8}$$

For $r\geqslant n+1$ we have $mr+r-\alpha_r\geqslant m(n+1)+n+1-\alpha_{n+1}\geqslant mn+n+2-\alpha_{n+1}$. Thus, if $r\geqslant n+1$, by (7.8) we have

$$\sum_{s=0}^{r}\binom{r}{s}(-1)^sE_{2^mst+b}\equiv 0 \pmod{2^{mn+n+2-\alpha_{n+1}}}.\tag{7.9}$$

By (4.5) we have

$$E_{2^mkt+b}=\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}E_{2^mrt+b}+\sum_{r=n}^{k}\binom{k}{r}(-1)^r\sum_{s=0}^{r}\binom{r}{s}(-1)^sE_{2^mst+b}.$$

Hence, applying (7.9) we obtain

$$\begin{aligned}&E_{2^mkt+b}-\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}E_{2^mrt+b}\\ &\quad\equiv\binom{k}{n}(-1)^n\sum_{s=0}^{n}\binom{n}{s}(-1)^sE_{2^mst+b} \pmod{2^{mn+n+2-\alpha_{n+1}}}.\end{aligned}\tag{7.10}$$

In view of (7.7), we get

$$E_{2^m kt+b} \equiv \sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}E_{2^m rt+b}+\binom{k}{n}(-1)^n\cdot 2^{mn}t^n e_n \pmod{2^{mn+n+1-\alpha_{n+1}}}.$$

Now assume $m\geqslant 2$. Then $(m-1)(n+1)+n\geqslant mn+1$. From the above we see that

$$\begin{aligned}&\sum_{r=0}^{n}\binom{n}{r}(-1)^r E_{2^m rt+b}\\ &\quad\equiv 2^{mn}A_n t^n+\sum_{r=n+1}^{2^{m-1}nt}(-2)^n(-1)^r A_r\cdot\frac{(-1)^{r-n}s(r,n)n!}{r!}2^{r-n}\cdot 2^{(m-1)n}t^n\\ &\quad\equiv 2^{mn}t^n\left(A_n+\sum_{r=n+1}^{2^{m-1}nt}\binom{n}{r-n}A_r\right)\equiv 2^{mn}t^n\sum_{r=n}^{n+2}\binom{n}{r-n}A_r\\ &\quad\equiv 2^{mn}t^n\left(e_n+ne_{n+1}+\binom{n}{2}e_{n+2}\right)\pmod{2^{mn+n+2-\alpha_{n+1}}}.\end{aligned}$$

This together with (7.10) yields the remaining result. Hence the proof is complete. □

As $2^{n-\alpha_n}\mid e_n$ and $n+1-\alpha_{n+1}\geqslant n-\alpha_n$, by Theorem 7.5 we have:

**Corollary 7.4.** *Let $k,m,n,t\in\mathbb{N}$ and $b\in\{0,2,4,\ldots\}$. Let $\alpha\in\mathbb{N}$ be given by $2^{\alpha-1}\leqslant n<2^{\alpha}$. Then*

$$E_{2^m kt+b}\equiv\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}E_{2^m rt+b}\pmod{2^{mn+n-\alpha}}.$$

**Corollary 7.5.** *Let $k,m\in\mathbb{N}$ and $b\in\{0,2,4,\ldots\}$. Then*

$$E_{2^m k+b}\equiv 2^m k+E_b\pmod{2^{m+1}}.$$

**Proof.** Observe that $e_1=1$ and $e_2=2$. For $m\geqslant 2$, taking $n=t=1$ in Theorem 7.5 we obtain

$$E_{2^m k+b}\equiv E_b+2^m(-k)(e_1+e_2)\equiv 2^m k+E_b\pmod{2^{m+1}}.$$

So the result holds for $m\geqslant 2$. Now taking $m=2$ and $b=0,2$ in the congruence we see that $E_{4k}\equiv 1+4k\pmod 8$ and $E_{4k+2}\equiv -1+4k\pmod 8$. Hence $E_{2k}\equiv(-1)^k\pmod 4$ and so $E_{2k+b}\equiv(-1)^{k+b/2}\equiv(-1)^{b/2}+2k\equiv E_b+2k\pmod 4$. So the result is also true for $m=1$. This completes the proof. □

**Remark 7.1.** Corollary 7.5 is equivalent to the following Stern's result (see [13]):

$$2^m\,\|\,E_{n_1}-E_{n_2}\Leftrightarrow 2^m\,\|\,n_1-n_2.$$

Putting $n=2$, $t=1$ in Theorem 7.5 and noting that $e_2=2$, $e_3=10$, $e_4=104$ we obtain the following result.

**Corollary 7.6.** *Let $k,m\in\mathbb{N}$, $m\geqslant 2$ and $b\in\{0,2,4,\ldots\}$. Then*

$$E_{2^m k+b}\equiv kE_{2^m+b}-(k-1)E_b+2^{2m}k(k-1)\pmod{2^{2m+2}}.$$

Taking $m = 2$ and $b = 0, 2$ in Corollary 7.6 we get:

**Corollary 7.7.** *For $k \in \mathbb{N}$ we have*

$$E_{4k} \equiv \begin{cases} 4k + 1 \pmod{64} & \text{if } k \equiv 0, 1 \pmod 4, \\ 4k + 33 \pmod{64} & \text{if } k \equiv 2, 3 \pmod 4 \end{cases}$$

*and*

$$E_{4k+2} \equiv \begin{cases} 4k - 1 \pmod{64} & \text{if } k \equiv 0, 1 \pmod 4, \\ 4k - 33 \pmod{64} & \text{if } k \equiv 2, 3 \pmod 4. \end{cases}$$

**Corollary 7.8.** *Let $k, m \in \mathbb{N}$, $m \geqslant 2$ and $b \in \{0, 2, 4, \ldots\}$. Let $\delta_k = 0$ or $1$ according as $4 \nmid k - 3$ or $4 \mid k - 3$. Then*

$$E_{2^m k+b} \equiv \binom{k}{2} E_{2^{m+1}+b} - k(k-2)E_{2^m+b} + \binom{k-1}{2} E_b + 2^{3m+1}\delta_k \pmod{2^{3m+2}}.$$

**Proof.** Observe that $e_3 = 10$, $e_4 = 104$, $e_5 = 1816$ and $\binom{k}{3} \equiv \delta_k \pmod 2$. Taking $n = 3$ and $t = 1$ in Theorem 7.5 we obtain the result. □

Taking $m = 2$, $b = 0, 2$ in Corollary 7.8 and noting that $E_8 \equiv 105 \pmod{256}$, $E_{10} \equiv -89 \pmod{256}$ we deduce:

**Corollary 7.9.** *Let $k \in \mathbb{N}$ and $\delta_k = 0$ or $1$ according to $4 \nmid k - 3$ or $4 \mid k - 3$. Then*

$$E_{4k} \equiv 48k^2 - 44k + 1 + 128\delta_k \pmod{256} \quad \text{and} \quad E_{4k+2} \equiv 16k^2 - 76k - 1 + 128\delta_k \pmod{256}.$$

**Remark 7.2.** Let $\{S_n\}$ be given by (3.1). From Remark 3.1 we know that $(-1)^k S_k$ is a 2-regular function and hence $f(k) = (-1)^{k+b} S_{k+b}$ is also a 2-regular function, where $b \in \{0, 1, 2, \ldots\}$. Thus, by Corollary 4.2, for $m \geqslant 2$, $k \geqslant 1$ and $b \geqslant 0$ we have $S_{2^{m-1}k+b} \equiv S_b \pmod{2^m}$ and $S_{2^{m-1}k+b} \equiv S_b - 2^{m-2}k(S_{b+2} + 4S_{b+1} + 3S_b) \pmod{2^{m+1}}$.

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