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Some Generalizations and Basic (or q-) Extensions of the Bernoulli, Euler and Genocchi Polynomials

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In the vast literature in *Analytic Number Theory*, one can find systematic and extensive investigations not only of the classical Bernoulli, Euler and Genocchi polynomials and their corresponding numbers, but also of their many generalizations and basic (or q-) extensions. Our main object in this presentation is to introduce and investigate some of the principal generalizations and unifications of each of these polynomials by means of suitable generating functions. We present several interesting properties of these general polynomial systems including some explicit series representations in terms of the Hurwitz (or generalized) zeta function and the familiar Gauss hypergeometric function. By introducing an analogue of the Stirling numbers of the second kind, that is, the so-called λ-Stirling numbers of the second kind, we derive several properties and formulas and consider some of their interesting applications to the family of the Apostol type polynomials. We also give a brief expository and historial account of the various basic (or q-) extensions of the classical Bernoulli polynomials and numbers, the classical Euler polynomials and numbers, the classical Genocchi polynomials and numbers, and also of their such generalizations as (for example) the above-mentioned families of the Apostol type polynomials and numbers. Relevant connections of the definitions and results presented here with those in earlier as well as forthcoming investigations will be indicated.

Keywords: Bernoulli polynomials and numbers; Euler polynomials and numbers; Taylor-Maclaurin series expansion; Basic (or q-) extensions.

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1 Introduction, Definitions and Motivation

Throughout this presentation, we use the following standard notations: $\mathbb{N} := \{1,2,3,\cdots\}$, $\mathbb{N}_0 := \{0,1,2,3,\cdots\} = \mathbb{N} \cup \{0\}$ and $\mathbb{Z}^- := \{-1,-2,-3,\cdots\} = \mathbb{Z}_0^- \setminus \{0\}$. Also, as usual, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers. Furthermore,

$$\{\lambda\}_0 = 1 \quad \text{and} \quad \{\lambda\}_k = \lambda(\lambda-1)\cdots(\lambda-k+1) \qquad (k \in \mathbb{N}_0;\ \lambda \in \mathbb{C})$$

denotes the *falling* factorial and

$$(\lambda)_0 = 1 \quad \text{and} \quad (\lambda)_k = \lambda(\lambda+1)\cdots(\lambda+k-1) \qquad (k \in \mathbb{N}_0;\ \lambda \in \mathbb{C})$$

denotes the *rising* factorial.

The classical Bernoulli polynomials $B_n(x)$, the classical Euler polynomials $E_n(x)$ and the classical Genocchi polynomials $G_n(x)$, together with their familiar generalizations $B_n^{(\alpha)}(x)$, $E_n^{(\alpha)}(x)$ and $G_n^{(\alpha)}(x)$ of (real or complex) order α, are usually defined by means of the following generating functions (see, for details, [62, p. 532–533] and [68, p. 61 *et seq.*]; see also [72, p. 397, Problem 27] and [73] and the references cited therein):

$$\left(\frac{z}{e^z-1}\right)^{\alpha} \cdot e^{xz} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\,\frac{z^n}{n!} \qquad (|z| < 2\pi;\ 1^{\alpha} := 1), \tag{1.1}$$

$$\left(\frac{2}{e^z+1}\right)^{\alpha} \cdot e^{xz} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x)\,\frac{z^n}{n!} \qquad (|z| < \pi;\ 1^{\alpha} := 1) \tag{1.2}$$

and

$$\left(\frac{2z}{e^z+1}\right)^{\alpha} \cdot e^{xz} = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x)\,\frac{z^n}{n!} \qquad (|z| < \pi; 1^{\alpha} := 1), \tag{1.3}$$

so that, obviously, the classical Bernoulli polynomials $B_n(x)$, the classical Euler polynomials $E_n(x)$ and the classical Genocchi polynomials $G_n(x)$ are given, respectively, by

$$B_n(x) := B_n^{(1)}(x),\ E_n(x) := E_n^{(1)}(x) \text{ and } G_n(x) := G_n^{(1)}(x) \qquad (n \in \mathbb{N}_0). \tag{1.4}$$

For the classical Bernoulli numbers B_n, the classical Euler numbers E_n and the classical Genocchi numbers G_n of order n, we have

$$B_n := B_n(0) = B_n^{(1)}(0),$$
$$E_n := E_n(0) = E_n^{(1)}(0),$$
$$G_n := G_n(0) = G_n^{(1)}(0),$$

respectively.

Some interesting analogues of the classical Bernoulli polynomials and numbers were first investigated by Apostol [2, p. 165, Eq. (3.1)] and (more recently) by Srivastava [66, pp. 83–84]. We begin by recalling here Apostol's definitions as follows.

Definition 1 (Apostol [2]; see also Srivastava [66]). The Apostol-Bernoulli polynomials

$$\mathcal{B}_n(x;\lambda) \quad (\lambda \in \mathbb{C})$$

are defined by means of the following generating function:

$$\frac{ze^{xz}}{\lambda e^z - 1} = \sum_{n=0}^{\infty} \mathcal{B}_n(x;\lambda)\,\frac{z^n}{n!} \tag{1.5}$$

$$(|z| < 2\pi \quad upwhen \quad \lambda = 1;\ |z| < |\log \lambda| \quad upwhen \quad \lambda \neq 1)$$

with, of course,

$$B_n(x) = \mathcal{B}_n(x;1) \qquad \text{and} \qquad \mathcal{B}_n(\lambda) := \mathcal{B}_n(0;\lambda)\,, \tag{1.6}$$

where $\mathcal{B}_n(\lambda)$ denotes the so-called Apostol-Bernoulli numbers.

Apostol [2] not only gave elementary properties of the polynomials $\mathcal{B}_n(x;\lambda)$, but also obtained the following interesting recursion formula for the numbers $\mathcal{B}_n(\lambda)$ (see [2, p. 166, Eq. (3.7)]):

$$\mathcal{B}_n(\lambda) = n\sum_{k=0}^{n-1} \frac{k!\,(-\lambda)^k}{(\lambda-1)^{k+1}}\,S(n-1,k) \qquad \big(n \in \mathbb{N}_0;\ \lambda \in \mathbb{C}\setminus\{1\}\big), \tag{1.7}$$

where $S(n,k)$ denotes the Stirling numbers of the second kind defined by means of the following expansion (see [15, p. 207, Theorem B]):

$$x^n = \sum_{k=0}^{n} \binom{x}{k} k!\,S(n,k), \tag{1.8}$$

so that

$$S(n,0) = \delta_{n,0},\ S(n,1) = S(n,n) = 1 \text{ and } S(n,n-1) = \binom{n}{2}, \tag{1.9}$$

$\delta_{n,k}$ being the Kronecker symbol.

Recently, Luo and Srivastava [52] further extended the Apostol-Bernoulli polynomials as the so-called Apostol-Bernoulli polynomials of order α. Luo [45], on the other hand, gave an analogous extension of the generalized Euler polynomials as the so-called Apostol-Euler polynomials of order α.

Definition 2 (*cf.* Luo and Srivastava [52]). The Apostol-Bernoulli polynomials

$$\mathcal{B}_n^{(\alpha)}(x;\lambda) \quad (\lambda \in \mathbb{C})$$

of (real or complex) order α are defined by means of the following generating function:

$$\left(\frac{z}{\lambda e^z - 1}\right)^{\alpha} \cdot e^{xz} = \sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha)}(x;\lambda)\,\frac{z^n}{n!} \tag{1.10}$$

$$(|z| < 2\pi \quad upwhen \quad \lambda = 1;\ |z| < |\log \lambda| \quad upwhen \quad \lambda \neq 1)$$

with, of course,

$$B_n^{(\alpha)}(x) = \mathcal{B}_n^{(\alpha)}(x;1) \qquad \text{and} \qquad \mathcal{B}_n^{(\alpha)}(\lambda) := \mathcal{B}_n^{(\alpha)}(0;\lambda), \tag{1.11}$$

where $\mathcal{B}_n^{(\alpha)}(\lambda)$ denotes the so-called Apostol-Bernoulli numbers of order α.

Definition 3 (*cf.* Luo [45]). The Apostol-Euler polynomials

$$\mathcal{E}_n^{(\alpha)}(x;\lambda) \quad (\lambda \in \mathbb{C})$$

of (real or complex) order α are defined by means of the following generating function:

$$\left(\frac{2}{\lambda e^z + 1}\right)^{\alpha} \cdot e^{xz} = \sum_{n=0}^{\infty} \mathcal{E}_n^{(\alpha)}(x;\lambda)\,\frac{z^n}{n!} \qquad \big(|z| < |\log(-\lambda)|\big) \tag{1.12}$$

with, of course,

$$E_n^{(\alpha)}(x) = \mathcal{E}_n^{(\alpha)}(x;1) \text{ and } \mathcal{E}_n^{(\alpha)}(\lambda) := \mathcal{E}_n^{(\alpha)}(0;\lambda), \tag{1.13}$$

where $\mathcal{E}_n^{(\alpha)}(\lambda)$ denotes the so-called Apostol-Euler numbers of order α.

Remark 1. The constraints on $|z|$, which we have used in Definitions 1, 2 and 3 above, are meant to ensure that the generating functions in (1.5), (1.10) and (1.12) are *analytic* throughout the prescribed *open* disks in the complex z-plane (centred at the origin $z = 0$) in order to have the corresponding convergent Taylor-Maclaurin series expansions (about the origin $z = 0$) occurring on their right-hand sides (each with a *positive* radius of convergence). Moreover, throughout this investigation, $\log z$ is tacitly assumed to denote the *principal* branch of the many-valued function $\log z$ *with* the imaginary part $\Im\big(\log z\big)$ constrained by $-\pi < \Im\big(\log z\big) \leqq \pi$. More importantly, throughout this presentation, wherever $|\log \lambda|$ and $|\log(-\lambda)|$ appear as the radii of the open disks in the complex z-plane (centred at the origin $z = 0$) in which the defining generating functions are analytic, it is *tacitly* assumed that the obviously exceptional cases when $\lambda = 1$ and $\lambda = -1$, respectively, are to be treated *separately*. Naturally, therefore, the corresponding constraints on $|z|$ in the earlier investigations (see, for example, [45], [52], [53] and [66]) should also be modified accordingly.

Remark 2. The *classical* Euler numbers $\widetilde{E}_n$ are usually defined by means of the following generating function (see, for example, [68, p. 64, Eq. 1.6 (40)]):

$$\frac{2e^z}{e^{2z}+1} = \operatorname{sech} z = \sum_{n=0}^{\infty} \widetilde{E}_n\,\frac{z^n}{n!} \qquad \left(|z| < \frac{\pi}{2}\right), \tag{1.14}$$

which, when compared with the generating function in (1.2), yields the following relationships [*cf.* Equation (1)]:

$$\widetilde{E}_n = 2^n\ E_n\left(\frac{1}{2}\right) = 2^n\ E_n^{(1)}\left(\frac{1}{2}\right) \tag{1.15}$$

with the Euler numbers E_n and the Euler polynomials $E_n^{(\alpha)}(x)$ used in this paper. For the Apostol-Euler numbers $\widetilde{\mathcal{E}}_n^{(\alpha)}(\lambda)\quad(\lambda\in\mathbb{C})$ of order α, which correspond to the *classical* Euler numbers $\widetilde{E}_n$, Luo [45] made use of the following definition:

$$\left(\frac{2e^z}{\lambda e^{2z}+1}\right)^{\alpha} = \sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_n^{(\alpha)}(\lambda)\ \frac{z^n}{n!} \qquad \left(|z| < \frac{1}{2}\ |\log(-\lambda)|\right). \tag{1.16}$$

However, for the sake of simplicity of the results presented in this paper, we find it to be convenient to use the Apostol-Euler numbers $\mathcal{E}_n^{(\alpha)}(\lambda)\quad(\lambda\in\mathbb{C})$ of order α, corresponding to the Euler numbers E_n, which are defined by means of the following generating function [*cf.* Equation (1.13)]:

$$\left(\frac{2}{\lambda e^{z}+1}\right)^{\alpha} = \sum_{n=0}^{\infty} \mathcal{E}_n^{(\alpha)}(\lambda)\ \frac{z^n}{n!} \qquad \left(|z| < |\log(-\lambda)|\right). \tag{1.17}$$

Of course, if and when it is needed, the interested reader will find it to be fairly straightforward to apply the following *explicit* relationships between the Apostol-Euler numbers

$$\mathcal{E}_n^{(\alpha)}(\lambda)\quad(\lambda\in\mathbb{C}) \qquad \text{and} \qquad \widetilde{\mathcal{E}}_n^{(\alpha)}(\lambda)\quad(\lambda\in\mathbb{C})$$

in order to convert any of these results into their desired forms.

$\mathcal{E}_n^{(\alpha)}(\lambda) = E_n^{(\alpha)}(0;\lambda)$	$\widetilde{\mathcal{E}}_n^{(\alpha)}(\lambda) = 2^n\ E_n^{(\alpha)}\left(\frac{\alpha}{2};\lambda\right)$	$\widetilde{\mathcal{E}}_n^{(\alpha)}(\lambda) = \sum_{k=0}^{n}\binom{n}{k}2^k\ \alpha^{n-k}\ \mathcal{E}_k^{(\alpha)}(\lambda)$
$\mathcal{E}_n(\lambda) = E_n(0;\lambda)$	$\widetilde{\mathcal{E}}_n(\lambda) = 2^n\ E_n\left(\frac{1}{2};\lambda\right)$	$\widetilde{\mathcal{E}}_n(\lambda) = \sum_{k=0}^{n}\binom{n}{k}2^k\ \mathcal{E}_k(\lambda)$
$E_n^{(\alpha)} = E_n^{(\alpha)}(0)$	$\widetilde{E}_n^{(\alpha)} = 2^n\ E_n^{(\alpha)}\left(\frac{\alpha}{2}\right)$	$\widetilde{E}_n^{(\alpha)} = \sum_{k=0}^{n}\binom{n}{k}2^k\ \alpha^{n-k}\ E_k^{(\alpha)}$
$E_n = E_n(0)$	$\widetilde{E}_n = 2^n\ E_n\left(\frac{1}{2}\right)$	$\widetilde{E}_n = \sum_{k=0}^{n}\binom{n}{k}2^k\ E_k$

Since the publication of the works by Luo and Srivastava (see [44], [45], [52], and [53]), many further investigations of the above-mentioned Apostol type polynomials have appeared in the literature. Boyadzhiev [4] gave some properties and representations of the Apostol-Bernoulli polynomials and the Eulerian polynomials. Garg *et al.* [17] studied the Apostol-Bernoulli polynomials of order α and obtained some new relations and formulas involving the Apostol type polynomials and the Hurwitz (or generalized) zeta function $\zeta(s,a)$ defined by (1.20) below. Luo (see [46] and [47]) obtained the Fourier expansions and integral representations for the Apostol-Bernoulli and the Apostol-Euler polynomials, and gave the multiplication formulas for the Apostol-Bernoulli and the Apostol-Euler polynomials of order α. Prévost [59] investigated the Apostol-Bernoulli and the Apostol-Euler polynomials by using the Padé approximation methods. Wang *et al.* (see [78] and [79]) further developed some results of Luo and Srivastava [53] and obtained some formulas involving power sums of the Apostol type polynomials. Zhang and Yang [81] gave several identities for the generalized Apostol-Bernoulli polynomials. On the other hand, Cenkci and Can [8] gave a q-analogue of the Apostol-Bernoulli polynomials $\mathcal{B}_n(x;\lambda)$. Choi *et al.* [11] gave the q-extensions of the Apostol-Bernoulli polynomials of order α and the Apostol-Euler polynomials of order α (see also [12]). Hwang *et al.* [24] and Kim *et al.* [35] also gave q-extensions of Apostol's type Euler polynomials.

On the subject of the Genocchi polynomials $G_n(x)$ and their various extensions, a remarkably large number of investigations have appeared in the literature (see, for example, [9], [12], [21], [22], [23], [25], [28], [30], [34], [38], [39], [40], [43], [41] [48], [49], [50], [58] and [80]; see also the references cited in each of these works). Moreover, Luo (see [48] and [50]) introduced and investigated the Apostol-Genocchi polynomials of a (real or complex) order α, which are defined as follows.

Definition 4. The Apostol-Genocchi polynomials

$$\mathcal{G}_n^{(\alpha)}(x;\lambda) \quad (\lambda \in \mathbb{C})$$

of (real or complex) order α are defined by means of the following generating function:

$$\left(\frac{2z}{\lambda e^z+1}\right)^{\alpha} \cdot e^{xz} = \sum_{n=0}^{\infty} \mathcal{G}_n^{(\alpha)}(x;\lambda)\frac{z^n}{n!} \qquad \big(|z| < |\log(-\lambda)|\big) \tag{1.18}$$

with, of course,

$$\begin{aligned} G_n^{(\alpha)}(x) = \mathcal{G}_n^{(\alpha)}(x;1), &\qquad \mathcal{G}_n^{(\alpha)}(\lambda) := \mathcal{G}_n^{(\alpha)}(0;\lambda), \\ \mathcal{G}_n(x;\lambda) := \mathcal{G}_n^{(1)}(x;\lambda) \qquad \text{and} &\qquad \mathcal{G}_n(\lambda) := \mathcal{G}_n^{(1)}(\lambda), \end{aligned} \tag{1.19}$$

where $\mathcal{G}_n(\lambda)$, $\mathcal{G}_n^{(\alpha)}(\lambda)$ and $\mathcal{G}_n(x;\lambda)$ denote the so-called Apostol-Genocchi numbers, the Apostol-Genocchi numbers of order α and the Apostol-Genocchi polynomials, respectively.

The main object of this presentation is to first present some elementary properties of the Apostol-Genocchi polynomials $\mathcal{G}_n^{(\alpha)}(x;\lambda)$ of order α in Section 2. We derive several explicit series representations of $\mathcal{G}_n^{(\alpha)}(x;\lambda)$ in terms of the Gaussian hypergeometric function in Section 3. We find some relationships between the various Apostol type polynomials in Section 4. We obtain the series representations for the Apostol type polynomials involving the Hurwitz (or generalized) zeta function $\zeta(s,a)$ in Section 5. We introduce the λ-Stirling numbers $\mathcal{S}(n,k;\lambda)$ of the second kind, which aid us to prove some basic properties and formulas in Section 6 in which we also pose two interesting *open* problems related to our present investigation. Finally, in Section 7, we give some interesting applications of the λ-Stirling numbers $\mathcal{S}(n,k;\lambda)$ of the second kind to the family of the Apostol type polynomials. For example, by closely following the work of Srivastava [66] dealing with the special case $\alpha = 1$, we will derive various explicit series representations for

$$\mathcal{G}_n^{(\alpha)}\left(\frac{p}{q}; e^{2\pi i \xi}\right), \quad \mathcal{G}_n^{(l)}\left(\frac{p}{q}; e^{2\pi i \xi}\right), \quad \mathcal{E}_n^{(\alpha)}\left(\frac{p}{q}; e^{2\pi i \xi}\right) \text{ and } \mathcal{E}_n^{(l)}\left(\frac{p}{q}; e^{2\pi i \xi}\right)$$
$$(q, l \in \mathbb{N};\ p \in \mathbb{Z};\ \xi \in \mathbb{R};\ \alpha \in \mathbb{C}),$$

involving *either* the Stirling numbers $S(n,k)$ of the second kind defined by (1.8) *or* the λ-Stirling numbers $\mathcal{S}(n,k;\lambda)$ of the second kind defined below by (6.1) and the Hurwitz (or generalized) zeta function $\zeta(s,a)$ defined by (*cf.* [3, p. 249] and [68, p. 88])

$$\zeta(s,a) := \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \qquad \left(\Re(s) > 1;\ a \in \mathbb{C} \setminus \mathbb{Z}_0^-\right), \tag{1.20}$$

so that

$$\zeta(s,1) =: \zeta(s) = \frac{1}{2^s - 1}\, \zeta\left(s, \frac{1}{2}\right) \tag{1.21}$$

for the Riemann zeta function $\zeta(s)$.

2 Elementary Properties of the Apostol-Genocchi Polynomials $\mathcal{G}_n^{(\alpha)}(x;\lambda)$ of Order α

The following elementary properties of the Apostol-Genocchi polynomials $\mathcal{G}_n^{(\alpha)}(x;\lambda)$ of order α are readily derived from (1.18). We, therefore, choose to omit the details involved.

Property 1. *Special values of the Apostol-Genocchi polynomials* (*or the Apostol-Genocchi numbers*) *of order* α:

$$\begin{aligned} \mathcal{G}_n^{(\alpha)}(\lambda) &= \mathcal{G}_n^{(\alpha)}(0;\lambda), \quad \mathcal{G}_n^{(0)}(x;\lambda) = x^n, \\ \mathcal{G}_n^{(0)}(\lambda) &= \delta_{n,0} \text{ and } \mathcal{G}_0^{(\alpha)}(x;\lambda) = \mathcal{G}_0^{(\alpha)}(\lambda) = \delta_{\alpha,0} \quad (n \in \mathbb{N}_0;\ \alpha \in \mathbb{C}), \end{aligned} \tag{2.1}$$

where $\delta_{n,k}$ *denotes the Kronecker symbol.*

Property 2. *Summation formulas for the Apostol-Genocchi polynomials of order* α:

$$\mathcal{G}_n^{(\alpha)}(x;\lambda) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{G}_k^{(\alpha)}(\lambda)\, x^{n-k} \tag{2.2}$$

and

$$\mathcal{G}_n^{(\alpha)}(x;\lambda) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{G}_{n-k}^{(\alpha-1)}(\lambda) \mathcal{G}_k(x;\lambda). \tag{2.3}$$

Property 3. *Difference equation*:

$$\lambda \mathcal{G}_n^{(\alpha)}(x+1;\lambda) + \mathcal{G}_n^{(\alpha)}(x;\lambda) = 2n\, \mathcal{G}_{n-1}^{(\alpha-1)}(x;\lambda) \qquad (n \in \mathbb{N}). \tag{2.4}$$

Property 4. *Differential relations*:

$$\frac{\partial}{\partial x}\left\{\mathcal{G}_n^{(\alpha)}(x;\lambda)\right\} = n\mathcal{G}_{n-1}^{(\alpha)}(x;\lambda) \qquad (n \in \mathbb{N}) \tag{2.5}$$

and

$$\frac{\partial^p}{\partial x^p}\left\{\mathcal{G}_n^{(\alpha)}(x;\lambda)\right\} = \frac{n!}{(n-p)!}\,\mathcal{G}_{n-p}^{(\alpha)}(x;\lambda),\ (n,p \in \mathbb{N}_0;\ 0 \leqq p \leqq n). \tag{2.6}$$

Property 5. *Integral formulas*:

$$\int_a^b \mathcal{G}_n^{(\alpha)}(x;\lambda)up\,dx = \frac{\mathcal{G}_{n+1}^{(\alpha)}(b;\lambda) - \mathcal{G}_{n+1}^{(\alpha)}(a;\lambda)}{n+1} \tag{2.7}$$

and

$$\int_a^b \mathcal{G}_n^{(\alpha)}(x;\lambda)up\,dx = \sum_{k=0}^{n} \frac{1}{n-k+1}\binom{n}{k}\mathcal{G}_k^{(\alpha)}(\lambda)\,(b^{n-k+1} - a^{n-k+1}). \tag{2.8}$$

Property 6. *Addition theorem of the argument*:

$$\mathcal{G}_n^{(\alpha+\beta)}(x+y;\lambda) = \sum_{k=0}^{n}\binom{n}{k}\mathcal{G}_k^{(\alpha)}(x;\lambda)\mathcal{G}_{n-k}^{(\beta)}(y;\lambda). \tag{2.9}$$

Property 7. *Complementary addition theorems*:

$$\mathcal{G}_n^{(\alpha)}(\alpha - x;\lambda) = \frac{(-1)^{n+\alpha}}{\lambda^{\alpha}}\mathcal{G}_n^{(\alpha)}(x;\lambda^{-1}) \tag{2.10}$$

and

$$\mathcal{G}_n^{(\alpha)}(\alpha + x;\lambda) = \frac{(-1)^{n+\alpha}}{\lambda^{\alpha}}\mathcal{G}_n^{(\alpha)}(-x;\lambda^{-1}). \tag{2.11}$$

Property 8. *Recursion formulas*:

$$(n-\alpha)\,\mathcal{G}_n^{(\alpha)}(x;\lambda) = nx\,\mathcal{G}_{n-1}^{(\alpha)}(x;\lambda) - \frac{\alpha\lambda}{2}\,\mathcal{G}_n^{(\alpha+1)}(x+1;\lambda) \tag{2.12}$$

and

$$\frac{\alpha}{2}\,\mathcal{G}_n^{(\alpha+1)}(x;\lambda) = n(\alpha - x)\,\mathcal{G}_{n-1}^{(\alpha)}(x;\lambda) + (n-\alpha)\,\mathcal{G}_n^{(\alpha)}(x;\lambda). \tag{2.13}$$

When we set $\alpha = 1$, $\lambda = 1$ and $\alpha = \lambda = 1$ in the formulas (2.1) to (2.13), we get the corresponding formulas for the Apostol-Genocchi polynomials $\mathcal{G}_n(x;\lambda)$, the generalized Genocchi polynomials $\mathcal{G}_n^{(\alpha)}(x)$ and the classical Genocchi polynomials $G_n(x)$, respectively.

3 Explicit Representations Involving the Gaussian Hypergeometric Function

By using Definition 4 in conjunction with the generating function (1.3), we have

$$\begin{aligned}\sum_{n=0}^{\infty} \mathcal{G}_n^{(l)}(x;\lambda)\frac{z^n}{n!} &= e^{-x\log\lambda}\left(\frac{2(z+\log\lambda)}{e^{z+\log\lambda}+1}\right)^l\left(\frac{z}{z+\log\lambda}\right)^l e^{x(z+\log\lambda)} \\ &= e^{-x\log\lambda}\sum_{k=0}^{\infty} G_k^{(l)}(x)\frac{(z+\log\lambda)^{k-l}z^l}{k!} \\ &= e^{-x\log\lambda}\sum_{k=0}^{\infty} G_k^{(l)}(x)\sum_{n=0}^{k}\binom{k-l}{n-l}\frac{z^n(\log\lambda)^{k-n}}{k!} \\ &= e^{-x\log\lambda}\sum_{n=0}^{\infty}\frac{z^n}{n!}\sum_{k=0}^{\infty}\binom{n+k-l}{k}\binom{n+k}{k}^{-1}G_{n+k}^{(l)}(x)\frac{(\log\lambda)^k}{k!},\end{aligned}$$

which yields Lemma 1 below asserting a relationship between the Apostol-Genocchi polynomials $\mathcal{G}_n^{(l)}(x;\lambda)$ of order $l \in \mathbb{N}_0$ and the Genocchi polynomials $G_n^{(l)}(x)$ of order $l \in \mathbb{N}_0$.

Lemma 1. *The following relationship holds true*:

$$\mathcal{G}_n^{(l)}(x;\lambda) = e^{-x\log\lambda}\sum_{k=0}^{\infty}\binom{n+k-l}{k}\binom{n+k}{k}^{-1}G_{n+k}^{(l)}(x)\frac{(\log\lambda)^k}{k!} \qquad (n,l\in\mathbb{N}_0;\ \lambda\in\mathbb{C}). \tag{3.1}$$

By (1.12) and (1.18) (with $\alpha = l \in \mathbb{N}_0$), we readily obtain Lemma 2 below.

Lemma 2. *The following relationship holds true*:

$$\mathcal{G}_n^{(l)}(x;\lambda) = \{n\}_l\,\mathcal{E}_{n-l}^{(l)}(x;\lambda) = \frac{n!}{(n-l)!}\,\mathcal{E}_{n-l}^{(l)}(x;\lambda) \qquad (n,l\in\mathbb{N}_0;\ 0\leqq l\leqq n;\ \lambda\in\mathbb{C}) \tag{3.2}$$

or, *equivalently*,

$$\mathcal{E}_n^{(l)}(x;\lambda) = \frac{1}{\{n+l\}_l}\,\mathcal{G}_{n+l}^{(l)}(x;\lambda) = \frac{n!}{(n+l)!}\,\mathcal{G}_{n+l}^{(l)}(x;\lambda) \qquad (n,l\in\mathbb{N}_0;\ \lambda\in\mathbb{C}) \tag{3.3}$$

between the Apostol-Genocchi polynomial of order l and the Apostol-Euler polynomial of order $n-l$.

Moreover, since the parameter $\lambda \in \mathbb{C}$, by comparing Definition 4 with our Definition, we are led easily to Lemma 3 below.

Lemma 3. *The following relationship holds true*:

$$\mathcal{G}_n^{(\alpha)}(x;\lambda) = (-2)^{\alpha}\ \mathcal{B}_n^{(\alpha)}(x;-\lambda) \qquad (\alpha,\lambda\in\mathbb{C};\ 1^{\alpha}:=1) \tag{3.4}$$

or, *equivalently*,

$$\mathcal{B}_n^{(\alpha)}(x;\lambda) = \frac{1}{(-2)^{\alpha}}\,\mathcal{G}_n^{(\alpha)}(x;-\lambda) \qquad (\alpha\in\mathbb{C};\ 1^{\alpha}:=1) \tag{3.5}$$

between the Apostol-Genocchi polynomials $\mathcal{G}_n^{(\alpha)}(x;\lambda)$ and the Apostol-Bernoulli polynomials $\mathcal{B}_n^{(\alpha)}(x;\lambda)$.

Lemma 4 below follows easily from Lemma 2 and Lemma 3.

Lemma 4. *The following relationship holds true:*

$$\mathcal{B}_n^{(l)}(x;\lambda) = \frac{n!}{(n-l)!(-2)^{l}}\,\mathcal{E}_{n-l}^{(l)}(x;-\lambda) \qquad (n,l\in\mathbb{N}_0;\ 0\leqq l\leqq n;\ \lambda\in\mathbb{C}) \tag{3.6}$$

or, *equivalently*,

$$\mathcal{E}_n^{(l)}(x;\lambda) = \frac{n!(-2)^{l}}{(n+l)!}\,\mathcal{B}_{n+l}^{(l)}(x;-\lambda) \qquad (n,l\in\mathbb{N}_0;\ \lambda\in\mathbb{C}) \tag{3.7}$$

between the Apostol-Bernoulli polynomial of order l and Apostol-Euler polynomial of order l.

In order to prove the main assertions in this section, we recall each of the following known results (see also the earlier investigations on the subject of explicit hypergeometric representations by Todorov [77] and Srivastava and Todorov [76]).

Lemma 5 (Luo and Srivastava [52, p. 293, Lemma 1 (13)])**.** *The Apostol-Euler polynomials $\mathcal{E}_n^{(\alpha)}(x;\lambda)$ of order α are represented by*

$$\mathcal{E}_n^{(\alpha)}(x;\lambda) = e^{-x\log\lambda}\sum_{k=0}^{\infty} E_{n+k}^{(\alpha)}(x)\,\frac{(\log\lambda)^k}{k!} \qquad (n\in\mathbb{N}_0;\ \lambda,\alpha\in\mathbb{C}) \tag{3.8}$$

in terms of the Euler polynomials of order α.

Theorem A (Luo [45, p. 920, Theorem 1])**.** *Each of the following explicit series representations holds true*:

$$\begin{aligned}\mathcal{E}_n^{(\alpha)}(x;\lambda) = 2^{\alpha}\sum_{k=0}^{n}\binom{n}{k}\binom{\alpha+k-1}{k}\frac{\lambda^k}{(\lambda+1)^{\alpha+k}}\sum_{j=0}^{k}(-1)^j\binom{k}{j}&\\ \cdot\, j^k(x+j)^{n-k}\,{}_2F_1\left(k-n,k;k+1;\frac{j}{x+j}\right)&\\ \big(n\in\mathbb{N}_0;\ \alpha\in\mathbb{C};\ \lambda\in\mathbb{C}\setminus\{-1\}\big)&\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}\mathcal{E}_n^{(\alpha)}(x;\lambda) = e^{-x\log\lambda}\sum_{k=0}^{\infty}\frac{(\log\lambda)^k}{k!}\sum_{r=0}^{n+k}\frac{1}{2^r}\binom{n+k}{r}\binom{\alpha+r-1}{r}&\\ \cdot\sum_{j=0}^{r}(-1)^j\binom{r}{j}j^r(x+j)^{n+k-r}\,{}_2F_1\left(r-n-k,r;r+1;\frac{j}{x+j}\right)&\end{aligned} \tag{3.10}$$

$$(n \in \mathbb{N}_0;\ \alpha, \lambda \in \mathbb{C}),$$

where ${}_2F_1(a,b;c;z)$ *denotes the Gaussian hypergeometric function defined by* (*cf*., *e.g*., [1, p. 556 *et seq*.])

$$ {}_2F_1\left(a,b;c;z\right) = {}_2F_1\left(b,a;c;z\right) := \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n}\,\frac{z^n}{n!}$$

$$\begin{aligned}\big(a,b &\in \mathbb{C};\ c\in\mathbb{C}\ \setminus\ \mathbb{Z}_0^-;|z|<1;\ z=1,\ \textit{and}\ \Re\left(c-a-b\right)>0;\\ z &= -1,\ \textit{and}\ \Re\left(c-a-b\right)>-1\big).\end{aligned}$$

We now state the main result in this section as Theorem 1 below.

Theorem 1. *The following explicit series representations hold true*:

$$\begin{aligned}\mathcal{G}_n^{(l)}\left(x;\lambda\right) = {} & 2^l\ l!\binom{n}{l}\sum_{k=0}^{n-l}\binom{n-l}{k}\binom{l+k-1}{k}\frac{\lambda^k}{\left(\lambda+1\right)^{l+k}}\sum_{j=0}^{k}\left(-1\right)^j\binom{k}{j}\\ & \cdot j^k\left(x+j\right)^{n-k-l}\ {}_2F_1\left(l+k-n,k;k+1;\frac{j}{x+j}\right)\end{aligned} \tag{3.11}$$

$$(n,l\in\mathbb{N}_0;\ \lambda\in\mathbb{C}\setminus\{-1\})$$

and

$$\begin{aligned}\mathcal{G}_n^{(l)}\left(x;\lambda\right) = {} & e^{-x\log\lambda}\sum_{k=0}^{\infty}\binom{n+k-l}{k}\binom{n+k}{k}^{-1}\binom{n+k}{l}\frac{l!\left(\log\lambda\right)^k}{k!}\\ & \cdot\sum_{r=0}^{n+k-l}\frac{1}{2^r}\binom{n+k-l}{r}\binom{l+r-1}{r}\sum_{j=0}^{r}\left(-1\right)^j\binom{r}{j}\\ & \cdot j^r\left(x+j\right)^{n+k-r-l}\ {}_2F_1\left(r+l-n-k,r;r+1;\frac{j}{x+j}\right)\end{aligned} \tag{3.12}$$

$$(n,l\in\mathbb{N}_0;\ \lambda\in\mathbb{C}),$$

where ${}_2F_1\left(a,b;c;z\right)$ *denotes the Gaussian hypergeometric function defined by* (**??**).

Proof. We make use of the relationship (3.2) in conjunction with (3.9) and (3.10) with, of course,

$$\alpha = l \qquad \text{and} \qquad n \longmapsto n-l \qquad (n,l\in\mathbb{N}_0;\ 0\leqq l\leqq n).$$

We thus readily obtain the assertions (3.11) and (3.12) of Theorem 1.

Corollary 1. *The following explicit formula for the Apostol-Genocchi polynomials* $\mathcal{G}_n^{(\alpha)}(x;\lambda)$ *involving the Stirling numbers* $S(n,k)$ *of the second kind holds true*:

$$\mathcal{G}_n^{(l)}(x;\lambda) = 2^l\, l! \sum_{k=0}^{n} \binom{n}{k}\binom{k}{l} \sum_{j=0}^{k-l} \binom{l+j-1}{j} \frac{j!\,(-\lambda)^j}{(\lambda+1)^{j+l}}\, S(k-l,j)\, x^{n-k} \tag{3.13}$$

$$(n,l \in \mathbb{N}_0;\ \lambda \in \mathbb{C}\setminus\{-1\}).$$

Further, by setting $\lambda = 1$ in (3.13), we obtain the following explicit formula for the generalized Genocchi polynomials $G_n^{(l)}(x)\quad (l \in \mathbb{N}_0)$ involving the Stirling numbers $S(n,k)$ of the second kind:

$$G_n^{(l)}(x) = \sum_{k=0}^{n} \binom{n}{k}\binom{k}{l} \sum_{j=0}^{k-l} \binom{l+j-1}{j} l!\, j! \left(-\frac{1}{2}\right)^j S(k-l,j)x^{n-k} \qquad (n,l \in \mathbb{N}_0). \tag{3.14}$$

By setting $\lambda = 1$ in (3.11), we obtain an explicit formula for the Genocchi polynomials $G_n^{(l)}(x)$ of order $l \in \mathbb{N}_0$ in terms of the Gaussian hypergeometric function.

Corollary 2. *The following series representation holds true*:

$$\begin{aligned} G_n^{(l)}(x) = {} & l! \binom{n}{l} \sum_{k=0}^{n-l} \frac{1}{2^k} \binom{n-l}{k}\binom{l+k-1}{k} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \\ & \cdot j^k (x+j)^{n-k-l}\ {}_2F_1\left(k+l-n, k; k+1; \frac{j}{x+j}\right) \qquad (n,l \in \mathbb{N}_0). \end{aligned} \tag{3.15}$$

By setting $x = 0$ in (3.11), we obtain the explicit series representation given by Corollary 3 below.

Corollary 3. *The following explicit series representation holds true*:

$$\mathcal{G}_n^{(l)}(\lambda) = \frac{2^l\, n!}{(n-l)!} \sum_{k=0}^{n-l} \binom{l+k-1}{k} \frac{k!\,(-\lambda)^k}{(\lambda+1)^{k+l}}\, S(n-l,k), \tag{3.16}$$

$$\big(n,l \in \mathbb{N}_0;\ \lambda \in \mathbb{C}\setminus\{-1\}\big).$$

If we set $\lambda = 1$ in (3.16), then we obtain the following formula for the Genocchi numbers $G_n^{(l)}$ of order $l \in \mathbb{N}_0$ involving the Stirling numbers of the second kind:

$$G_n^{(l)} = \frac{n!}{(n-l)!} \sum_{k=0}^{n-l} \binom{l+k-1}{k} k! \left(-\frac{1}{2}\right)^k S(n-l,k) \qquad \big(n,l \in \mathbb{N}_0\big). \tag{3.17}$$

Corollary 4. *The following explicit series representation holds true for the Apostol-Genocchi polynomials* $\mathcal{G}_n(x;\lambda)$:

$$\mathcal{G}_n(x;\lambda) = 2n \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{\lambda^k}{(\lambda+1)^{k+1}} \sum_{j=0}^{k} (-1)^j \binom{k}{j} j^k (x+j)^{n-k-1} \cdot {}_2F_1\left(k-n+1, k; k+1; \frac{j}{x+j}\right), \ \left(n \in \mathbb{N}_0;\ \lambda \in \mathbb{C} \setminus \{-1\}\right). \tag{3.18}$$

Finally, we calculate a few values of the Apostol-Genocchi numbers $\mathcal{G}_n(\lambda)$ by applying the formula (3.16) (with $l = 1$) as follows:

$$\begin{aligned} &\mathcal{G}_0(\lambda) = 0, \quad \mathcal{G}_1(\lambda) = \frac{2}{\lambda+1}, \quad \mathcal{G}_2(\lambda) = -\frac{4\lambda}{(\lambda+1)^2}, \quad \mathcal{G}_3(\lambda) = \frac{6\lambda(\lambda-1)}{(\lambda+1)^3}, \\ &\mathcal{G}_4(\lambda) = -\frac{8\lambda(\lambda^2-4\lambda+1)}{(\lambda+1)^4}, \quad \mathcal{G}_5(\lambda) = \frac{10\lambda(\lambda^3-11\lambda^2+11\lambda-1)}{(\lambda+1)^5}, \\ &\mathcal{G}_6(\lambda) = -\frac{12\lambda(\lambda^4-26\lambda^3+66\lambda^2-26\lambda+1)}{(\lambda+1)^6}, \end{aligned} \tag{3.19}$$

and so on.

By applying (3.3) (with $l = 1$ and $x = 0$) in conjunction with (3.19), we have the corresponding values of the Apostol-Euler numbers $\mathcal{E}_n(\lambda)$ given by

$$\begin{aligned} &\mathcal{E}_0(\lambda) = \frac{2}{\lambda+1}, \qquad \mathcal{E}_1(\lambda) = -\frac{2\lambda}{(\lambda+1)^2}, \quad \mathcal{E}_2(\lambda) = \frac{2\lambda(\lambda-1)}{(\lambda+1)^3}, \\ &\mathcal{E}_3(\lambda) = -\frac{2\lambda(\lambda^2-4\lambda+1)}{(\lambda+1)^4}, \quad \mathcal{E}_4(\lambda) = \frac{2\lambda(\lambda^3-11\lambda^2+11\lambda-1)}{(\lambda+1)^5}, \\ &\mathcal{E}_5(\lambda) = -\frac{2\lambda(\lambda^4-26\lambda^3+66\lambda^2-26\lambda+1)}{(\lambda+1)^6}, \end{aligned} \tag{3.20}$$

and so on.

4 Relationships Involving the Apostol-Genocchi Polynomials $\mathcal{G}_n^{(\alpha)}(x;\lambda)$ of Order α

In this section, we prove an interesting relationship between the generalized Apostol-Genocchi polynomials $\mathcal{G}_n^{(\alpha)}(x;\lambda)$ and the Apostol-Bernoulli polynomials $\mathcal{B}_n(x;\lambda)$.

Theorem 2. *The following relationship holds true*:

$$\mathcal{G}_n^{(\alpha)}(x+y;\lambda) = \sum_{k=0}^{n} \frac{2}{k+1} \binom{n}{k} \left[(k+1)\mathcal{G}_k^{(\alpha-1)}(y;\lambda) - \mathcal{G}_{k+1}^{(\alpha)}(y;\lambda)\right] \mathcal{B}_{n-k}(x;\lambda) \tag{4.1}$$

$$\big(\alpha, \lambda \in \mathbb{C};\ n \in \mathbb{N}_0\big)$$

between the generalized Apostol-Genocchi polynomials and the Apostol-Bernoulli polynomials.

Proof. By applying an analogous method (see the proof given by Luo and Srivastava [53, p. 636, Theorem 1]), we can obtain the explicit formula (4.1) asserted by Theorem 2. The details involved are being omitted here.

In terms of the generalized Apostol-Genocchi numbers $\left\{\mathcal{G}_n^{(\alpha)}(\lambda)\right\}_{n=0}^{\infty}$, by setting $y=0$ in Theorem 2, we obtain the following explicit relationship between the generalized Apostol-Genocchi polynomials $\mathcal{G}_n^{(\alpha)}(x;\lambda)$ of order α and the Apostol-Bernoulli polynomials $\mathcal{B}_k(x;\lambda)$.

Corollary 5. *The following relationship holds true*:

$$\mathcal{G}_n^{(\alpha)}(x;\lambda) = \sum_{k=0}^{n} \frac{2}{k+1}\binom{n}{k}\left[(k+1)\mathcal{G}_k^{(\alpha-1)}(\lambda) - \mathcal{G}_{k+1}^{(\alpha)}(\lambda)\right]\mathcal{B}_{n-k}(x;\lambda) \tag{4.2}$$

$$\big(\alpha, \lambda \in \mathbb{C};\ n \in \mathbb{N}_0\big)$$

between the Apostol-Genocchi polynomials of order α and the Apostol-Bernoulli polynomials.

By noting that

$$\mathcal{G}_n^{(0)}(y;\lambda) = y^n \qquad \big(n \in \mathbb{N}_0;\ \lambda \in \mathbb{C}\big)$$

and using the assertion (4.1) (with $\alpha = 1$), we deduce Corollary 6 below.

Corollary 6. *The following relationship holds true*:

$$\mathcal{G}_n(x+y;\lambda) = \sum_{k=0}^{n} \frac{2}{k+1}\binom{n}{k}\left[(k+1)y^k - \mathcal{G}_{k+1}(y;\lambda)\right]\mathcal{B}_{n-k}(x;\lambda). \tag{4.3}$$

$$\big(n \in \mathbb{N}_0;\ \lambda \in \mathbb{C}\big)$$

between the Apostol-Genocchi polynomials and the Apostol-Bernoulli polynomials.

By taking $y=0$ in (4.3), and in view of the fact that

$$\mathcal{G}_1(y;\lambda) = \mathcal{G}_1(\lambda) = \frac{2}{\lambda+1},$$

we get the following relationship:

$$\mathcal{G}_n(x;\lambda) = -\sum_{k=1}^{n} \frac{2}{k+1}\binom{n}{k}\mathcal{G}_{k+1}(\lambda)\mathcal{B}_{n-k}(x;\lambda) + 2\left(\frac{\lambda-1}{\lambda+1}\right)\mathcal{B}_n(x;\lambda) \tag{4.4}$$

$$(\lambda \in \mathbb{C}\setminus\{-1\};\ n \in \mathbb{N}).$$

By setting $\lambda = 1$ in the formula (4.4), we obtain the following relationship between the classical Genocchi numbers and the classical Bernoulli polynomials:

$$G_n(x) = -\sum_{k=1}^{n} \frac{2}{k+1}\binom{n}{k} G_{k+1} B_{n-k}(x) \qquad (n \in \mathbb{N}), \tag{4.5}$$

which, in its further special case when $x = 0$, yields the following relationship between the classical Genocchi numbers and the classical Bernoulli numbers:

$$G_n = -\sum_{k=1}^{n} \frac{2}{k+1}\binom{n}{k} G_{k+1} B_{n-k} \qquad (n \in \mathbb{N}). \tag{4.6}$$

By setting $\lambda = 1$ in (4.1), we obtain an addition theorem for the Genocchi polynomials of order α given by Corollary 7 below.

Corollary 7. *The following relationship holds true*:

$$G_n^{(\alpha)}(x+y) = \sum_{k=0}^{n} \frac{2}{k+1}\binom{n}{k} \left[(k+1)G_k^{(\alpha-1)}(y) - G_{k+1}^{(\alpha)}(y)\right] B_{n-k}(x) \tag{4.7}$$

$$(\alpha \in \mathbb{C};\ n \in \mathbb{N}_0).$$

Letting $y = 0$ in (4.7), we get the following relationship between the Genocchi polynomials of order α and the classical Bernoulli polynomials:

$$G_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{2}{k+1}\binom{n}{k} \left[(k+1)G_k^{(\alpha-1)} - G_{k+1}^{(\alpha)}\right] B_{n-k}(x) \qquad (\alpha \in \mathbb{C};\ n \in \mathbb{N}_0). \tag{4.8}$$

We next recall a potentially useful result due to Luo and Srivastava [53, p. 638, Theorem 2].

Theorem B (Luo and Srivastava [53, p. 638, Theorem 2])**.** *The following relationship holds true*:

$$\mathcal{E}_n^{(\alpha)}(x+y;\lambda) = \sum_{k=0}^{n} \frac{2}{k+1}\binom{n}{k} \left[\mathcal{E}_{k+1}^{(\alpha-1)}(y;\lambda) - \mathcal{E}_{k+1}^{(\alpha)}(y;\lambda)\right] \mathcal{B}_{n-k}(x;\lambda) \tag{4.9}$$

$$+ \left(\frac{\lambda-1}{n+1}\right)\left(\frac{2}{\lambda+1}\right)^{\alpha} \mathcal{B}_{n+1}(x;\lambda) \qquad \big(\alpha \in \mathbb{C};\ \lambda \in \mathbb{C}\setminus\{-1\};\ n \in \mathbb{N}_0\big) \tag{4.10}$$

between the generalized Apostol-Euler polynomials and the Apostol-Bernoulli polynomials.

Remark 3. The following *additional* term in (4.9):

$$\left(\frac{\lambda-1}{n+1}\right)\left(\frac{2}{\lambda+1}\right)^{\alpha} \mathcal{B}_{n+1}(x;\lambda),$$

was first found by Wang *et al.* (see [78, Corollary 2.6 (2.13)]).

In terms of the generalized Apostol-Euler numbers $\left\{\mathcal{E}_n^{(\alpha)}(\lambda)\right\}_{n=0}^{\infty}$, by setting $y=0$ in Theorem B, we obtain the following explicit relationship between the generalized Apostol-Euler polynomials and the Apostol-Bernoulli polynomials.

Corollary 8. *The following relationship holds true*:

$$\mathcal{E}_n^{(\alpha)}(x;\lambda)=\sum_{k=0}^{n}\frac{2}{k+1}\binom{n}{k}\left[\mathcal{E}_{k+1}^{(\alpha-1)}(\lambda)-\mathcal{E}_{k+1}^{(\alpha)}(\lambda)\right]\mathcal{B}_{n-k}(x;\lambda)$$
$$+\left(\frac{\lambda-1}{n+1}\right)\left(\frac{2}{\lambda+1}\right)^{\alpha}\mathcal{B}_{n+1}(x;\lambda)\qquad\big(\alpha\in\mathbb{C};\ \lambda\in\mathbb{C}\setminus\{-1\};\ n\in\mathbb{N}_0\big)\tag{4.11}$$

between the generalized Apostol-Euler polynomials and the Apostol-Bernoulli polynomials.

Corollary 9 below provides the *corrected* version of each of the *five* known formulas due to Luo and Srivastava [53, pp. 638–639, Eqs. (56), (57), (60), (63) and (64)].

Corollary 9. *Each of the following relationships holds true*:

$$\mathcal{E}_n(x+y;\lambda)=\sum_{k=0}^{n}\frac{2}{k+1}\binom{n}{k}\left[y^{k+1}-\mathcal{E}_{k+1}(y;\lambda)\right]\mathcal{B}_{n-k}(x;\lambda)$$
$$+\left(\frac{\lambda-1}{n+1}\right)\left(\frac{2}{\lambda+1}\right)\mathcal{B}_{n+1}(x;\lambda),\tag{4.12}$$

$$\mathcal{E}_n(x;\lambda)=-\sum_{k=0}^{n}\frac{2}{k+1}\binom{n}{k}\mathcal{E}_{k+1}(0;\lambda)\mathcal{B}_{n-k}(x;\lambda)+\left(\frac{\lambda-1}{n+1}\right)\left(\frac{2}{\lambda+1}\right)\mathcal{B}_{n+1}(x;\lambda),\tag{4.13}$$

$$\mathcal{E}_{n-2}(x;\lambda)=2\binom{n}{2}^{-1}\sum_{k=0}^{n-2}\binom{n}{k}\left[2^{n-k}\mathcal{B}_{n-k}(\lambda^2)-\mathcal{B}_{n-k}(\lambda)\right]\mathcal{B}_k(x;\lambda)$$
$$+\left(\frac{\lambda-1}{n+1}\right)\left(\frac{2}{\lambda+1}\right)\mathcal{B}_{n+1}(x;\lambda),\tag{4.14}$$

$$\mathcal{E}_n^{(\alpha)}(x;\lambda)=\sum_{k=0}^{n}\frac{2}{k+1}\binom{n}{k}\left[\mathcal{E}_{k+1}^{(\alpha-1)}(x;\lambda)-\mathcal{E}_{k+1}^{(\alpha)}(x;\lambda)\right]\mathcal{B}_{n-k}(\lambda)$$
$$+\left(\frac{\lambda-1}{n+1}\right)\left(\frac{2}{\lambda+1}\right)^{\alpha}\mathcal{B}_{n+1}(\lambda)\tag{4.15}$$

and

$$\mathcal{E}_n^{(\alpha)}(\lambda)=\sum_{k=0}^{n}\frac{2^{n-k}}{k+1}\binom{n}{k}\left[2^{k+1}\mathcal{E}_{k+1}^{(\alpha-1)}\left(\frac{\alpha}{2};\lambda\right)-\mathcal{E}_{k+1}^{(\alpha)}(\lambda)\right]\mathcal{B}_{n-k}(\lambda)$$
$$+\left(\frac{\lambda-1}{n+1}\right)\left(\frac{2}{\lambda+1}\right)^{\alpha}\mathcal{B}_{n+1}(\lambda)\tag{4.16}$$
$$\big(\alpha\in\mathbb{C};\ \lambda\in\mathbb{C}\setminus\{-1\};\ n\in\mathbb{N}_0\big).$$

5 Explicit Representations Involving the Hurwitz (or Generalized) Zeta Function $\zeta(s, a)$

The Hurwitz-Lerch zeta function $\Phi(z, s, a)$ defined by (see, for example, [68, p. 121 *et seq.*])

$$\Phi(z, s, a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \tag{5.1}$$

$$\left(a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ s \in \mathbb{C} \ \text{ when } \ |z| < 1\ ;\ \Re(s) > 1 \ \text{ when } \ |z| = 1\right),$$

which can indeed be continued *meromorphically* to the whole complex s-plane, except for a simple pole at $s = 1$ with its residue 1, contains (as its special cases) not only the Hurwitz (or generalized) zeta function $\zeta(s, a)$ defined by (1.20) and the Riemann zeta function $\zeta(s)$ defined by (1.21), but also such other important functions of *Analytic Number Theory* as (for example) the Lipschitz-Lerch zeta function $\phi(\xi, a, s)$ or $L(\xi, s, a)$ defined by (*cf.* [68, p. 122, Eq. 2.5 (11)]):

$$\phi(\xi, a, s) := \sum_{n=0}^{\infty} \frac{e^{2n\pi i\xi}}{(n+a)^s} = \Phi\left(e^{2\pi i\xi}, s, a\right) =: L(\xi, s, a) \tag{5.2}$$

$$\left(a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ \Re(s) > 0 \quad \text{when} \quad \xi \in \mathbb{R} \setminus \mathbb{Z};\ \Re(s) > 1 \quad \text{when} \quad \xi \in \mathbb{Z}\right)$$

which was first studied by Rudolf Lipschitz (1832-1903) and Matyáš Lerch (1860-1922) in connection with Dirichlet's famous theorem on primes in arithmetic progressions. For various extensions and generalizations of the Hurwitz-Lerch zeta function $\Phi(z, s, a)$ defined by (5.1), the interested reader may be referred to several recent works including (for example) [19], [41] and [75] and the references cited in each of these works (see also [13] and [67]).

Precisely one decade ago, Srivastava [66] made use of Lerch's functional equation:

$$\begin{aligned}\phi(\xi, a, 1-s) = \frac{\Gamma(s)}{(2\pi)^s}\Bigg\{ &\exp\left[\left(\frac{1}{2}s - 2a\xi\right)\pi i\right]\phi(-a, \xi, s) \\ &+ \exp\left[\left(-\frac{1}{2}s + 2a(1-\xi)\right)\pi i\right]\phi(a, 1-\xi, s)\Bigg\} \qquad (s \in \mathbb{C};\ 0 < \xi < 1)\end{aligned} \tag{5.3}$$

in conjunction with Apostol's formula [2, p. 164]:

$$\phi(\xi, a, 1-n) = \Phi(e^{2\pi i\xi}, 1-n, a) = -\frac{\mathcal{B}_n\left(a; e^{2\pi i\xi}\right)}{n} \qquad (n \in \mathbb{N}), \tag{5.4}$$

in order to obtain an *elegant* formula for the Apostol-Bernoulli polynomials $\mathcal{B}_n(x; \lambda)$, which we recall here as Theorem C below.

Theorem C (Srivastava's formula [66, p. 84, Eq. (4.6)]). *The Apostol-Bernoulli polynomials $\mathcal{B}_n(x;\lambda)$ at rational arguments are given by*

$$\begin{aligned}\mathcal{B}_n\left(\frac{p}{q};e^{2\pi i\xi}\right) &= -\frac{n!}{(2q\pi)^n}\Bigg\{\sum_{j=1}^{q}\zeta\left(n,\frac{\xi+j-1}{q}\right)\\ &\quad\times\exp\left[\left(\frac{n}{2}-\frac{2(\xi+j-1)p}{q}\right)\pi i\right]+\sum_{j=1}^{q}\zeta\left(n,\frac{j-\xi}{q}\right)\\ &\quad\exp\left[\left(-\frac{n}{2}+\frac{2(j-\xi)p}{q}\right)\pi i\right]\Bigg\},\\ &\quad\big(n\in\mathbb{N}\setminus\{1\};\ p\in\mathbb{Z};\ q\in\mathbb{N};\ \xi\in\mathbb{R}\big),\end{aligned}\tag{5.5}$$

in terms of the Hurwitz (*or generalized*) *zeta function $\zeta(s,a)$.*

Two analogous formulas for the Apostol-Euler polynomials $\mathcal{E}_n(x;\lambda)$ and the Apostol-Genocchi polynomials $\mathcal{G}_n(x;\lambda)$ at rational arguments are asserted by Theorem 3 and Theorem 4, respectively.

Theorem 3. *The following representation of the Apostol-Euler polynomials at rational arguments holds true*:

$$\begin{aligned}\mathcal{E}_n\left(\frac{p}{q};e^{2\pi i\xi}\right) &= \frac{2\cdot n!}{(2q\pi)^{n+1}}\Bigg\{\sum_{j=1}^{q}\zeta\left(n+1,\frac{2\xi+2j-1}{2q}\right)\\ \times\exp\left[\left(\frac{n+1}{2}-\frac{(2\xi+2j-1)p}{q}\right)\pi i\right]&+\sum_{j=1}^{q}\zeta\left(n+1,\frac{2j-2\xi-1}{2q}\right)\\ \times\exp\left[\left(-\frac{n+1}{2}+\frac{(2j-2\xi-1)p}{q}\right)\pi i\right]\Bigg\}&\\ \big(n,q\in\mathbb{N};\ p\in\mathbb{Z};\ \xi\in\mathbb{R}\big)&\end{aligned}\tag{5.6}$$

in terms of the Hurwitz (*or generalized*) *zeta function $\zeta(s,a)$.*

Proof. First of all, we recall a *useful* relationship between the Apostol-Euler polynomials and the Apostol-Bernoulli polynomials given by (see [53, p. 636, Eq. (38)])

$$\mathcal{E}_{n-1}(x;\lambda)=\frac{2}{n}\left[\mathcal{B}_n(x;\lambda)-2^n\mathcal{B}_n\left(\frac{x}{2};\lambda^2\right)\right]\qquad(n\in\mathbb{N})\tag{5.7}$$

or, equivalently, by

$$\mathcal{E}_n(x;\lambda)=\frac{2}{n+1}\left[\mathcal{B}_{n+1}(x;\lambda)-2^{n+1}\mathcal{B}_{n+1}\left(\frac{x}{2};\lambda^2\right)\right]\qquad(n\in\mathbb{N}_0).\tag{5.8}$$

Taking

$$x=\frac{p}{q}\qquad\text{and}\qquad\lambda=e^{2\pi i\xi}\qquad\big(p\in\mathbb{Z};\ q\in\mathbb{N};\ \xi\in\mathbb{R}\big)$$

in the last formula (5.8), we find from Srivastava's formula (**??**) with

$$n \mapsto n+1, \quad q \mapsto 2q \quad \text{and} \quad \xi \mapsto 2\xi$$

that

$$\begin{aligned}
&\mathcal{E}_n\left(\frac{p}{q}; e^{2\pi i\xi}\right) \\
&\quad = \frac{2}{n+1}\Bigg\{-\frac{(n+1)!}{(2q\pi)^{n+1}}\left[\sum_{j=1}^{q}\zeta\left(n+1, \frac{\xi+j-1}{q}\right)\exp\left[\left(\frac{n+1}{2}-\frac{2(\xi+j-1)p}{q}\right)\pi i\right]\right. \\
&\qquad \left. + \sum_{j=1}^{q}\zeta\left(n+1, \frac{j-\xi}{q}\right)\exp\left[\left(-\frac{n+1}{2}+\frac{2(j-\xi)p}{q}\right)\pi i\right]\right] \\
&\quad + 2^{n+1}\cdot\frac{(n+1)!}{(4q\pi)^{n+1}}\left[\sum_{j=1}^{2q}\zeta\left(n+1, \frac{2\xi+j-1}{2q}\right)\exp\left[\left(\frac{n+1}{2}-\frac{(2\xi+j-1)p}{q}\right)\pi i\right]\right. \\
&\qquad \left. + \sum_{j=1}^{2q}\zeta\left(n+1, \frac{j-2\xi}{2q}\right)\exp\left[\left(-\frac{n+1}{2}+\frac{(j-2\xi)p}{q}\right)\pi i\right]\right]\Bigg\} \\
&\quad = \frac{2\cdot n!}{(2q\pi)^{n+1}}\Bigg\{\sum_{j=1}^{2q}\zeta\left(n+1, \frac{2\xi+j-1}{2q}\right)\exp\left[\left(\frac{n+1}{2}-\frac{(2\xi+j-1)p}{q}\right)\pi i\right] \\
&\qquad - \sum_{j=1}^{q}\zeta\left(n+1, \frac{\xi+j-1}{q}\right)\exp\left[\left(\frac{n+1}{2}-\frac{2(\xi+j-1)p}{q}\right)\pi i\right] \\
&\qquad + \sum_{j=1}^{2q}\zeta\left(n+1, \frac{j-2\xi}{2q}\right)\exp\left[\left(-\frac{n+1}{2}+\frac{(j-2\xi)p}{q}\right)\pi i\right] \\
&\qquad - \sum_{j=1}^{q}\zeta\left(n+1, \frac{j-\xi}{q}\right)\exp\left[\left(-\frac{n+1}{2}+\frac{2(j-\xi)p}{q}\right)\pi i\right]\Bigg\}. \qquad (5.9)
\end{aligned}$$

The first sum in (5.9) can obviously be rewritten the following form:

$$\begin{aligned}
&\sum_{j=1}^{2q}\zeta\left(n+1, \frac{2\xi+j-1}{2q}\right)\exp\left[\left(\frac{n+1}{2}-\frac{(2\xi+j-1)p}{q}\right)\pi i\right] \\
&\quad = \sum_{j=1}^{q}\zeta\left(n+1, \frac{\xi+j-1}{q}\right)\exp\left[\left(\frac{n+1}{2}-\frac{2(\xi+j-1)p}{q}\right)\pi i\right] \\
&\qquad + \sum_{j=1}^{q}\zeta\left(n+1, \frac{2\xi+2j-1}{2q}\right)\exp\left[\left(\frac{n+1}{2}-\frac{(2\xi+2j-1)p}{q}\right)\pi i\right].
\end{aligned} \tag{5.10}$$

The third sum in (5.9) can also be rewritten the following form:

$$\begin{aligned}\sum_{j=1}^{2q}\zeta\left(n+1,\frac{j-2\xi}{2q}\right)\exp\left[\left(-\frac{n+1}{2}+\frac{(j-2\xi)p}{q}\right)\pi i\right]\\ =\sum_{j=1}^{q}\zeta\left(n+1,\frac{2j-2\xi-1}{2q}\right)\exp\left[\left(-\frac{n+1}{2}+\frac{(2j-2\xi-1)p}{q}\right)\pi i\right]\\ +\sum_{j=1}^{q}\zeta\left(n+1,\frac{j-\xi}{q}\right)\exp\left[\left(-\frac{n+1}{2}+\frac{2(j-\xi)p}{q}\right)\pi i\right].\end{aligned}\tag{5.11}$$

Upon first separating the even and odd terms in (5.10) and (5.11), and then substituting from (5.10) and (5.11) into (5.9), we are led eventually to the formula (**??**) asserted by Theorem 3.

Theorem 4. *The following representation of the Apostol-Genocchi polynomials at rational arguments holds true*:

$$\begin{aligned}\mathcal{G}_n\left(\frac{p}{q};e^{2\pi i\xi}\right)=\frac{2\cdot n!}{(2q\pi)^n}\left\{\sum_{j=1}^{q}\zeta\left(n,\frac{2\xi+2j-1}{2q}\right)\exp\left[\left(\frac{n}{2}-\frac{(2\xi+2j-1)p}{q}\right)\pi i\right]\right.\\ \left.+\sum_{j=1}^{q}\zeta\left(n,\frac{2j-2\xi-1}{2q}\right)\exp\left[\left(-\frac{n}{2}+\frac{(2j-2\xi-1)p}{q}\right)\pi i\right]\right\}\end{aligned}\tag{5.12}$$

$$\big(n\in\mathbb{N}\setminus\{1\};\ p\in\mathbb{Z};\ q\in\mathbb{N};\ \xi\in\mathbb{R}\big)$$

in terms of the Hurwitz (*or generalized*) *zeta function* $\zeta(s,a)$.

Proof. We apply the relationship:

$$\mathcal{G}_n(x;\lambda)=n\,\mathcal{E}_{n-1}(x;\lambda)\tag{5.13}$$

with, of course,

$$x=\frac{p}{q}\qquad\text{and}\qquad\lambda=e^{2\pi i\xi}\qquad\big(p\in\mathbb{Z};\ q\in\mathbb{N};\ \xi\in\mathbb{R}\big)$$

in conjunction with the formula (**??**). We thus obtain the assertion (5.12) of Theorem 4.

For $\xi\in\mathbb{Z}$, the formula (**??**) can easily be shown to reduce to the following known result given earlier by Cvijović and Klinowski [16, p. 1529, Theorem B] (see also [66, p. 78, Theorem B]).

Corollary 10. *The following representation of the classical Euler polynomials holds true*:

$$E_n\left(\frac{p}{q}\right)=\frac{4\cdot n!}{(2q\pi)^{n+1}}\sum_{j=1}^{q}\zeta\left(n+1,\frac{2j-1}{2q}\right)\sin\left(\frac{(2j-1)p\pi}{q}-\frac{n\pi}{2}\right),\ (n,q\in\mathbb{N};\ p\in\mathbb{Z})\tag{5.14}$$

in terms of the Hurwitz (*or generalized*) *zeta function* $\zeta(s,a)$.

A special case for the formula (5.12) when $\xi \in \mathbb{Z}$ is stated here as Corollary 10 below.

Corollary 11. *The following representation of the classical Genocchi polynomials holds true*: $G_n\left(\frac{p}{q}\right) = \frac{4\cdot n!}{(2q\pi)^n}\sum_{j=1}^{q}\zeta\left(n, \frac{2j-1}{2q}\right)$
$\times \cos\left(\frac{(2j-1)p\pi}{q} - \frac{n\pi}{2}\right)$

$$\left(n \in \mathbb{N}\setminus\{1\};\ p \in \mathbb{Z};\ q \in \mathbb{N}\right)$$

in terms of the Hurwitz (*or generalized*) *zeta function* $\zeta(s,a)$.

The following formula for the Apostol-Bernoulli polynomials $\mathcal{B}_n^{(\alpha)}(x;\lambda)$ of order α was proven by Luo and Srivastava [52].

Theorem D (Luo and Srivastava [52, p. 300, Theorem 2])**.** *The Apostol-Bernoulli polynomials* $\mathcal{B}_n^{(\alpha)}(x;\lambda)$ *of order* α *at rational arguments are given by*

$$\begin{aligned}\mathcal{B}_n^{(\alpha)}\left(\frac{p}{q};e^{2\pi i\xi}\right) &= n\left(e^{2\pi i\xi}-1\right)^{-1}\mathcal{B}_{n-1}^{(\alpha-1)}\left(e^{2\pi i\xi}\right) - \sum_{k=2}^{n}\frac{k!}{(2q\pi)^k}\binom{n}{k}\mathcal{B}_{n-k}^{(\alpha-1)}\left(e^{2\pi i\xi}\right)\\ &\quad\cdot\left\{\sum_{j=1}^{q}\zeta\left(k,\frac{\xi+j-1}{q}\right)\exp\left[\left(\frac{k}{2}-\frac{2(\xi+j-1)p}{q}\right)\pi i\right]\right.\\ &\quad\left.+\sum_{j=1}^{q}\zeta\left(k,\frac{j-\xi}{q}\right)\exp\left[\left(-\frac{k}{2}+\frac{2(j-\xi)p}{q}\right)\pi i\right]\right\}\end{aligned} \tag{5.15}$$

$$\left(n \in \mathbb{N}\setminus\{1\};\ p \in \mathbb{Z};\ q \in \mathbb{N};\ \xi \in \mathbb{R}\setminus\mathbb{Z};\ \alpha \in \mathbb{C}\right)$$

holds true in terms of the Hurwitz (*or generalized*) *zeta function* $\zeta(s,a)$.

For $\alpha = 1$, the formula (5.15) reduces to Srivastava's formula (**??**). When $\xi \in \mathbb{Z}$ in (**??**), Srivastava's formula (5.15) can easily be shown to reduce to a known result given earlier by Cvijović and Klinowski [16, p. 1529, Theorem A] (see also [66, p. 78, Theorem A]):

$$B_n\left(\frac{p}{q}\right) = -\frac{2\cdot n!}{(2q\pi)^n}\sum_{j=1}^{q}\zeta\left(n,\frac{j}{q}\right)\cos\left(\frac{2jp\pi}{q}-\frac{n\pi}{2}\right),\ \ (n \in \mathbb{N}\setminus\{1\};\ p \in \mathbb{Z};\ q \in \mathbb{N}). \tag{5.16}$$

The following formula is a *complement* of (5.15) (when $\xi \in \mathbb{Z}$):

$$\begin{aligned}B_n^{(\alpha)}\left(\frac{p}{q}\right) &= B_n^{(\alpha-1)} + n\left(\frac{p}{q}-\frac{1}{2}\right)B_{n-1}^{(\alpha-1)}\\ &\quad-\sum_{k=2}^{n}\frac{2\cdot k!}{(2q\pi)^k}\binom{n}{k}B_{n-k}^{(\alpha-1)}\sum_{j=1}^{q}\zeta\left(k,\frac{j}{q}\right)\cos\left(\frac{2jp\pi}{q}-\frac{k\pi}{2}\right)\end{aligned} \tag{5.17}$$

$$(n \in \mathbb{N} \setminus \{1\}\,;\ p \in \mathbb{Z};\ q \in \mathbb{N};\ \alpha \in \mathbb{C})\,.$$

By applying (**??**) and (5.12), we now derive the following represenation formulas for the Apostol-Euler polynomials of order α and the Apostol-Genocchi polynomials of order α, respectively.

Theorem 5. *The following representation of the Apostol-Euler polynomials of order α holds true*:

$$\begin{aligned}\mathcal{E}_n^{(\alpha)}\left(\frac{p}{q};e^{2\pi i\xi}\right) &= \frac{2}{e^{2\pi i\xi}+1}\,\mathcal{E}_n^{(\alpha-1)}(e^{2\pi i\xi}) + \sum_{k=1}^{n}\frac{2\cdot k!}{(2q\pi)^{k+1}}\binom{n}{k}\mathcal{E}_{n-k}^{(\alpha-1)}(e^{2\pi i\xi})\\ &\cdot\left\{\sum_{j=1}^{q}\zeta\left(k+1,\frac{2\xi+2j-1}{2q}\right)\exp\left[\left(\frac{k+1}{2}-\frac{(2\xi+2j-1)p}{q}\right)\pi i\right]\right.\\ &\left.+\sum_{j=1}^{q}\zeta\left(k+1,\frac{2j-2\xi-1}{2q}\right)\exp\left[\left(-\frac{k+1}{2}+\frac{(2j-2\xi-1)p}{q}\right)\pi i\right]\right\}\end{aligned} \tag{5.18}$$

$$\left(n,q \in \mathbb{N};\ p \in \mathbb{Z};\ \xi \in \mathbb{R}\setminus\Lambda\ \left(\Lambda := \left\{k+\frac{1}{2} : k \in \mathbb{Z}\right\}\right);\ \alpha \in \mathbb{C}\right)$$

in terms of the Hurwitz (*or generalized*) *zeta function* $\zeta(s,a)$.

Proof. We apply the known result [45, p. 919, Eq. (9) *with* $\alpha \mapsto \alpha-1$ and $\beta = 1$]:

$$\mathcal{E}_n^{(\alpha)}(x;\lambda) = \sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_{n-k}^{(\alpha-1)}(\lambda)\mathcal{E}_k(x;\lambda)$$

and the special values of $\mathcal{E}_n(x;\lambda)$ given by

$$\mathcal{E}_0(x;\lambda) = \mathcal{E}_0(\lambda) = \frac{2}{\lambda+1}.$$

Upon separating the $k = 0$ term in conjunction with the formula (**??**), the representation formula (5.18) follows readily.

Theorem 6. *The following representation of the Apostol-Genocchi polynomials of order α at rational arguments holds true*:

$$\begin{aligned}\mathcal{G}_n^{(\alpha)}\left(\frac{p}{q};e^{2\pi i\xi}\right) &= \frac{2n}{e^{2\pi i\xi}+1}\,\mathcal{G}_{n-1}^{(\alpha-1)}\left(e^{2\pi i\xi}\right) + \sum_{k=2}^{n}\frac{2\cdot k!}{(2q\pi)^{k}}\binom{n}{k}\mathcal{G}_{n-k}^{(\alpha-1)}(e^{2\pi i\xi})\\ &\cdot\left\{\sum_{j=1}^{q}\zeta\left(k,\frac{2\xi+2j-1}{2q}\right)\exp\left[\left(\frac{k}{2}-\frac{(2\xi+2j-1)p}{q}\right)\pi i\right]\right.\\ &\left.+\sum_{j=1}^{q}\zeta\left(k,\frac{2j-2\xi-1}{2q}\right)\exp\left[\left(-\frac{k}{2}+\frac{(2j-2\xi-1)p}{q}\right)\pi i\right]\right\}\end{aligned} \tag{5.19}$$

$$\left(n \in \mathbb{N} \setminus \{1\};\ p \in \mathbb{Z};\ q \in \mathbb{N};\ \xi \in \mathbb{R} \setminus \Lambda\ \left(\Lambda := \left\{k + \frac{1}{2} : k \in \mathbb{Z}\right\}\right);\ \alpha \in \mathbb{C}\right)$$

in terms of the Hurwitz (*or generalized*) *zeta function* $\zeta(s, a)$.

Proof. We apply the formula (2.3) and note that

$$\mathcal{G}_0(x; \lambda) = \mathcal{G}_0(\lambda) = 0 \qquad \text{and} \qquad \mathcal{G}_1(x; \lambda) = \mathcal{G}_1(\lambda) = \frac{2}{\lambda + 1}. \tag{5.20}$$

Upon first separating the $k = 0$ and $k = 2$ terms, and then using the formula (5.12), we arrive at the representation (5.19) asserted by Theorem 6.

In their special cases when $\xi \in \mathbb{Z}$, Theorems 5 and 6 readily yield Corollaries 12 and 13, respectively, which provide the corresponding representations of the Euler polynomials of order α and the Genocchi polynomials of order α at rational arguments.

Corollary 12. *The following representation of the generalized Euler polynomials at rational arguments holds true*:

$$E_n^{(\alpha)}\left(\frac{p}{q}\right) = E_n^{(\alpha-1)} + \sum_{k=1}^{n} \frac{4 \cdot k!}{(2q\pi)^{k+1}} \binom{n}{k} E_{n-k}^{(\alpha-1)} \sum_{j=1}^{q} \zeta\left(k+1, \frac{2j-1}{2q}\right) \cdot \sin\left(\frac{(2j-1)p\pi}{q} - \frac{k\pi}{2}\right),\ \left(n, q \in \mathbb{N};\ p \in \mathbb{Z};\ \alpha \in \mathbb{C}\right) \tag{5.21}$$

in terms of the Hurwitz (*or generalized*) *zeta function* $\zeta(s, a)$.

Corollary 13. *The following representation of the generalized Genocchi polynomials at rational arguments holds true*:

$$G_n^{(\alpha)}\left(\frac{p}{q}\right) = nG_{n-1}^{(\alpha-1)} + \sum_{k=2}^{n} \frac{4 \cdot k!}{(2q\pi)^{k}} \binom{n}{k} G_{n-k}^{(\alpha-1)} \sum_{j=1}^{q} \zeta\left(k, \frac{2j-1}{2q}\right) \cos\left(\frac{(2j-1)p\pi}{q} - \frac{k\pi}{2}\right) \tag{5.22}$$

$$\left(n \in \mathbb{N} \setminus \{1\};\ p \in \mathbb{Z};\ q \in \mathbb{N};\ \alpha \in \mathbb{C}\right)$$

in terms of the Hurwitz (*or generalized*) *zeta function* $\zeta(s, a)$.

Clearly, by setting $\alpha = 1$ in (5.21) and (5.22), we again obtain the formulas (5.14) and (11), respectively. On the other hand, if we apply the formulas (3.4) of Lemma 3 and (3.7) of Lemma 4 in conjunction with the assertion (5.15) of Theorem D of Luo and Srivastava [52], we obtain the series representations of $\mathcal{G}_n^{(\alpha)}(x; \lambda)$ and $\mathcal{E}_n^{(\alpha)}(x; \lambda)$, respectively, which are given by Theorems 7 and 8 below.

Theorem 7. *The following series representation holds true for the Apostol-Genocchi polynomials of reder* α:

$$\mathcal{G}_n^{(\alpha)}\left(\frac{p}{q};e^{2\pi i}\right)=-\frac{n\,(-2)^\alpha}{e^{2\pi i\xi}+1}\mathcal{B}_{n-1}^{(\alpha-1)}\left(-e^{2\pi i\xi)}\right)-\sum_{k=2}^{n}\frac{k!}{(2q\pi)^k}\binom{n}{k}\mathcal{B}_{n-k}^{(\alpha-1)}\left(-e^{2\pi i\xi}\right)$$
$$\cdot\left\{\sum_{j=1}^{q}\zeta\left(k,\frac{2\xi+2j-1}{2q}\right)\exp\left[\left(\frac{k}{2}-\frac{(2\xi+2j-1)\,p}{q}\right)\pi i\right]\right.$$
$$\left.+\sum_{j=1}^{q}\zeta\left(k,\frac{2j-2\xi-1}{2q}\right)\exp\left[\left(-\frac{k}{2}+\frac{(2j-2\xi-1)\,p}{q}\right)\pi i\right]\right\} \tag{5.23}$$

$$\left(n\in\mathbb{N}\setminus\{1\};\ p\in\mathbb{Z};\ q\in\mathbb{N};\ \xi\in\mathbb{R}\setminus\Lambda\quad\left(\Lambda:=\left\{k+\frac{1}{2}:k\in\mathbb{Z}\right\}\right);\ \alpha\in\mathbb{C}\right)$$

in terms of the Hurwitz (*or generalized*) *zeta function.*

Theorem 8. *The following series representation holds true for the Apostol-Euler polynomials of order* l:

$$\mathcal{E}_n^{(l)}\left(\frac{p}{q};e^{2\pi i\xi}\right)=-\frac{n!\,(-2)^l}{(n+l-1)!(e^{2\pi i\xi}+1)}\mathcal{B}_{n+l-1}^{(l-1)}\left(-e^{2\pi i\xi)}\right)-\sum_{k=2}^{n+l}\frac{k!}{(2q\pi)^k}\binom{n+l}{k}$$
$$\cdot\mathcal{B}_{n+l-k}^{(l-1)}\left(-e^{2\pi i\xi}\right)\left\{\sum_{j=1}^{q}\zeta\left(k,\frac{2\xi+2j-1}{2q}\right)\exp\left[\left(\frac{k}{2}-\frac{(2\xi+2j-1)\,p}{q}\right)\pi i\right]\right.$$
$$\left.+\sum_{j=1}^{q}\zeta\left(k,\frac{2j-2\xi-1}{2q}\right)\exp\left[\left(-\frac{k}{2}+\frac{(2j-2\xi-1)\,p}{q}\right)\pi i\right]\right\} \tag{5.24}$$

$$\left(n\in\mathbb{N}\setminus\{1\};\ p\in\mathbb{Z};\ l,q\in\mathbb{N};\ \xi\in\mathbb{R}\setminus\Lambda\quad\left(\Lambda:=\left\{k+\frac{1}{2}:k\in\mathbb{Z}\right\}\right)\right)$$

in terms of the Hurwitz (*or generalized*) *zeta function.*

Remark 4. It is not difficult to apply the relationships (3.5) of Lemma 3 and (3.6) of Lemma 4 in conjunction with the above formulas (5.19) and (5.18), respectively, in order to obtain the corresponding series representations for the Apostol-Bernoulli polynomials $\mathcal{B}_n^{(\alpha)}(x;\lambda)$ of order $\alpha\in\mathbb{C}$..

6 The λ-Stirling Numbers of the Second Kind and Their Elementary Properties

In this section, we first introduce an analogue of the familiar Stirling numbers $S(n,k)$ of the second kind, which we choose to call the *λ-Stirling numbers of the second kind*. We then derive several elementary properties including recurrence relations for them. We also pose two *open* problems relevant to our present investigation.

Definition 5. The λ-Stirling numbers $\mathcal{S}(n,k;\lambda)$ of the second kind is defined by means of the following generating function:

$$\frac{(\lambda e^z-1)^k}{k!}=\sum_{n=0}^{\infty}\mathcal{S}(n,k;\lambda)\,\frac{z^n}{n!}\qquad (k\in\mathbb{N}_0;\ \lambda\in\mathbb{C}), \tag{6.1}$$

so that, obviously,

$$S(n,k):=\mathcal{S}(n,k;1)$$

for the Stirling numbers $S(n,k)$ of the second kind defined by (1.8) (see [15, p. 206, Theorem A]).

Theorem 9. *The λ-Stirling numbers $\mathcal{S}(n,k;\lambda)$ of the second kind can also be defined as follows*:

$$\lambda^x\,x^n=\sum_{k=0}^{\infty}\binom{x}{k}k!\,\mathcal{S}(n,k;\lambda)\qquad (k\in\mathbb{N}_0;\ \lambda\in\mathbb{C}). \tag{6.2}$$

Proof. By using (6.1) and the binomial theorem, we easily obtain the assertion (6.2) of Theorem 7.

Theorem 10. *The following explicit representation formulas hold true*:

$$\mathcal{S}(n,k;\lambda)=\frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\lambda^j j^n\qquad (n,k\in\mathbb{N}_0;\ \lambda\in\mathbb{C}) \tag{6.3}$$

and

$$\mathcal{S}(n,k;\lambda)=\frac{1}{k!}\sum_{j=0}^{k}(-1)^{j}\binom{k}{j}\lambda^{k-j}(k-j)^n\qquad (n,k\in\mathbb{N}_0;\ \lambda\in\mathbb{C}). \tag{6.4}$$

Proof. Just as in our demonstration of Theorem 7, we can easily derive (6.3) and (6.4) by using (6.1) and the binomial theorem.

Theorem 11. *The λ-Stirling numbers $\mathcal{S}(n,k;\lambda)$ of the second kind satisfy the following triangular and vertical recurrence relations*:

$$\mathcal{S}(n,k;\lambda)=\mathcal{S}(n-1,k-1;\lambda)+k\,\mathcal{S}(n-1,k;\lambda)\qquad (n,k\in\mathbb{N}) \tag{6.5}$$

and

$$\mathcal{S}(n,k;\lambda)=\sum_{j=0}^{n-1}\binom{n-1}{j}\lambda^{n-j-1}\,\mathcal{S}(j,k-1;\lambda)\qquad(n,k\in\mathbb{N}),\tag{6.6}$$

respectively.

Proof. By differentiating both sides of (6.1) with respect to the variable z, we readily arrive at the recursion formulas (6.5) and (6.6) asserted by Theorem 9.

Theorem 12. *The following explicit relationships hold true*:

$$\mathcal{S}(n,k;\lambda)=n!\sum_{j=n}^{\infty}\binom{j}{n}\frac{(\log\lambda)^{j-n}}{j!}\,S(j,k)\qquad(n,k\in\mathbb{N}_0;\ \lambda\in\mathbb{C})\tag{6.7}$$

and

$$\mathcal{S}(n,k;\lambda)=\sum_{j=0}^{k}\frac{\lambda^{j}(\lambda-1)^{k-j}}{(k-j)!}\,S(n,j)\qquad(n,k\in\mathbb{N}_0;\ \lambda\in\mathbb{C})\tag{6.8}$$

between the λ-Stirling numbers $\mathcal{S}(n,k;\lambda)$ of the second kind and the Stirling numbers $S(n,k)$ of the second kind.

Proof. By applying (6.1), it is failrly straightforward to derive the formulas (6.7) and (6.8).

By means of the formula (6.3) or (6.8) in conjunction with (6.1), we can compute several values of $\mathcal{S}(n,k;\lambda)$ given by

$$\begin{aligned}
&\mathcal{S}(0,0;\lambda)=1,\quad \mathcal{S}(1,0;\lambda)=0,\quad \mathcal{S}(1,1;\lambda)=\lambda,\quad \mathcal{S}(2,0;\lambda)=0,\quad \mathcal{S}(2,1;\lambda)=\lambda,\\
&\mathcal{S}(2,2;\lambda)=\lambda(2\lambda-1),\quad \mathcal{S}(3,0;\lambda)=0,\quad \mathcal{S}(3,1;\lambda)=\lambda,\\
&\mathcal{S}(3,2;\lambda)=\lambda(4\lambda-1),\quad \mathcal{S}(3,3;\lambda)=\frac{1}{2}\lambda(9\lambda^2-8\lambda+1),\\
&\mathcal{S}(4,0;\lambda)=0,\quad \mathcal{S}(4,1;\lambda)=\lambda,\quad \mathcal{S}(4,2;\lambda)=\lambda(8\lambda-1),\\
&\mathcal{S}(4,3;\lambda)=\frac{1}{2}\lambda(27\lambda^2-16\lambda+1),\quad \mathcal{S}(4,4;\lambda)=\frac{1}{6}\lambda(64\lambda^3-81\lambda^2+24\lambda-1),\\
&\mathcal{S}(5,0;\lambda)=0,\quad \mathcal{S}(5,1;\lambda)=\lambda,\quad \mathcal{S}(5,2;\lambda)=\lambda(16\lambda-1),\\
&\mathcal{S}(5,3;\lambda)=\frac{1}{2}\lambda(81\lambda^2-32\lambda+1),\quad \mathcal{S}(5,4;\lambda)=\frac{1}{6}\lambda(256\lambda^3-243\lambda^2+48\lambda-1),\\
&\mathcal{S}(5,5;\lambda)=\frac{1}{24}\lambda(625\lambda^5-1024\lambda^3+486\lambda^2-64\lambda+1),
\end{aligned}\tag{6.9}$$

and

$$\mathcal{S}(0,k;\lambda)=\frac{(\lambda-1)^k}{k!},\quad \mathcal{S}(n,0;\lambda)=\delta_{n,0}\quad\text{and}\quad \mathcal{S}(n,1;\lambda)=\lambda\qquad(n,k\in\mathbb{N}_0),\tag{6.10}$$

and so on, $\delta_{m,n}$ being the Kronecker symbol.

When $\lambda = 1$, (6.1) and (6.2) become the corresponding (rather familiar) definitions for the Stirling numbers $S(n,k)$ of the second kind (see, for details, [15, p. 206, Theorem A; p. 207 Theorem B]). Similarly, in their special case when $\lambda = 1$, the formulas (6.3), (6.4), (6.5) and (6.6) would yield the corresponding well-known results for the Stirling numbers $S(n,k)$ of the second kind (see, for details, [15, p. 204, Theorem A; p. 208, Theorem A; p. 209, Theorem B]).

Each of the following special values of $S(n,k)$ is known (see [15, pp. 226–227, Ex. 16] and [60, p. 231]):

$$\begin{aligned} &S(n,n) = 1, && S(n,n-1) = \binom{n}{2}, \\ &S(n,n-2) = \frac{1}{4}\binom{n}{3}(3n-5) \quad \text{and} && S(n,n-3) = \frac{1}{2}\binom{n}{4}(n^2-5n+6), \end{aligned} \tag{6.11}$$

so that, if we make use of the formula (6.3) (with $\lambda = 1$) in conjunction with these special values of $S(n,k)$, we obtain the following interesting summation formulas:

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}j^n = (-1)^n\ n!, \tag{6.12}$$

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}j^{n+1} = (-1)^n(n+1)!\cdot\frac{n}{2}, \tag{6.13}$$

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}j^{n+2} = \frac{(-1)^n(n+2)!}{2}\cdot\frac{n(3n+1)}{12} \tag{6.14}$$

and

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}j^{n+3} = \frac{(-1)^n(n+3)!}{6}\cdot\frac{n^2(n+1)}{8}. \tag{6.15}$$

More generally, we have the following formula recorded by Gould [18, p. 3, Entry (1.17)]:

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}j^{n+k} = (-1)^n\ (n+k)!\sum_{j=0}^{k}\binom{k-n}{k-j}\binom{n}{j}\frac{1}{(k+j)!}\ S(k+j,j). \tag{6.16}$$

Open Problem 1. Does there exist an analogue of the sum given below?

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}\lambda^j\ j^{n+k} \qquad (n\in\mathbb{N};\ k\in\mathbb{N}_0;\ \lambda\in\mathbb{C}).$$

Open Problem 2. Can we find a *rational* generating function for the λ-Stiling numbers $\mathcal{S}(n,k;\lambda)$ of the second kind analogous to a known result [15, p. 207, Theorem C]?

7 Applications of the λ-Stirling Numbers $\mathcal{S}(n,k;\lambda)$ of the Second Kind to the Family of the Apostol Type Polynomials

In the section, we give some applications of the λ-Stirling numbers $\mathcal{S}(n,k;\lambda)$ of the second kind to the Apostol type polynomials and Apostol type numbers. We obtain some interesting series representations for the Apostol-Genocchi polynomials involving the λ-Stirling numbers $\mathcal{S}(n,k;\lambda)$ of the second kind and the Hurwitz (or generalized) zeta function $\zeta(s,a)$. We begin by recalling that Wang *et al.* [78] gave the following results for the Apostol-Euler polynomials of order α using the λ-Stirling numbers $\mathcal{S}(n,k;\lambda)$ of the second kind defined by (6.1).

$$\begin{aligned}\mathcal{E}_n^{(\alpha)}(x+y;\lambda) &= \sum_{l=-j}^{n}\sum_{k=0}^{n-l}\frac{n!j!}{k!(l+j)!(n-k-l)!}\\ &\quad\cdot \mathcal{S}(l+j,j;\lambda)\mathcal{E}_{n-k-l}^{(\alpha)}(y;\lambda)\mathcal{B}_k^{(j)}(x;\lambda) \qquad (n,j\in\mathbb{N}_0;\ \alpha,\lambda\in\mathbb{C}) \end{aligned}\tag{7.1}$$

and

$$x^n = \sum_{l=-j}^{n}\frac{n!j!}{(l+j)!(n-l)!}\,\mathcal{S}(l+j,j;\lambda)\mathcal{B}_{n-l}^{(j)}(x;\lambda) \qquad (n,j\in\mathbb{N}_0;\ \lambda\in\mathbb{C}). \tag{7.2}$$

Application 1. First of all, we give some recurrence relationships for the Apostol-Bernoulli numbers of order l ($l\in\mathbb{N}$) by using the λ-Stirling numbers of the second kind.

Theorem 13. *Let $\mathcal{S}(n,k;\lambda)$ denote the λ-Stirling numbers of the second kind defined by* (6.1). *Then*

$$\sum_{k=0}^{n+l}\binom{n+l}{k}\mathcal{S}(n+l-k,l;\lambda)\mathcal{B}_k^{(l)}(\lambda)=0 \qquad (n,l\in\mathbb{N};\ \lambda\in\mathbb{C}). \tag{7.3}$$

Proof. By applying (1.10) (with $\alpha=l\in\mathbb{N}$ and $x=0$) and (6.1), we find that

$$\begin{aligned}1 &= z^{-l}(\lambda e^z-1)^l\sum_{n=0}^{\infty}\mathcal{B}_n^{(l)}(\lambda)\,\frac{z^n}{n!}\\ &= z^{-l}\ l!\,\sum_{n=0}^{\infty}\mathcal{S}(n,l;\lambda)\,\frac{z^n}{n!}\cdot\sum_{n=0}^{\infty}\mathcal{B}_n^{(l)}(\lambda)\frac{z^n}{n!}\\ &= \sum_{n=0}^{\infty}\left[\binom{n+l}{l}^{-1}\sum_{k=0}^{n+l}\binom{n+l}{k}\mathcal{S}(n+l-k,l;\lambda)\mathcal{B}_k^{(l)}(\lambda)\right]\frac{z^n}{n!}.\end{aligned}\tag{7.4}$$

Now, by comparing the coefficients of z^n ($n\in\mathbb{N}$) on both sides of (7.4), we easily obtain the assertion (7.3) of Theorem 11.

Remark 5. By setting $\lambda = 1$ in (7.3) and observing that

$$\sum_{k=0}^{n+l} = \sum_{k=0}^{n} + \sum_{k=n+1}^{n+l} \qquad \text{and} \qquad S(n+l-k,l) = 0 \quad (n+1 \leqq k \leqq n+l),$$

we have the following recurrence relation for the Bernoulli numbers of order l (or, equivalently, the *Nörlund numbers* [56]):

$$B_n^{(l)} = -\binom{n+l}{n}^{-1} \sum_{k=0}^{n-1} \binom{n+l}{k} S(n+l-k,l) B_k^{(l)}. \tag{7.5}$$

Remark 6. When $\lambda \neq 1$ in (7.3), if we apply the following values for the λ-Stirling numbers $\mathcal{S}(n,k;\lambda)$:

$$\mathcal{S}(0,k;\lambda) = \frac{(\lambda-1)^k}{k!} \qquad \text{and} \qquad \mathcal{S}(n,1;\lambda) = \lambda \qquad (n,k \in \mathbb{N}_0) \tag{7.6}$$

in conjunction with (7.3), we have the following recurrence relation for the Apostol-Bernoulli numbers of order l (or, equivalently, the *generalized Nörlund numbers* [56]):

$$\mathcal{B}_{n+l}^{(l)}(\lambda) = -\frac{l!}{(\lambda-1)^l} \sum_{k=0}^{n+l-1} \binom{n+l}{k} \mathcal{S}(n+l-k,l;\lambda) \mathcal{B}_k^{(l)}(\lambda). \tag{7.7}$$

Remark 7. By setting $l = 1$ in (7.5) and noting that $S(n,1) = 1$, we deduce the following familiar recurrence relations for the classical Bernoulli numbers B_n:

$$B_0 = 1 \quad \text{and} \quad B_n = -\frac{1}{n+1} \sum_{k=0}^{n-1} \binom{n+1}{k} B_k \qquad (n \in \mathbb{N}). \tag{7.8}$$

Remark 8. By setting $l = 1$ in (7.7) and noting that $\mathcal{S}(n,1;\lambda) = \lambda$, we deduce the following known recurrence relations for the Apostol-Bernoulli numbers $\mathcal{B}_n(\lambda)$:

$$\mathcal{B}_0(\lambda) = 0, \quad \mathcal{B}_1(\lambda) = \frac{1}{\lambda-1} \quad \text{and} \quad \mathcal{B}_n(\lambda) = \frac{\lambda}{1-\lambda} \sum_{k=0}^{n-1} \binom{n}{k} \mathcal{B}_k(\lambda) \quad (n \in \mathbb{N} \setminus \{1\}). \tag{7.9}$$

Application 2. If we take $\alpha = -l \quad (l \in \mathbb{N})$ in (1.10), then Definition 2 assumes the following form:

$$\left(\frac{\lambda e^z - 1}{z}\right)^l \cdot e^{xz} = \sum_{n=0}^{\infty} \mathcal{B}_n^{(-l)}(x;\lambda) \, \frac{z^n}{n!}. \tag{7.10}$$

By (7.10) and (6.1), we thus have

$$\begin{aligned}\sum_{n=0}^{\infty} \mathcal{B}_n^{(-l)}(x;\lambda)\,\frac{z^n}{n!} &= z^{-l}\left(\lambda e^z - 1\right)^l \cdot e^{xz} \\ &= z^{-l}\, l! \sum_{n=0}^{\infty} \mathcal{S}(n,l;\lambda)\,\frac{z^n}{n!} \cdot \sum_{n=0}^{\infty} \frac{(zx)^n}{n!} \\ &= \sum_{n=0}^{\infty} \left[\binom{n+l}{l}^{-1} \sum_{k=0}^{n+l} \binom{n+l}{k} \mathcal{S}(k,l;\lambda) x^{n+l-k}\right] \frac{z^n}{n!}, \end{aligned} \tag{7.11}$$

which leads us to Theorem 12 below.

Theorem 14. *The following relationship holds true*:

$$\mathcal{B}_n^{(-l)}(x;\lambda) = \binom{n+l}{l}^{-1} \sum_{k=0}^{n+l} \binom{n+l}{k} \mathcal{S}(k,l;\lambda) x^{n+l-k} \qquad (n, l \in \mathbb{N}) \tag{7.12}$$

between the generalized Apostol-Bernoulli polynomials of order $-l$ $(l \in \mathbb{N})$ *and the* λ-*Stirling numbers of the second kind.*

Remark 9. Taking $\lambda = 1$ in (7.12), we have

$$B_n^{(-l)}(x) = \binom{n+l}{l}^{-1} \sum_{k=0}^{n+l} \binom{n+l}{k} S(k,l) x^{n+l-k}, \tag{7.13}$$

which upon setting $l = n$, yields

$$B_n^{(-n)}(x) = \frac{(n!)^2}{(2n)!} \sum_{k=0}^{2n} \binom{2n}{k} S(k,n) x^{2n-k} \tag{7.14}$$

or, equivalently,

$$B_n^{(-n)}(x) = \frac{(n!)^2}{(2n)!} \sum_{k=0}^{n} \binom{2n}{n+k} S(n+k,n) x^{n-k}. \tag{7.15}$$

Remark 10. Putting $x = 0$ in (7.12), we have

$$\mathcal{B}_n^{(-l)}(\lambda) = \binom{n+l}{l}^{-1} \mathcal{S}(n+l,l;\lambda). \tag{7.16}$$

Further, upon letting $\lambda = 1$ in (7.16) or setting $x = 0$ in (7.13), we obtain

$$B_n^{(-l)} = \binom{n+l}{l}^{-1} S(n+l,l), \tag{7.17}$$

which, for $l = n$, yields

$$B_n^{(-n)} = \binom{2n}{n}^{-1} S(2n,n), \tag{7.18}$$

or, equivalently,

$$B_n^{(-n)} = \frac{(n!)^2}{(2n)!}\, S(2n,n). \tag{7.19}$$

Applying the recursion formula (7.19) and the known formulas in [56, p. 146]), we can calculate the first five values of $B_n^{(-n)}$ and $B_n^{(n)}$ $(n \in \mathbb{N})$ as given below:

$$\begin{aligned}
&B_1^{(-1)} = \frac{1}{2},\ \ B_2^{(-2)} = \frac{7}{6},\ \ B_3^{(-3)} = \frac{9}{2},\ \ B_4^{(-4)} = \frac{243}{10},\ \ B_5^{(-5)} = \frac{6075}{36},\\
&B_1^{(1)} = -\frac{1}{2},\ \ B_2^{(2)} = \frac{5}{6},\ \ B_3^{(3)} = -\frac{9}{4},\ \ B_4^{(4)} = \frac{251}{30} \ \text{ and } \ B_5^{(5)} = \frac{475}{12}\\
&B_1^{(1)} = -\frac{1}{2},\ \ B_2^{(2)} = \frac{5}{6},\ \ B_3^{(3)} = -\frac{9}{4},\ \ B_4^{(4)} = \frac{251}{30} \ \text{ and } \ B_5^{(5)} = \frac{475}{12}.
\end{aligned} \tag{7.20}$$

Next, by applying (1.10) (with $\alpha = l \in \mathbb{N}$) and (6.1), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} x^n \frac{z^{n+l}}{n!} &= (\lambda e^z - 1)^l \cdot \sum_{n=0}^{\infty} \mathcal{B}_n^{(l)}(x;\lambda)\, \frac{z^n}{n!}\\
&= l! \sum_{n=0}^{\infty} \mathcal{S}(n,l;\lambda)\, \frac{z^n}{n!} \cdot \sum_{n=0}^{\infty} \mathcal{B}_n^{(l)}(x;\lambda)\, \frac{z^n}{n!}\\
&= \sum_{n=0}^{\infty} \left[l! \sum_{k=0}^{n} \binom{n}{k} \mathcal{S}(k,l;\lambda)\mathcal{B}_{n-k}^{(l)}(x;\lambda) \right] \frac{z^n}{n!},
\end{aligned} \tag{7.21}$$

which leads us to an equivalent version of (7.2) given by Theorem 13 below.

Theorem 15. *The following expansion formula holds true*:

$$x^{n-l} = \binom{n}{l}^{-1} \sum_{k=0}^{n} \binom{n}{k} \mathcal{S}(k,l;\lambda)\mathcal{B}_{n-k}^{(l)}(x;\lambda) \qquad (n,l \in \mathbb{N}_0;\ n \geqq l). \tag{7.22}$$

Remark 11. When $\lambda = 1$ in (7.22), we have

$$x^{n-l} = \binom{n}{l}^{-1} \sum_{k=0}^{n} \binom{n}{k} S(k,l) B_{n-k}^{(l)}(x) \qquad (n,l \in \mathbb{N}_0;\ n \geqq l). \tag{7.23}$$

Remark 12. Upon setting $l = 1$ in (7.22), if we apply (6.10), we deduce the following known difference equation:

$$nx^{n-1} = \lambda \mathcal{B}_n(x+1;\lambda) - \mathcal{B}_n(x;\lambda), \tag{7.24}$$

which, in the further special case when $\lambda = 1$, is a well-known (rather classical) result.

Application 3. We here obtain some series representations of the Apostol-Genocchi polynomials of higher order by applying the λ-Stirling numbers of the second kind. Indeed, by using (1.10), (1.18) and (6.1), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{G}_n^{(l)}(x;\lambda)\frac{z^n}{n!}&=\left(\frac{2z}{\lambda^2e^{2z}-1}\right)^l\cdot e^{xz}\cdot(\lambda e^z-1)^l\\&=\sum_{n=0}^{\infty}\mathcal{B}_n^{(l)}\left(\frac{x}{2};\lambda^2\right)\frac{(2z)^n}{n!}\cdot l!\sum_{n=0}^{\infty}\mathcal{S}(n,l;\lambda)\frac{z^n}{n!}\\&=\sum_{n=0}^{\infty}\left[l!\sum_{r=0}^{n}\binom{n}{r}2^r\mathcal{S}(n-r,l;\lambda)\mathcal{B}_r^{(l)}\left(\frac{x}{2};\lambda^2\right)\right]\frac{z^n}{n!},\end{aligned}\tag{7.25}$$

which leads us to the following lemma.

Lemma 6. *The following relationship holds true*:

$$\mathcal{G}_n^{(l)}(x;\lambda)=l!\sum_{r=0}^{n}\binom{n}{r}2^r\mathcal{S}(n-r,l;\lambda)\mathcal{B}_r^{(l)}\left(\frac{x}{2};\lambda^2\right)\qquad(n,l\in\mathbb{N}_0;\ \lambda\in\mathbb{C})\tag{7.26}$$

between the generalized Apostol-Genocchi polynomials and the λ-Stirling numbers of the second kind.

By applying (5.15) and (7.26), we easily obtain the following series representation for the generalized Genocchi polynomials $\mathcal{G}_n^{(l)}(x)$.

Theorem 16. *The Apostol-Genocchi polynomials $\mathcal{G}_n^{(l)}(x;\lambda)$ of order l at rational arguments are given by*

$$\begin{aligned}\mathcal{G}_n^{(l)}\left(\frac{p}{q};e^{2\pi i\xi}\right)&=\sum_{r=2}^{n}\frac{r\cdot l!\cdot 2^r}{e^{4\pi i\xi}-1}\binom{n}{r}\mathcal{B}_{r-1}^{(l-1)}\left(e^{4\pi i\xi}\right)\mathcal{S}\left(n-r,l;e^{2\pi i\xi}\right)\\&\quad-\sum_{r=2}^{n}\sum_{k=2}^{r}\frac{k!\cdot l!\cdot 2^r}{(4q\pi)^k}\binom{n}{r}\binom{r}{k}\mathcal{B}_{r-k}^{(l-1)}\left(e^{4\pi i\xi}\right)\mathcal{S}(n-r,l;e^{2\pi i\xi})\\&\quad\cdot\left\{\sum_{j=1}^{2q}\zeta\left(k,\frac{2\xi+j-1}{2q}\right)\exp\left[\left(\frac{k}{2}-\frac{(2\xi+j-1)\,p}{q}\right)\pi i\right]\right.\\&\qquad\left.+\sum_{j=1}^{2q}\zeta\left(k,\frac{j-2\xi}{2q}\right)\exp\left[\left(-\frac{k}{2}+\frac{(j-2\xi)\,p}{q}\right)\pi i\right]\right\}\end{aligned}\tag{7.27}$$

$$\left(n,l\in\mathbb{N}\setminus\{1\};\ p\in\mathbb{Z};\ q\in\mathbb{N};\ \xi\in\mathbb{R}\setminus\Lambda\ \ \left(\Lambda:=\left\{\frac{k}{2}:k\in\mathbb{Z}\right\}\right)\right)$$

in terms of the λ-Stirling numbers $\mathcal{S}(n,k;\lambda)$ of the second kind and the Hurwitz (*or generalized*) *zeta function $\zeta(s,a)$.*

By applying (7.26) (with $\lambda = 1$) and (5.17), we can obtain the following series representation for the generalized Genocchi polynomials $G_n^{(l)}(x)$, which is actually a *complement* of (7.27) for $\xi \in \mathbb{Z}$.

Corollary 14. *The generalized Genocchi polynomials $G_n^{(l)}(x)$ of rational arguments are given by*

$$\begin{aligned} G_n^{(l)}\left(\frac{p}{q}\right) &= l! \cdot S(n,l) + n \cdot l! \cdot S(n-1,l)\left(\frac{p}{q} - l\right) \\ &\quad + \sum_{r=2}^{n} \binom{n}{r} 2^r \cdot S(n-r,l)\left[B_r^{(l-1)} + r\left(\frac{p}{2q} - \frac{1}{2}\right) B_{r-1}^{(l-1)}\right] \\ &\quad - \sum_{r=2}^{n} \sum_{k=2}^{r} \frac{2^{r+1} \cdot k!}{(4q\pi)^k} \binom{n}{r}\binom{r}{k} B_{r-k}^{(l-1)}\ S(n-r,l) \sum_{j=1}^{2q} \zeta\left(k, \frac{j}{2q}\right) \cos\left(\frac{jp\pi}{q} - \frac{k\pi}{2}\right) \end{aligned} \tag{7.28}$$

$$(n \in \mathbb{N} \setminus \{1\};\ \ q, l \in \mathbb{N};\ p \in \mathbb{Z})$$

in terms of the Stirling numbers $S(n,k)$ of the second kind and the Hurwitz (or generalized) zeta function $\zeta(s,a)$.

By letting $l = 1$ in (7.28), we obtain the following explicit series representation for the classical Genocchi polynomials.

Corollary 15. *The classical Genocchi polynomials $G_n(x)$ at rational arguments are given by*

$$G_n\left(\frac{p}{q}\right) = 1 + n\left(\frac{p}{q} - 1\right) - \sum_{k=2}^{n} \frac{2 \cdot k!}{(2q\pi)^k} \binom{n}{k} \sum_{j=1}^{2q} \zeta\left(k, \frac{j}{2q}\right) \cos\left(\frac{jp\pi}{q} - \frac{k\pi}{2}\right) \tag{7.29}$$

$$(n \in \mathbb{N} \setminus \{1\};\ \ q, l \in \mathbb{N};\ p \in \mathbb{Z})$$

in terms of the Hurwitz (or generalized) zeta function $\zeta(s,a)$.

Remark 13. It is not difficult to derive the corresponding formulas for the Apostol-Euler polynomials and the Apostol-Bernoulli polynomials at rational arguments by applying the relationships (3.3) and (3.5) in conjunction with (7.27), (7.28) and (7.29). The details are being omitted here.

8 Further Results and Observations

In this section, we apply Srivastava's formula (Theorem C above) and some relationships in order to obtain several different series representations for the Genocchi polynomials of order α and the Euler polynomials of order α.

We first rewrite the formulas (4.2) and (4.11) in convenient forms given by Lemmas 7 and 8, respectively.

Lemma 7. *The following series representation holds true*:

$$\mathcal{G}_n^{(\alpha)}(x;\lambda) = \sum_{k=0}^{n} \frac{2}{n-k+1}\binom{n}{k}\left[(n-k+1)\mathcal{G}_{n-k}^{(\alpha-1)}(\lambda) - \mathcal{G}_{n-k+1}^{(\alpha)}(\lambda)\right]\mathcal{B}_k(x;\lambda) \tag{8.1}$$

$$\left(\alpha, \lambda \in \mathbb{C};\ n \in \mathbb{N}_0\right).$$

Lemma 8. *The following series representation holds true*:

$$\mathcal{E}_n^{(\alpha)}(x;\lambda) = \sum_{k=0}^{n} \frac{2}{n-k+1}\binom{n}{k}\left[\mathcal{E}_{n-k+1}^{(\alpha-1)}(\lambda) - \mathcal{E}_{n-k+1}^{(\alpha)}(\lambda)\right]\mathcal{B}_k(x;\lambda) + \left(\frac{\lambda-1}{n+1}\right)\left(\frac{2}{\lambda+1}\right)^{\alpha}\mathcal{B}_{n+1}(x;\lambda) \qquad \left(\alpha, \lambda \in \mathbb{C};\ n \in \mathbb{N}_0\right). \tag{8.2}$$

Theorem 17. *The Apostol-Genocchi polynomials $\mathcal{G}_n^{(\alpha)}(x;\lambda)$ of order α at rational arguments are given by*

$$\begin{aligned}\mathcal{G}_n^{(\alpha)}\left(\frac{p}{q};e^{2\pi i\xi}\right) &= \frac{2}{e^{2\pi i\xi}-1}\left[n\,\mathcal{G}_{n-1}^{(\alpha-1)}(e^{2\pi i\xi}) - \mathcal{G}_n^{(\alpha)}(e^{2\pi i\xi})\right] \\ &- \sum_{k=2}^{n}\frac{2}{n-k+1}\frac{k!}{(2q\pi)^k}\binom{n}{k}\cdot\left[(n-k+1)\mathcal{G}_{n-k}^{(\alpha-1)}(e^{2\pi i\xi}) - \mathcal{G}_{n-k+1}^{(\alpha)}(e^{2\pi i\xi})\right] \\ &\cdot\left\{\sum_{j=1}^{q}\zeta\left(k,\frac{\xi+j-1}{q}\right)\exp\left[\left(\frac{k}{2}-\frac{2(\xi+j-1)p}{q}\right)\pi i\right]\right. \\ &\left.+\sum_{j=1}^{q}\zeta\left(k,\frac{j-\xi}{q}\right)\exp\left[\left(-\frac{k}{2}+\frac{2(j-\xi)p}{q}\right)\pi i\right]\right\}\end{aligned} \tag{8.3}$$

$$\left(n \in \mathbb{N}\setminus\{1\};\ q \in \mathbb{N};\ p \in \mathbb{Z};\ \xi \in \mathbb{R}\setminus\mathbb{Z};\ \alpha \in \mathbb{C}\right)$$

in terms of the Hurwitz (*or generalized*) *zeta function $\zeta(s,a)$*.

Proof. Upon separating the $k = 0$ and $k = 1$ terms in (8.1) and applying Srivastava's formula (**??**) (with $n \in \mathbb{N}\setminus\{1\}$), if we note that

$$\mathcal{B}_0(x;\lambda) = \mathcal{B}_0(\lambda) = 0 \qquad \text{and} \qquad \mathcal{B}_1(x;\lambda) = \mathcal{B}_1(\lambda) = \frac{1}{\lambda-1}, \tag{8.4}$$

we arrive at the formula (8.3) asserted by Theorem 15.

Theorem 18. *The Apostol-Euler polynomials $\mathcal{E}_n^{(\alpha)}(x;\lambda)$ of order α at rational arguments are given by*

$$\begin{aligned}\mathcal{E}_n^{(\alpha)}\left(\frac{p}{q};e^{2\pi i\xi}\right) &= \frac{2}{e^{2\pi i\xi}-1}\left[\mathcal{E}_n^{(\alpha-1)}(e^{2\pi i\xi})-\mathcal{E}_n^{(\alpha)}(e^{2\pi i\xi})\right]\\ &\quad -\sum_{k=2}^{n}\frac{2}{n-k+1}\frac{k!}{(2q\pi)^k}\binom{n}{k}\left[\mathcal{E}_{n-k+1}^{(\alpha-1)}(e^{2\pi i\xi})-\mathcal{E}_{n-k+1}^{(\alpha)}(e^{2\pi i\xi})\right]\\ &\quad \cdot\left\{\sum_{j=1}^{q}\zeta\left(k,\frac{\xi+j-1}{q}\right)\exp\left[\left(\frac{k}{2}-\frac{2(\xi+j-1)p}{q}\right)\pi i\right]\right.\\ &\quad \left.+\sum_{j=1}^{q}\zeta\left(k,\frac{j-\xi}{q}\right)\exp\left[\left(-\frac{k}{2}+\frac{2(j-\xi)p}{q}\right)\pi i\right]\right\}\\ &\quad -\frac{e^{2\pi i\xi}-1}{n+1}\frac{(n+1)!}{(2q\pi)^{n+1}}\left(\frac{2}{e^{2\pi i\xi}+1}\right)^{\alpha}\left\{\sum_{j=1}^{q}\zeta\left(n+1,\frac{\xi+j-1}{q}\right)\right.\\ &\quad \cdot\exp\left[\left(\frac{n+1}{2}-\frac{2(\xi+j-1)p}{q}\right)\pi i\right]\\ &\quad \left.+\sum_{j=1}^{q}\zeta\left(n+1,\frac{j-\xi}{q}\right)\exp\left[\left(-\frac{n+1}{2}+\frac{2(j-\xi)p}{q}\right)\pi i\right]\right\} \qquad (8.5)\end{aligned}$$

$$\left(n\in\mathbb{N}\setminus\{1\};\ q\in\mathbb{N};\ p\in\mathbb{Z};\ \xi\in\mathbb{R}\setminus\{\mathbb{Z}\cup\Lambda\}\ \ \left(\Lambda:=\left\{k+\frac{1}{2}:k\in\mathbb{Z}\right\}\right);\ \alpha\in\mathbb{C}\right)$$

in terms of the Hurwitz (*or generalized*) *zeta function* $\zeta(s,a)$.

Proof. Just as in our demonstration of Theorem 16, the representation formula (8.5) can be proven by applying (8.2) and (**??**).

By means of (8.1) (with $\lambda=1$) and (8.2) (with $\lambda=1$) in conjunction with the formula (5.16), we can deduce Corollaries 16 and 17 below asserting series representations for the Genocchi polynomials of order α and the Euler polynomials of order α, respectively.

Corollary 16. *The generalized Genocchi polynomials $G_n^{(\alpha)}(x)$ of order α at rational arguments are given by*

$$\begin{aligned}G_n^{(\alpha)}\left(\frac{p}{q}\right) &= \frac{2}{n+1}\left[(n+1)G_n^{(\alpha-1)}-G_{n+1}^{(\alpha)}\right]+\left(\frac{2p}{q}-1\right)\left[nG_{n-1}^{(\alpha-1)}-G_n^{(\alpha)}\right]\\ &\quad -\sum_{k=2}^{n}\frac{4}{n-k+1}\frac{k!}{(2q\pi)^k}\binom{n}{k}\left[(n-k+1)G_{n-k}^{(\alpha-1)}-G_{n-k+1}^{(\alpha)}\right]\\ &\quad \cdot\sum_{j=1}^{q}\zeta\left(k,\frac{j}{q}\right)\cos\left(\frac{2jp\pi}{q}-\frac{k\pi}{2}\right) \qquad (n\in\mathbb{N}\setminus\{1\};\ q\in\mathbb{N};\ p\in\mathbb{Z};\ \alpha\in\mathbb{C})\end{aligned}$$

(8.6)

in terms of the Hurwitz (*or generalized*) *zeta function* $\zeta(s,a)$.

Corollary 17. *The generalized Euler polynomials* $E_n^{(\alpha)}(x)$ *of order* α *at rational arguments are given by*

$$\begin{aligned}E_n^{(\alpha)}\left(\frac{p}{q}\right) &= \frac{2}{n+1}\left[E_{n+1}^{(\alpha-1)} - E_{n+1}^{(\alpha)}\right] + \left(\frac{2p}{q}-1\right)\left[E_n^{(\alpha-1)} - E_n^{(\alpha)}\right] \\ &\quad - \sum_{k=2}^{n} \frac{4}{n-k+1}\frac{k!}{(2q\pi)^k}\binom{n}{k}\left[E_{n-k+1}^{(\alpha-1)} - E_{n-k+1}^{(\alpha)}\right] \\ &\quad \cdot \sum_{j=1}^{q} \zeta\left(k,\frac{j}{q}\right)\cos\left(\frac{2jp\pi}{q}-\frac{k\pi}{2}\right) \qquad (n\in\mathbb{N}\setminus\{1\};\ q\in\mathbb{N};\ p\in\mathbb{Z};\ \alpha\in\mathbb{C})\end{aligned} \tag{8.7}$$

in terms of the Hurwitz (*or generalized*) *zeta function* $\zeta(s,a)$.

Remark 14. The series representation formulas (8.6) and (8.7) are, respectively, the *complement* of (8.3) and (8.5) for $\xi\in\mathbb{Z}$. Furthermore, by letting $\alpha=1$ in (8.6) and (8.7), we obtain the following explicit series representations for the classical Genocchi polynomials and the classical Euler polynomials, respectively.

Corollary 18. *The classical Genocchi polynomials* $G_n(x)$ *at rational arguments are given by*

$$\begin{aligned}G_n\left(\frac{p}{q}\right) &= -\frac{2}{n+1}\,G_{n+1} - \left(\frac{2p}{q}-1\right)G_n + \sum_{k=2}^{n-1}\frac{4G_{n-k+1}}{n-k+1}\frac{k!}{(2q\pi)^k}\binom{n}{k} \\ &\quad \cdot \sum_{j=1}^{q} \zeta\left(k,\frac{j}{q}\right)\cos\left(\frac{2jp\pi}{q}-\frac{k\pi}{2}\right) \qquad (n\in\mathbb{N}\setminus\{1\};\ q\in\mathbb{N};\ p\in\mathbb{Z})\end{aligned} \tag{8.8}$$

in terms of the Hurwitz (*or generalized*) *zeta function* $\zeta(s,a)$.

Corollary 19. *The classical Euler polynomials* $E_n(x)$ *at rational arguments are given by*

$$\begin{aligned}E_n\left(\frac{p}{q}\right) &= -\frac{2}{n+1}\,E_{n+1} - \left(\frac{2p}{q}-1\right)E_n + \sum_{k=2}^{n}\frac{4\,E_{n-k+1}}{n-k+1}\frac{k!}{(2q\pi)^k}\binom{n}{k} \\ &\quad \cdot \sum_{j=1}^{q} \zeta\left(k,\frac{j}{q}\right)\cos\left(\frac{2jp\pi}{q}-\frac{k\pi}{2}\right) \qquad (n\in\mathbb{N}\setminus\{1\};\ q\in\mathbb{N};\ p\in\mathbb{Z})\end{aligned} \tag{8.9}$$

in terms of the Hurwitz (*or generalized*) *zeta function* $\zeta(s,a)$.

Remark 15. It is fairly easy to apply the relationships (3.5) of Lemma 3 and (3.6) of Lemma 4 in conjunction with the above formulas (8.3) and (8.5), respectively, in order to obtain the corresponding series representations for the Apostol-Bernoulli polynomials $\mathcal{B}_n^{(\alpha)}(x;\lambda)$ of order α at rational arguments. The details involved are being left as an exercise for the interested reader.

We now separate the even and odd terms of the formula (8.9). By noting that

$$E_{2n} = 0 \qquad \text{and} \qquad E_{2n-1} = \frac{1}{2n}\, G_{2n} \qquad (n \in \mathbb{N}),$$

we thus obtain

$$E_{2n-1}\left(\frac{p}{q}\right) = -\left(\frac{2p}{q} - 1\right) E_{2n-1} + \sum_{k=2}^{n} \frac{4\, E_{2n-2k+1}}{2n-2k+1} \frac{(-1)^{k+1}(2k-1)!}{(2q\pi)^{2k-1}} \binom{2n-1}{2k-1} \cdot \sum_{j=1}^{q} \zeta\left(2k-1, \frac{j}{q}\right) \sin\left(\frac{2jp\pi}{q}\right) \qquad (n \in \mathbb{N} \setminus \{1\};\ q \in \mathbb{N};\ p \in \mathbb{Z}) \tag{8.10}$$

and

$$E_{2n}\left(\frac{p}{q}\right) = -\left(\frac{2}{2n+1}\right) E_{2n+1} + \sum_{k=1}^{n} \frac{4\, E_{2n-2k+1}}{2n-2k+1} \frac{(-1)^{k}(2k)!}{(2q\pi)^{2k}} \binom{2n}{2k} \cdot \sum_{j=1}^{q} \zeta\left(2k, \frac{j}{q}\right) \cos\left(\frac{2jp\pi}{q}\right) \qquad (n \in \mathbb{N} \setminus \{1\};\ q \in \mathbb{N};\ p \in \mathbb{Z}). \tag{8.11}$$

On the other hand, by separating the even and odd terms of the formula (5.14), we get (see [16, p. 1529, Theorem B] and [66, p. 78, Theorem B]; see also Corollary 10 above)

$$E_{2n-1}\left(\frac{p}{q}\right) = (-1)^n\, \frac{4(2n-1)!}{(2q\pi)^{2n}} \sum_{j=1}^{q} \zeta\left(2n, \frac{2j-1}{2q}\right) \cos\left(\frac{(2j-1)p\pi}{q}\right) \tag{8.12}$$

$$(n \in \mathbb{N} \setminus \{1\};\ q \in \mathbb{N};\ p \in \mathbb{Z})$$

and

$$E_{2n}\left(\frac{p}{q}\right) = (-1)^n\, \frac{4(2n)!}{(2q\pi)^{2n+1}} \sum_{j=1}^{q} \zeta\left(2n+1, \frac{2j-1}{2q}\right) \sin\left(\frac{(2j-1)p\pi}{q}\right) \tag{8.13}$$

$$(n \in \mathbb{N} \setminus \{1\};\ q \in \mathbb{N};\ p \in \mathbb{Z}).$$

Finally, by comparing the formulas (8.12) and (8.10) and the formulas (8.13) and (8.11), respectively, we obtain the following interesting relationships involving the even and odd Hurwitz (or generalized) zeta functions:

$$\sum_{k=2}^{n} \frac{(-1)^{k+1}(2q\pi)^{2n-2k+1}}{(2n-2k+1)!} \sum_{j=1}^{q} \zeta\left(2k-1, \frac{j}{q}\right) \sin\left(\frac{2jp\pi}{q}\right) E_{2n-2k+1} = (-1)^n \sum_{j=1}^{q} \zeta \times \left(2n, \frac{2j-1}{2q}\right) \cos\left(\frac{(2j-1)p\pi}{q}\right) + \frac{(2q\pi)^{2n}}{2 \cdot (2n-1)!} \left(\frac{p}{q} - \frac{1}{2}\right) E_{2n-1} \tag{8.14}$$

$$(n \in \mathbb{N} \setminus \{1\};\ q \in \mathbb{N};\ p \in \mathbb{Z})$$

and

$$\sum_{k=1}^{n} \frac{(-1)^k (2q\pi)^{2n-2k+1}}{(2n-2k+1)!} \sum_{j=1}^{q} \zeta\left(2k, \frac{j}{q}\right) \cos\left(\frac{2jp\pi}{q}\right) E_{2n-2k+1} = (-1)^n \sum_{j=1}^{q} \zeta$$
$$\times \left(2n+1, \frac{2j-1}{2q}\right) \sin\left(\frac{(2j-1)p\pi}{q}\right) + \frac{(2q\pi)^{2n+1}}{2\cdot(2n+1)!} E_{2n+1} \qquad (8.15)$$
$$(n \in \mathbb{N} \setminus \{1\};\ q \in \mathbb{N};\ p \in \mathbb{Z}).$$

We now recall the following interesting integral representations for the Apostol-Bernoulli polynomials and the Apostol-Euler polynomials, which were given recently by Luo [46].

Lemma 9 (Luo [46, p. 2198, Theorem 3.1 (3.1); p. 2199, Theorem 3.2 (3.3)]). *The following integral representation holds true for the Apostol-Bernoulli polynomials*:

$$\mathcal{B}_n(z; e^{2\pi i\xi}) = -\Delta_n(z;\xi) - ne^{-2\pi iz\xi}$$
$$\cdot \int_0^\infty \left(\frac{U(n;z,t)\cosh(2\pi\xi t) + i\,V(n;z,t)\sinh(2\pi\xi t)}{\cosh(2\pi t) - \cos(2\pi x)}\right) t^{n-1}\ up\, dt, \qquad (8.16)$$

$$\big(n \in \mathbb{N};\ 0 \leqq \Re(z) \leqq 1;\ |\xi| < 1\ \ (\xi \in \mathbb{R})\big),$$

where $\Delta_n(z;\xi)$ *is given by*

$$\Delta_n(z;\xi) = \begin{cases} 0 & (\xi = 0) \\ \\ \dfrac{(-1)^n\ n!}{(2\pi i\xi)^n\ e^{2\pi iz\xi}} & (\xi \neq 0), \end{cases}$$

$$U(n;z,t) = \left[\cos\left(2\pi z - \frac{n\pi}{2}\right) - \cos\left(\frac{n\pi}{2}\right) e^{-2\pi t}\right]$$

and

$$V(n;z,t) = \left[\sin\left(2\pi z - \frac{n\pi}{2}\right) + \sin\left(\frac{n\pi}{2}\right) e^{-2\pi t}\right].$$

Furthermore, *the following integral representation holds true for the Apostol-Euler polynomials*:

$$\mathcal{E}_n(z; e^{2\pi i\xi}) = 2e^{-2\pi iz\xi} \qquad (8.17)$$
$$\times \int_0^\infty \left(\frac{X(n;x,t)\cosh(2\pi\xi t) + i\,Y(n;z,t)\sinh(2\pi\xi t)}{\cosh(2\pi t) - \cos(2\pi z)}\right) t^n\ up\,updt$$

$$\big(n \in \mathbb{N};\ 0 \leqq \Re(z) \leqq 1;\ |\xi| < 1\ \ (\xi \in \mathbb{R})\big),$$

where

$$X(n;z,t)=\left[e^{-\pi t}\sin\left(\pi z+\frac{n\pi}{2}\right)+e^{\pi t}\sin\left(\pi z-\frac{n\pi}{2}\right)\right]$$

and

$$Y(n;z,t)=\left[e^{-\pi t}\cos\left(\pi z+\frac{n\pi}{2}\right)-e^{\pi t}\cos\left(\pi z-\frac{n\pi}{2}\right)\right].$$

We apply the relationships (3.2) of Lemma 2 and (3.4) of Lemma 3 in conjunction with the above formulas (8.16) and (8.17), respectively. We thus obtain the corresponding integral representations for the Apostol-Genocchi polynomials $\mathcal{G}_n^{(\alpha)}(z;\lambda)$.

Theorem 19. *The following integral representation holds true for the Apostol-Genocchi polynomials*:

$$\begin{aligned}\mathcal{G}_n(z;e^{2\pi i\xi}) &= 2ne^{-2\pi iz\xi} \qquad (8.18)\\ &\quad\times\int_0^\infty\left(\frac{M(n;z,t)\cosh(2\pi\xi t)+i\,N(n;x,t)\sinh(2\pi\xi t)}{\cosh(2\pi t)-\cos(2\pi z)}\right)t^{n-1}\,up\,dt\end{aligned}$$

$$\left(n\in\mathbb{N};\ 0\leqq\Re(z)\leqq 1;\ |\xi|<\frac{1}{2}\ \ (\xi\in\mathbb{R})\right),$$

where

$$M(n;z,t)=\left[e^{\pi t}\cos\left(\pi z-\frac{n\pi}{2}\right)-e^{-\pi t}\cos\left(\pi z+\frac{n\pi}{2}\right)\right]$$

and

$$N(n;z,t)=\left[e^{\pi t}\sin\left(\pi z-\frac{n\pi}{2}\right)+e^{-\pi t}\sin\left(\pi z+\frac{n\pi}{2}\right)\right].$$

Remark 16. Upon letting $\xi\in\mathbb{Z}$ in (8.16) and (8.17), we easily deduce that

$$B_n(z)=-n\int_0^\infty\left(\frac{\cos\left(2\pi z-\frac{n\pi}{2}\right)-e^{-2\pi t}\cos\left(\frac{n\pi}{2}\right)}{\cosh(2\pi t)-\cos(2\pi z)}\right)t^{n-1}\,up\,dt \qquad (8.19)$$

$$\left(n\in\mathbb{N};\ 0\leqq\Re(z)\leqq 1\right)$$

and

$$E_n(z)=2\int_0^\infty\left(\frac{e^{\pi t}\sin\left(\pi z-\frac{n\pi}{2}\right)+e^{-\pi t}\sin\left(\pi x+\frac{n\pi}{2}\right)}{\cosh(2\pi t)-\cos(2\pi z)}\right)t^{n}\,up\,dt \qquad (8.20)$$

$$\left(n\in\mathbb{N};\ 0\leqq\Re(z)\leqq 1\right)$$

for the classical Bernoulli polynomials and the classical Euler polynomials, respectively. Moreover, by setting $z=\frac{p}{q}$ in (8.19) and (8.20), and noting the formulas (5.16) and

(5.14), we can get the following integral representations for the Hurwitz (or generalized) zeta function $\zeta(s,a)$:

$$\sum_{j=1}^{q}\zeta\left(n,\frac{j}{q}\right)\cos\left(\frac{2jp\pi}{q}-\frac{n\pi}{2}\right)$$
$$=\frac{(2q\pi)^n}{2\cdot(n-1)!}\int_0^\infty\left(\frac{\cos\left(\frac{2p\pi}{q}-\frac{n\pi}{2}\right)-e^{-2\pi t}\cos\left(\frac{n\pi}{2}\right)}{\cosh(2\pi t)-\cos\left(\frac{2p\pi}{q}\right)}\right)t^{n-1}\,up\,dt \tag{8.21}$$

$$\left(n\in\mathbb{N}\setminus\{1\}\,;\ p\in\mathbb{N}_0;\ q\in\mathbb{N},\ p\leqq q\right)$$

and

$$\sum_{j=1}^{q}\zeta\left(n+1,\frac{2j-1}{2q}\right)\sin\left(\frac{(2j-1)p\pi}{q}-\frac{n\pi}{2}\right)$$
$$=\frac{(2q\pi)^{n+1}}{2\cdot n!}\int_0^\infty\left(\frac{e^{\pi t}\sin\left(\frac{p\pi}{q}-\frac{n\pi}{2}\right)+e^{-\pi t}\sin\left(\frac{p\pi}{q}+\frac{n\pi}{2}\right)}{\cosh(2\pi t)-\cos\left(\frac{2p\pi}{q}\right)}\right)t^{n}\,up\,dt, \tag{8.22}$$

$$\left(n\in\mathbb{N};\ p\in\mathbb{N}_0;\ q\in\mathbb{N},\ p\leqq q\right).$$

Remark 17. By letting $n\longmapsto 2n$ in (8.21) and (8.22), we obtain the following interesting integral representations involving the *even* Hurwitz (or generalized) zeta function $\zeta(2n,a)$ and the *odd* Hurwitz (or generalized) zeta function $\zeta(2n+1,a)$, respectively:

$$\sum_{j=1}^{q}\zeta\left(2n,\frac{j}{q}\right)\cos\left(\frac{2jp\pi}{q}\right)$$
$$=\frac{(2q\pi)^{2n}}{2\cdot(2n-1)!}\int_0^\infty\left(\frac{\cos\left(\frac{2p\pi}{q}\right)-e^{-2\pi t}}{\cosh(2\pi t)-\cos\left(\frac{2p\pi}{q}\right)}\right)t^{2n-1}\,up\,dt \tag{8.23}$$

$$\left(n\in\mathbb{N}\setminus\{1\}\,;\ p\in\mathbb{N}_0;\ q\in\mathbb{N};\ p\leqq q\right)$$

and

$$\sum_{j=1}^{q}\zeta\left(2n+1,\frac{2j-1}{2q}\right)\sin\left(\frac{(2j-1)p\pi}{q}\right)$$
$$=\frac{(2q\pi)^{2n+1}\sin\left(\frac{p\pi}{q}\right)}{(2n)!}\int_0^\infty\left(\frac{\cosh(\pi t)}{\cosh(2\pi t)-\cos\left(\frac{2p\pi}{q}\right)}\right)t^{2n}\,up\,dt \tag{8.24}$$

$$\left(n\in\mathbb{N};\ p\in\mathbb{N}_0;\ q\in\mathbb{N};\ p\leqq q\right).$$

Remark 18. The formulas (**??**), (**??**) and (5.12) lead us easily to the following representations for the Apostol-Bernoulli polynomials, the Apostol-Euler polynomials and the Apostol-Genocchi polynomials at rational arguments:

$$\begin{aligned}\mathcal{B}_n\left(\frac{p}{q};-e^{2\pi i\xi}\right) &= -\frac{n!}{(2q\pi)^n}\left\{\sum_{j=1}^{q}\zeta\left(n,\frac{2\xi+2j-1}{2q}\right)\right.\\ &\times\exp\left[\left(\frac{n}{2}-\frac{(2\xi+2j-1)p}{q}\right)\pi i\right]+\sum_{j=1}^{q}\zeta\left(n,\frac{2j-2\xi-1}{2q}\right)\\ &\left.\times\exp\left[\left(-\frac{n}{2}+\frac{(2j-2\xi-1)p}{q}\right)\pi i\right]\right\}\end{aligned}\tag{8.25}$$

$$(n\in\mathbb{N}\setminus\{1\}\,;\ p\in\mathbb{Z};\ q\in\mathbb{N};\ \xi\in\mathbb{R}),$$

$$\begin{aligned}\mathcal{E}_n\left(\frac{p}{q};-e^{2\pi i\xi}\right) &= \frac{2\cdot n!}{(2q\pi)^{n+1}}\left\{\sum_{j=1}^{q}\zeta\left(n+1,\frac{\xi+j}{q}\right)\exp\left[\left(\frac{n+1}{2}-\frac{2(\xi+j)p}{q}\right)\pi i\right]\right.\\ &\left.+\sum_{j=1}^{q}\zeta\left(n+1,\frac{j-\xi-1}{q}\right)\exp\left[\left(-\frac{n+1}{2}+\frac{2(j-\xi-1)p}{q}\right)\pi i\right]\right\}\end{aligned}\tag{8.26}$$

$$(n,q\in\mathbb{N};\ p\in\mathbb{Z};\ \xi\in\mathbb{R})$$

and

$$\begin{aligned}\mathcal{G}_n\left(\frac{p}{q};-e^{2\pi i\xi}\right) &= \frac{2\cdot n!}{(2q\pi)^{n}}\left\{\sum_{j=1}^{q}\zeta\left(n,\frac{\xi+j}{q}\right)\exp\left[\left(\frac{n}{2}-\frac{2(\xi+j)p}{q}\right)\pi i\right]\right.\\ &\left.+\sum_{j=1}^{q}\zeta\left(n,\frac{j-\xi-1}{q}\right)\exp\left[\left(-\frac{n}{2}+\frac{2(j-\xi-1)p}{q}\right)\pi i\right]\right\}\end{aligned}\tag{8.27}$$

$$(n\in\mathbb{N}\setminus\{1\};\ p\in\mathbb{Z};\ q\in\mathbb{N};\ \xi\in\mathbb{R}),$$

respectively.

By applying Lemma 2, Lemma 3 and Lemma 4 and the above formulas (**??**), (8.26) and (8.27) in conjunction with the results of this paper and of the earlier works (see, for example, [44], [52], [45], [53], [50], [46], [47], [48] and [49]), we can also derive a large number of interesting formulas and relationships. For example, if we apply the relationships (3.2) and (3.4) in conjunction with the known Fourier expansions of the Apostol-Bernoulli polynomials and the Apostol-Euler polynomials (see, for details, [46, p. 2195, Theorem 2.1 (2.2) and (2.3); p. 2196, Theorem 2.2 (2.8) and Theorem 2.2 (2.9)]), we obtain the corresponding Fourier *exponential* series expansions for the Apostol-Genocchi polynomials $\mathcal{G}_n(x;\lambda)$ as follows.

Theorem 20. *The following Fourier exponential series expansions hold true for the Apostol-Genocchi polynomials* $\mathcal{G}_n(x;\lambda)$:

$$\mathcal{G}_n(x;\lambda) = \frac{2\cdot n!}{\lambda^x} \sum_{k=-\infty}^{k=\infty} \frac{e^{(2k-1)\pi i x}}{[(2k-1)\pi i - \log\lambda]^n} = \frac{(2\cdot n!)i^n}{\lambda^x}$$
$$\times \left(\sum_{k=0}^{\infty} \frac{\exp\left[\left(\frac{n\pi}{2} - (2k+1)\pi x\right) i\right]}{[(2k+1)\pi i + \log\lambda]^n} + \frac{\exp\left[\left(-\frac{n\pi}{2} + (2k+1)\pi x\right) i\right]}{[(2k+1)\pi i - \log\lambda]^n} \right) \tag{8.28}$$
$$(n \in \mathbb{N};\ 0 \leqq x \leqq 1;\ \lambda \in \mathbb{C} \setminus \{0,-1\}).$$

9 Unified Presentations of the Generalized Apostol Type Polynomials

The mutual relationships among the families of the generalized Apostol-Bernoulli polynomials, the generalized Apostol-Euler polynomials and the generalized Apostol-Genocchi polynomials, which are already asserted by Lemmas Lemma 2, Lemma 3 and Lemma 4, can be appropriately applied with a view to translating various formulas involving one family of these generalized polynomials into the corresponding results involving each of the other two families of these generalized polynomials. Nevertheless, we find it to be useful to investigate properties and results involving these three families of generalized Apostol type polynomials in a *unified* manner. In fact, the following interesting unification (and generalization) of the generating functions of the three families of Apostol type polynomials was recently investigated rather systematically by Ozden *et al.* (*cf.* [58, p. 2779, Equation (1.1)]):

$$\frac{2^{1-\kappa}\, z^{\kappa}\, e^{xz}}{\beta^b e^z - a^b} = \sum_{n=0}^{\infty} \mathcal{Y}_{n,\beta}(x;\kappa,a,b)\, \frac{z^n}{n!} \tag{9.1}$$

$$\left(|z| < 2\pi \quad \text{when} \quad \beta = a;\ |z| < \left| b \log\left(\frac{\beta}{a}\right)\right| \right.$$
$$\left. \text{when} \quad \beta \neq a;\ 1^{\alpha} := 1;\ \kappa,\beta \in \mathbb{C};\ a,b \in \mathbb{C}\setminus\{0\} \right),$$

where we have not only suitably relaxed the constraints on the parameters κ, a and b, but we have also strictly followed Remark 1 regarding the open disk in the complex z-plane (centred at the origin $z = 0$) within which the generating function in (9.1) is analytic in order to have the corresponding convergent Taylor-Maclaurin series expansion (about the origin $z = 0$) occurring on the right-hand side (with a *positive* radius of convergence).

Here, in conclusion of our present investigation, we define the following unification (and generalization) of the generating functions of the above-mentioned three families of the generalized Apostol type polynomials.

Definition 6. The *generalized* Apostol type polynomials

$$\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu;\nu) \qquad (\alpha,\lambda,\mu,\nu \in \mathbb{C})$$

of (real or complex) order α are defined by means of the following generating function:

$$\left(\frac{2^{\mu}\, z^{\nu}}{\lambda e^z+1}\right)^{\alpha} \cdot e^{xz} = \sum_{n=0}^{\infty} \mathcal{F}_n^{(\alpha)}(x;\lambda;\mu;\nu)\,\frac{z^n}{n!} \qquad \big(|z|<|\log(-\lambda)|\,;\; 1^{\alpha}:=1\big), \quad (9.2)$$

so that, by comparing Definition 6 with Definitions 2, 3 and 4, we have

$$\mathcal{B}_n^{(\alpha)}(x;\lambda) = (-1)^{\alpha}\, \mathcal{F}_n^{(\alpha)}(x;-\lambda;0;1)\,, \tag{9.3}$$

$$\mathcal{E}_n^{(\alpha)}(x;\lambda) = \mathcal{F}_n^{(\alpha)}(x;\lambda;1;0) \tag{9.4}$$

and

$$\mathcal{G}_n^{(\alpha)}(x;\lambda) = \mathcal{F}_n^{(\alpha)}(x;\lambda;1;1)\,. \tag{9.5}$$

Furthermore, if we compare the generating functions (9.1) and (9.2), we have

$$\mathcal{Y}_{n,\beta}(x;\kappa,a,b) = -\frac{1}{a^b}\,\mathcal{F}_n^{(1)}\left(x;-\left(\frac{\beta}{a}\right)^{b};1-\kappa;\kappa\right). \tag{9.6}$$

We thus see from the relationships (9.3), (9.4), (9.5) and (9.6) that the generating function of $\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu;\nu)$ in (9.2) includes, as its special cases, not only the generating function of the polynomials $\mathcal{Y}_{n,\beta}(x;\kappa,a,b)$ in (9.1) and the generating functions of *all* three of the generalized Apostol type polynomials $\mathcal{B}_n^{(\alpha)}(x;\lambda)$, $\mathcal{E}_n^{(\alpha)}(x;\lambda)$ and $\mathcal{G}_n^{(\alpha)}(x;\lambda)$, but indeed also the generating functions of their special cases $B_n^{(\alpha)}(x)$, $E_n^{(\alpha)}(x)$ and $G_n^{(\alpha)}(x)$.

The various interesting properties and results involving the *new* family of generalized Apostol type polynomials $\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu;\nu)$ can also be derived in a manner analogous to that of our investigation in this presentation.

The following natural generalization and unification of the Apostol-Bernoulli polynomials $\mathcal{B}_n^{(\alpha)}(x;\lambda)$ of order α as well as the generalized Bernoulli numbers $B_n(a,b)$ studied by Guo and Qi [20] and the generalized Bernoulli polynomials $B_n(x;a,b)$ studied by Luo *et al.* [51] was introduced and investigated recently by Srivastava *et al.* [69] (see also Definition 2).

Definition 7 (*cf.* [69, p. 254, Equation (20)]). The *generalized Apostol-Bernoulli type polynomials* $\mathfrak{B}_n^{(\alpha)}(x;\lambda;a,b,c)$ *of order* $\alpha \in \mathbb{C}$ are defined by the following generating function:

$$\left(\frac{z}{\lambda b^z - a^z}\right)^{\alpha} \cdot c^{xz} = \sum_{n=0}^{\infty} \mathfrak{B}_n^{(\alpha)}(x;\lambda;a,b,c)\,\frac{z^n}{n!} \tag{9.7}$$

$$\left(|z| < \left|\frac{\log\lambda}{\log\left(\frac{b}{a}\right)}\right|;\ a \in \mathbb{C}\setminus\{0\};\ b,c \in \mathbb{R}^{+};\ a \neq b;\ 1^{\alpha} := 1\right).$$

In a forthcoming sequel to the work by Srivastava *et al.* [69], a similar generalization of each of the families of Euler and Genocchi polynomials were introduced and investigated (see, for details, [70, Section 4]; see also Definitions 3 and 4).

Definition 8 (*cf.* [70, Section 2]). The *generalized Apostol-Euler type polynomials* $\mathfrak{E}_n^{(\alpha)}(x;\lambda;a,b,c)$ *of order* $\alpha \in \mathbb{C}$ are defined by the following generating function:

$$\left(\frac{2}{\lambda b^z + a^z}\right)^{\alpha} \cdot c^{xz} = \sum_{n=0}^{\infty} \mathfrak{E}_n^{(\alpha)}(x;\lambda;a,b,c)\ \frac{z^n}{n!} \tag{9.8}$$

$$\left(|z| < \left|\frac{\log(-\lambda)}{\log\left(\frac{b}{a}\right)}\right|;\ a \in \mathbb{C}\setminus\{0\};\ b,c \in \mathbb{R}^{+};\ a \neq b;\ 1^{\alpha} := 1\right).$$

Definition 9 (*cf.* [70, Section 4]). The *generalized Apostol-Genocchi type polynomials* $\mathfrak{G}_n^{(\alpha)}(x;\lambda;a,b,c)$ *of order* $\alpha \in \mathbb{C}$ are defined by the following generating function:

$$\left(\frac{2z}{\lambda b^z + a^z}\right)^{\alpha} \cdot c^{xz} = \sum_{n=0}^{\infty} \mathfrak{G}_n^{(\alpha)}(x;\lambda;a,b,c)\ \frac{z^n}{n!} \tag{9.9}$$

$$\left(|z| < \left|\frac{\log(-\lambda)}{\log\left(\frac{b}{a}\right)}\right|;\ a \in \mathbb{C}\setminus\{0\};\ b,c \in \mathbb{R}^{+};\ a \neq b;\ 1^{\alpha} := 1\right).$$

Remark 19. In their special case when

$$a = 1 \qquad \text{and} \qquad b = c = e,$$

the generalized Apostol-Bernoulli type polynomials $\mathfrak{B}_n^{(\alpha)}(x;\lambda;a,b,c)$ defined by (9.7), the generalized Apostol-Euler type polynomials $\mathfrak{E}_n^{(\alpha)}(x;\lambda;a,b,c)$ defined by (9.8) and the generalized Apostol-Genocchi type polynomials $\mathfrak{G}_n^{(\alpha)}(x;\lambda;a,b,c)$ defined by (9.9) would reduce at once to the Apostol-Bernoulli polynomials $\mathcal{B}_n^{(\alpha)}(x;\lambda)$, the the Apostol-Euler polynomials $\mathcal{E}_n^{(\alpha)}(x;\lambda)$ and the Apostol-Genocchi polynomials $\mathcal{G}_n^{(\alpha)}(x;\lambda)$, respectively (see Definitions 2, 3 and 4).

Since the parameter $\lambda \in \mathbb{C}$, by comparing Definitions 7, 8 and 9, we can easily deduce the following potentially useful lemma (see also Lemmas 1, 2 and 3).

Lemma 10. *The families of the generalized Apostol-Bernoulli type polynomials* $\mathfrak{B}_n^{(l)}(x;\lambda;a,b,c)\,(l \in \mathbb{N}_0)$ *and the generalized Apostol-Euler type polynomials* $\mathfrak{E}_n^{(l)}(x;\lambda;a,b,c)\quad(l \in \mathbb{N}_0)$ *are related by*

$$\mathfrak{B}_n^{(l)}(x;\lambda;a,b,c) = \left(-\frac{1}{2}\right)^{l} \frac{n!}{(n-l)!}\ \mathfrak{E}_{n-l}^{(l)}(x;-\lambda;a,b,c) \qquad (n,l \in \mathbb{N}_0;\ n \geqq l) \tag{9.10}$$

or, *equivalently*, *by*

$$\mathfrak{E}_n^{(l)}(x;\lambda;a,b,c) = (-2)^l \frac{n!}{(n+l)!} \mathfrak{B}_{n+l}^{(l)}(x;-\lambda;a,b,c) \qquad (n,l \in \mathbb{N}_0). \tag{9.11}$$

Furthermore, *the families of the generalized Apostol-Bernoulli type polynomials* $\mathfrak{B}_n^{(l)}(x;\lambda;a,b,c)\,(l \in \mathbb{N}_0)$ *and the generalized Apostol-Euler type polynomials* $\mathfrak{E}_n^{(l)}(x;\lambda;a,b,c) \quad (l \in \mathbb{N}_0)$ *are related to the generalized Apostol-Genocchi type polynomials* $\mathfrak{G}_n^{(l)}(x;\lambda;a,b,c) \quad (l \in \mathbb{N}_0)$ *as follows*:

$$\mathfrak{G}_n^{(\alpha)}(x;\lambda;a,b,c) = (-2)^\alpha \; \mathfrak{B}_n^{(\alpha)}(x;-\lambda;a,b,c) \qquad (\alpha \in \mathbb{C};\ 1^\alpha := 1) \tag{9.12}$$

and

$$\mathfrak{G}_n^{(l)}(x;\lambda;a,b,c) = (-1)^l \,(-n)_l \; \mathfrak{E}_{n-l}^{(l)}(x;\lambda;a,b,c) = \frac{n!}{(n-l)!} \; \mathfrak{E}_{n-l}^{(l)}(x;\lambda;a,b,c) \tag{9.13}$$

$$(n,l \in \mathbb{N}_0;\ n \geqq l;\ \lambda \in \mathbb{C}).$$

The inter-relationships asserted by Lemma 10 do aid in translating the various properties and results involving anyone of these three families of generalized Apostol type polynomials in terms of the corresponding properties and results involving the other two families. Nonetheless, it would occasionally seem to be more appropriately convenient to investigate these three families in a *unified* manner by means of Definition 10 below.

Definition 10. A unification of the *generalized Apostol-Bernoulli type polynomials* $\mathfrak{B}_n^{(\alpha)}(x;\lambda;a,b,c)$, the *generalized Apostol-Euler type polynomials* $\mathfrak{E}_n^{(\alpha)}(x;\lambda;a,b,c)$ and the *generalized Apostol-Genocchi type polynomials* $\mathfrak{G}_n^{(\alpha)}(x;\lambda;a,b,c)$ *of order* $\alpha \in \mathbb{C}$ is defined by the following generating function:

$$\left(\frac{2^\mu \; z^\nu}{\lambda b^z + a^z}\right)^\alpha \cdot c^{xz} = \sum_{n=0}^\infty \mathcal{Z}_n^{(\alpha)}(x;\lambda;a,b,c;\mu;\nu) \; \frac{z^n}{n!} \tag{9.14}$$

$$\left(|z| < \left|\frac{\log(-\lambda)}{\log\left(\frac{b}{a}\right)}\right|;\ a \in \mathbb{C}\setminus\{0\};\ b,c \in \mathbb{R}^+;\ a \neq b;\ \alpha,\lambda,\mu,\nu \in \mathbb{C};\ 1^\alpha := 1\right),$$

so that, by comparing Definition 10 with Definitions 6 to 9, we have

$$\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu;\nu) = \mathcal{Z}_n^{(\alpha)}(x;\lambda;1,e,e;\mu;\nu), \tag{9.15}$$

$$\mathfrak{B}_n^{(\alpha)}(x;\lambda;a,b,c) = (-1)^\alpha \; \mathcal{Z}_n^{(\alpha)}(x;-\lambda;a,b.c;0;1), \tag{9.16}$$

$$\mathfrak{E}_n^{(\alpha)}(x;\lambda;a,b,c) = \mathcal{Z}_n^{(\alpha)}(x;\lambda;a,b.c;1;0) \tag{9.17}$$

and

$$\mathfrak{G}_n^{(\alpha)}(x;\lambda;a,b,c) = \mathcal{Z}_n^{(\alpha)}(x;-\lambda;a,b.c;1;1). \tag{9.18}$$

Thus, clearly, Definitions 6 and 10 above provide us with remarkably powerful and extensive generalizations of the various families of the Apostol type polynomials and Apostol type numbers. Properties and results involving ach of these generalizations deserve to be investigated further (see also [55], [58], [69] and [70]).

10 Basic (or q-) Extensions For $q \in \mathbb{C}\ \ (|q| < 1)$, the q-shifted factorial $(\lambda; q)_\mu$ is defined by (see, for example,

$$(\lambda; q)_\mu = \prod_{j=0}^{\infty} \left(\frac{1 - \lambda q^j}{1 - \lambda q^{\mu+j}} \right) \qquad (q, \lambda, \mu \in \mathbb{C};\ |q| < 1), \tag{10.1}$$

so that

$$(\lambda; q)_n = \begin{cases} 1 & (n = 0) \\ \\ (1-\lambda)(1-\lambda q)\cdots(1-\lambda q^{n-1}) & (n \in \mathbb{N}), \end{cases} \tag{10.2}$$

$$(\lambda; q)_\infty = \prod_{j=0}^{\infty} (1 - \lambda q^j) \tag{10.3}$$

and

$$\lim_{q \to 1} \left\{ \frac{(q^\lambda; q)_n}{(q^\mu; q)_n} \right\} = \frac{(\lambda)_n}{(\mu)_n} \qquad (n \in \mathbb{N}_0;\ \mu \notin \mathbb{Z}_0 := \{0, -1, -2, \cdots\}), \tag{10.4}$$

where $(\lambda)_\nu$ denotes the Pochammer symbol (or the *shifted* or *rising* factorial) defined, in terms of the familiar Gamma function, by

$$(\lambda)_\nu = \frac{\Gamma(\lambda + \nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0;\ \lambda \in \mathbb{C} \setminus \{0\}) \\ \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu = n \in \mathbb{N};\ \lambda \in \mathbb{C}), \end{cases} \tag{10.5}$$

it being understood *conventionally* that $(0)_0 := 1$.

The q-number $[\lambda]_q$, the q-number factorial $[\lambda]_q!$ and the q-number shifted factorial $([\lambda]_q)_n$ are defined by

$$[0]_q = 0 \quad \text{and} \quad [\lambda]_q = \frac{1 - q^\lambda}{1 - q} \qquad (q \neq 1;\ \lambda \in \mathbb{C} \setminus \{0\}), \tag{10.6}$$

$$[0]_q! = 1 \quad \text{and} \quad [n]_q! = [1]_q [2]_q [3]_q \cdots [n]_q \qquad (n \in \mathbb{N}) \tag{10.7}$$

and

$$([\lambda]_q)_n = [\lambda]_q [\lambda+1]_q \cdots [\lambda+n-1]_q \qquad (n \in \mathbb{N};\ \lambda \in \mathbb{C}), \tag{10.8}$$

respectively. Clearly, we have the following limit cases:

$$\lim_{q\to 1}\{[\lambda]_q\} = \lambda, \quad \lim_{q\to 1}\{[n]_q!\} = n! \qquad \text{and} \qquad \lim_{q\to 1}\{([\lambda]_q)_n\} = (\lambda)_n, \tag{10.9}$$

where the Pochhammer symbol $(\lambda)_n$ is given by (10.5).

Over seven decades ago, Leonard Carlitz (1907–1999) extended the classical Bernoulli and Euler polynomials and numbers and introduced the q-Bernoulli and the q-Euler polynomials as well as the q-Bernoulli and the q-Euler numbers (see [5], [6] and [7]). There are numerous recent investigations on this subject by, among many other authors, Cenki *et al.* ([8], [9] and [10]), Choi *et al.* ([11] and [12]), [14], Kim *et al.* ([26], [27], [28], [29], [31], [32], [33], [34], [36] and [37]), Luo and Srivastava [54], Ozden and Simsek [57], Ryoo *et al.* [61], Simsek ([63], [64] and [65]) and Srivastava *et al.* [74].

We choose to recall here the definitions of the q-Bernoulli and the q-Euler polynomials of higher order as follows (see [5], [6], [7], [11], [12] and [54]).

Definition 11 (q-Bernoulli Polynomials of Order α). For $q, \alpha \in \mathbb{C}$ $(|q| < 1)$, the q-Bernoulli polynomials $B_{n;q}^{(\alpha)}(x)$ of order α in q^x are defined by means of the following generating function:

$$(-z)^\alpha \sum_{n=0}^{\infty} \frac{([\alpha]_q)_n}{[n]_q!} q^{n+x} e^{z[n+x]_q} = \sum_{n=0}^{\infty} B_{n;q}^{(\alpha)}(x)\,\frac{z^n}{n!}. \tag{10.10}$$

Obviously, we have (see Section 1 above)

$$\lim_{q\to 1}\left\{B_{n;q}^{(\alpha)}(x)\right\} = B_n^{(\alpha)}(x) \qquad \text{and} \qquad \lim_{q\to 1}\left\{B_{n;q}^{(\alpha)}\right\} = B_n^{(\alpha)}. \tag{10.11}$$

We also write

$$B_{n;q}(x) := B_{n;q}^{(1)}(x) \qquad (n \in \mathbb{N}_0) \tag{10.12}$$

for the *ordinary* q-Bernoulli polynomials $B_{n;q}(x)$.

Definition 12 (q-Bernoulli Numbers of Order α). For $q, \alpha \in \mathbb{C}$ $(|q| < 1)$, the q-Bernoulli numbers $B_{n;q}^{(\alpha)}$ of order α are defined by

$$B_{n;q}^{(\alpha)} := B_{n;q}^{(\alpha)}(0)\,. \tag{10.13}$$

We also write

$$B_{n;q} := B_{n;q}(0) \qquad (n \in \mathbb{N}_0) \tag{10.14}$$

for the *ordinary* q-Bernoulli numbers.

Definition 13 (q-Euler Polynomials of Order α). For $q, \alpha \in \mathbb{C}$ ($|q| < 1$), the q-Euler polynomials $E_{n;q}^{(\alpha)}(x)$ of order α in q^x are defined by means of the following generating function:

$$2^\alpha \sum_{n=0}^{\infty} \frac{([\alpha]_q)_n}{[n]_q!}(-1)^n \; q^{n+x} e^{z[n+x]_q} = \sum_{n=0}^{\infty} E_{n;q}^{(\alpha)}(x) \; \frac{z^n}{n!}. \tag{10.15}$$

Obviously, we have (see Section 1 above)

$$\lim_{q \to 1} \left\{ E_{n;q}^{(\alpha)}(x) \right\} = E_n^{(\alpha)}(x) \qquad \text{and} \qquad \lim_{q \to 1} \left\{ E_{n;q}^{(\alpha)} \right\} = E_n^{(\alpha)}. \tag{10.16}$$

We also write

$$E_{n;q}(x) := E_{n;q}^{(1)}(x) \qquad (n \in \mathbb{N}_0) \tag{10.17}$$

for the *ordinary* q-Euler polynomials $E_{n;q}(x)$.

Definition 14 (q-Euler Numbers of Order α). For $q, \alpha \in \mathbb{C}$ ($|q| < 1$), the q-Euler numbers $\widetilde{E}_{n;q}^{(\alpha)}$ of order α are defined by (see Remark 2)

$$\widetilde{E}_{n;q}^{(\alpha)} := 2^n \; E_{n;q}^{(\alpha)} \left(\frac{\alpha}{2} \right). \tag{10.18}$$

We also write

$$\widetilde{E}_{n;q} := 2^n \; E_{n;q} \left(\frac{1}{2} \right) \qquad (n \in \mathbb{N}_0) \tag{10.19}$$

for the *ordinary* q-Euler numbers $\widetilde{E}_{n;q}$.

In a similar manner, the q-Genocchi Polynomials $G_{n;q}^{(\alpha)}(x)$ and the q-Genocchi Numbers $\widetilde{G}_{n;q}^{(\alpha)}$ of Order α can be introduced here as follows.

Definition 15 (q-Genocchi Polynomials of Order α). For $q, \alpha \in \mathbb{C}$ ($|q| < 1$), the q-Genocchi polynomials $G_{n;q}^{(\alpha)}(x)$ of order α in q^x are defined by means of the following generating function:

$$(2z)^\alpha \sum_{n=0}^{\infty} \frac{([\alpha]_q)_n}{[n]_q!}(-1)^n \; q^{n+x} e^{z[n+x]_q} = \sum_{n=0}^{\infty} G_{n;q}^{(\alpha)}(x) \; \frac{z^n}{n!}. \tag{10.20}$$

Obviously, we have (see Section 1 above)

$$\lim_{q \to 1} \left\{ G_{n;q}^{(\alpha)}(x) \right\} = G_n^{(\alpha)}(x) \qquad \text{and} \qquad \lim_{q \to 1} \left\{ G_{n;q}^{(\alpha)} \right\} = G_n^{(\alpha)}. \tag{10.21}$$

We also write

$$G_{n;q}(x) := G_{n;q}^{(1)}(x) \qquad (n \in \mathbb{N}_0) \tag{10.22}$$

for the *ordinary* q-Genocchi polynomials $G_{n;q}(x)$.

Definition 16 (q-Genocchi Numbers of Order α). For $q, \alpha \in \mathbb{C}$ $(|q| < 1)$, the q-Genocchi numbers $\widetilde{G}_{n;q}^{(\alpha)}$ of order α are defined by (see also Section 1)

$$\widetilde{G}_{n;q}^{(\alpha)} := 2^n \, G_{n;q}^{(\alpha)}\left(\frac{\alpha}{2}\right). \tag{10.23}$$

We also write

$$\widetilde{G}_{n;q} := 2^n \, G_{n;q}\left(\frac{1}{2}\right) \qquad (n \in \mathbb{N}_0) \tag{10.24}$$

for the *ordinary* q-Genocchi numbers $\widetilde{G}_{n;q}$.

In the existing literature on the subjects and topics, which we have touched upon in our presentation here, one can find many different families of basic (or q-) extensions, not only of some of the aforementioned Bernoulli, Euler and Genocchi polynomials and numbers and their Apostol-type generalizations, but also of such other important functions of *Analytic Number Theory* as (for example) the Riemann zeta function $\zeta(s)$, the Hurwitz (or generalized) zeta function $\zeta(s, a)$ and the Hurwitz-Lerch zeta function $\Phi(z, s, a)$. Many (but, by no means, all) of these readily accessible recent references are being cited here.

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