Some Identities Involving Bernoulli and Stirling Numbers

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In this paper we prove some identities involving Bernoulli and Stirling numbers, relation for two or three consecutive Bernoulli numbers, and various representations of Bernoulli numbers. © 2001 Academic Press

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INTRODUCTION

Let $f(x, a) = a/\{1 - (1 - x)^a\}$ $(a \neq 0)$. We evaluate the coefficients of $f(x, a)$ in two ways; that is, in Section 1 we calculate $f(x, a)$ by using a result of Carlitz and in Section 2 expand it into a Laurent series directly. Then comparing these results, we get some identities involving Bernoulli and Stirling numbers (Main Theorem 3). As an application, we obtain relation for two or three consecutive Bernoulli numbers (Theorem 4). In Section 3 we analyze in detail the main theorem and derive various representations of Bernoulli numbers (Theorem 6 and Corollary 7).

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1. PRELIMINARIES

Carlitz [2] studied the expansion

$$\frac{x}{(1+\lambda x)^{\mu}-1} = \sum_{m=0}^{\infty} \beta_m(\lambda)\frac{x^m}{m!} \qquad (\lambda\mu = 1), \tag{1.1}$$

where $\beta_m(\lambda)$ is a polynomial in λ with rational coefficients. Since

$$(1+\lambda x)^{\mu} = (1+\lambda x)^{1/\lambda} \to e^x \qquad (\lambda \to 0),$$

(1.1) may be considered a degenerate type of the generating function

$$\frac{x}{e^x-1} = \sum_{m=0}^{\infty} B_m \frac{x^m}{m!}$$

of Bernoulli numbers B_m, and $\beta_m(\lambda)$ is called the degenerate Bernoulli number (see [3, p. 56]). We have $\beta_m(0) = B_m$, and in general, we expect that $\beta_m(\lambda)$ has arithmetic properties analogous to those of B_m. Indeed, Carlitz [2] proved an analog of the Staudt-Clausen theorem for $\beta_m(\lambda)$. In particular, he gave the following explicit formula for $\beta_m(\lambda)$ (see [2, p. 31, (4.3)])

$$\beta_m(\lambda) = \sum_{s=0}^{m} B_s \lambda^{m-s} \sum_{r=s}^{m} \frac{1}{r+1}\binom{r+1}{s} S_m^r, \tag{1.2}$$

in which S_m^r denotes the Stirling number of the first kind defined by

$$x(x-1)(x-2)\cdots(x-m+1) = S_m^1 x + S_m^2 x^2 + \cdots + S_m^m x^m, \tag{1.3}$$
$$S_0^r = 0 \quad \text{if} \quad r \neq 0 \quad \text{and} \quad S_0^0 = 1.$$

We now define a function $f(x, a)$ by

$$f(x, a) = \frac{a}{1-(1-x)^a} \qquad (a \neq 0).$$

Since $f(x, a)$ has a pole of order 1 at $x = 0$ with residue 1, it can be expanded into a Laurent series

$$f(x, a) = \sum_{m=-1}^{\infty} A_m(a) x^m, \qquad A_{-1}(a) = 1, \tag{1.4}$$

where $A_m(a)$ stands for a polynomial in a with rational coefficients.

To determine the coefficients $A_m(a)$ explicitly, we replace x by $-ax$ in (1.1) and set $\mu = a$. Then we get

$$\frac{ax}{1-(1-x)^a} = \sum_{m=0}^{\infty} (-1)^m \, a^m \beta_m \left(\frac{1}{a}\right) \frac{x^m}{m!},$$

and therefore the coefficient of x^m in (1.4) is given by

$$A_m(a) = \frac{(-1)^{m+1} \, a^{m+1}}{(m+1)!} \, \beta_{m+1} \left(\frac{1}{a}\right) \qquad (m \geqslant -1).$$

By virtue of (1.2), we have

$$
\begin{aligned}
A_m(a) &= \frac{(-1)^{m+1}}{(m+1)!} \sum_{s=0}^{m+1} B_s a^s \sum_{r=s}^{m+1} \frac{1}{r+1} \binom{r+1}{s} S_{m+1}^r \\
&= \frac{(-1)^{m+1}}{(m+1)!} \left\{ \sum_{r=0}^{m+1} \frac{S_{m+1}^r}{r+1} + \sum_{s=1}^{m+1} \frac{B_s}{s} a^s \sum_{r=s}^{m+1} \binom{r}{s-1} S_{m+1}^r \right\}.
\end{aligned}
$$

It follows from Jordan [10, p. 261, (29)] or [11, p. 185, (1)] that if $s \geqslant 2$, then

$$\sum_{r=s-1}^{m+1} \binom{r}{s-1} S_{m+1}^r = S_m^{s-1} + S_m^{s-2}.$$

Subtracting $S_{m+1}^{s-1} = S_m^{s-2} - m S_m^{s-1}$ from both sides, we get

$$\sum_{r=s}^{m+1} \binom{r}{s-1} S_{m+1}^r = (m+1) \, S_m^{s-1},$$

and the last equality is valid also for $s = 1$ and $m \geqslant 0$ (for $m > 0$, see [10, p. 256, (4)] or [11, p. 145, (6)]). Thus we have proved the following

THEOREM 1.   *Let*

$$f(x, a) = \frac{a}{1-(1-x)^a} = \frac{1}{x} + \sum_{m=0}^{\infty} A_m(a) \, x^m \qquad (a \neq 0).$$

Then for $m \geqslant -1$, we have

$$A_m(a) = \frac{(-1)^{m+1}}{(m+1)!} \left\{ \sum_{r=0}^{m+1} \frac{S_{m+1}^r}{r+1} + (m+1) \sum_{s=1}^{m+1} \frac{B_s}{s} S_m^{s-1} a^s \right\}.$$

Here we give two applications of this theorem.

(1)   Since $f(x, 1) = 1/x$, we have $A_m(1) = 0$ for $m \geqslant 0$. Then by Theorem 1, we get

$$\sum_{k=1}^{m+1} \frac{B_k}{k} S_m^{k-1} = -\frac{1}{m+1} \sum_{r=0}^{m+1} \frac{S_{m+1}^r}{r+1} \qquad (m \geqslant 0).$$

This relation involving Bernoulli and Stirling numbers is known (see [11, p. 147, (7) and p. 249, (5)]); it implies

$$A_m(a) = \frac{(-1)^{m+1}}{m!} \sum_{k=1}^{m+1} \frac{B_k}{k} S_m^{k-1}(a^k - 1) \qquad (m \geqslant 0). \qquad (1.5)$$

(2)   Since $f(x, 2) = 1/(x(1-x/2))$, we have $A_m(2) = 2^{-m-1}$ for $m \geqslant -1$, and from Theorem 1, we obtain

COROLLARY 2.   *For $m \geqslant 0$, we have*

$$\sum_{k=1}^{m+1} 2^k \frac{B_k}{k} S_m^{k-1} = m! \left(-\frac{1}{2}\right)^{m+1} - \frac{1}{m+1} \sum_{r=0}^{m+1} \frac{S_{m+1}^r}{r+1}$$

and

$$\sum_{k=1}^{m+1} (2^k - 1) \frac{B_k}{k} S_m^{k-1} = m! \left(-\frac{1}{2}\right)^{m+1}.$$

2. MAIN THEOREM

In this section, we expand $f(x, a)$ into a Laurent series directly, and by comparing it with $f(x, a)$ in Theorem 1, we obtain some identities involving Bernoulli and Stirling numbers.

We have

$$f(x, a) = \frac{1}{x\left\{1 - \dfrac{(1-x)^a - 1 + ax}{ax}\right\}} = \frac{1}{x} \sum_{s=0}^{\infty} \left\{\frac{(1-x)^a - 1 + ax}{ax}\right\}^s$$

$$= \frac{1}{x} + \frac{1}{x} \sum_{s=1}^{\infty} \left\{\sum_{j=1}^{\infty} (-1)^{j+1} \frac{1}{a} \binom{a}{j+1} x^j\right\}^s.$$

Therefore when $m \geqslant 0$, the coefficient $A_m(a)$ of x^m in $f(x, a)$ is given by

$$\sum_{s=1}^{m+1} (-1)^{m+1+s} \sum_{\substack{p_1 + p_2 + \cdots + p_s = m+1 \\ p_i \geqslant 1}} \frac{1}{a} \binom{a}{p_1+1} \frac{1}{a} \binom{a}{p_2+1} \cdots \frac{1}{a} \binom{a}{p_s+1}.$$

Moreover by (1.3), we get

$$\frac{1}{a}\binom{a}{p+1} = \frac{(a-1)(a-2)\cdots(a-p)}{(p+1)!} = \frac{1}{(p+1)!}\sum_{q=0}^{p} S_{p+1}^{q+1} a^q.$$

Hence we obtain by (1.5),

$$\frac{1}{m!}\sum_{k=1}^{m+1}\frac{B_k}{k}S_m^{k-1}(a^k-1)$$

$$= \sum_{s=1}^{m+1}(-1)^s \sum_{\substack{p_1+p_2+\cdots+p_s=m+1 \\ p_i \geqslant 1}} \frac{1}{(p_1+1)!\,(p_2+1)!\cdots(p_s+1)!}$$

$$\times \sum_{k=0}^{m+1}\left(\sum_{\substack{q_1+q_2+\cdots+q_s=k \\ q_i \geqslant 0}} S_{p_1+1}^{q_1+1}S_{p_2+1}^{q_2+1}\cdots S_{p_s+1}^{q_s+1}\right)a^k \qquad (m \geqslant 0).$$

Notice that if $q > p$, then $S_{p+1}^{q+1} = 0$. Since both sides are polynomials of degree $m+1$ in a, the following main theorem is now obtained by equating coefficients of a^k, $k \geqslant 0$. For the case $k = 0$, we note that $S_{p_i+1}^1 = (-1)^{p_i} p_i!$ (see, e.g., [1, p. 824], [6, p. 260, (6.5)], [10, p. 256, 1] or [11, p. 147, 1]).

THEOREM 3.   (1)   *For $m \geqslant 0$, we have*

$$\sum_{k=1}^{m+1}\frac{B_k}{k}S_m^{k-1} = (-1)^m m!\sum_{s=1}^{m+1}(-1)^s$$

$$\times \sum_{\substack{p_1+p_2+\cdots+p_s=m+1 \\ p_i \geqslant 1}} \frac{1}{(p_1+1)(p_2+1)\cdots(p_s+1)}.$$

(2)   *For $1 \leqslant k \leqslant m+1$, we have*

$$\frac{B_k}{k}S_m^{k-1} = m!\sum_{s=1}^{m+1}(-1)^s$$

$$\times \sum_{\substack{p_1+p_2+\cdots+p_s=m+1 \\ p_i \geqslant 1}}\sum_{\substack{q_1+q_2+\cdots+q_s=k \\ q_i \geqslant 0}} \frac{S_{p_1+1}^{q_1+1}S_{p_2+1}^{q_2+1}\cdots S_{p_s+1}^{q_s+1}}{(p_1+1)!\,(p_2+1)!\cdots(p_s+1)!}.$$

Let $k = m + 1$ in (2) of Theorem 3. Then the conditions

$$\begin{cases} p_1 + p_2 + \cdots + p_s = m + 1, \\ q_1 + q_2 + \cdots + q_s = m + 1, \qquad 0 \leqslant q_i \leqslant p_i, \end{cases}$$

imply that $q_i = p_i$ for $1 \leqslant i \leqslant s$. Hence for $m \geqslant 0$, we have

$$B_{m+1} = (m+1)! \sum_{s=1}^{m+1} (-1)^s$$

$$\times \sum_{\substack{p_1 + p_2 + \cdots + p_s = m+1 \\ p_i \geqslant 1}} \frac{1}{(p_1+1)! \, (p_2+1)! \cdots (p_s+1)!}, \quad (2.1)$$

because of $S_n^n = 1$. This is the well-known formula for Bernoulli numbers (see [11, p. 247, (5), p. 251 or p. 599, (19)]). If we vary the value of k, we can produce analogous formulas. We treat here only the cases of $k = m$ and of $k = m - 1$.

Let $k = m \geqslant 1$ in (2) of Theorem 3. Then the conditions

$$\begin{cases} p_1 + p_2 + \cdots + p_s = m + 1, \\ q_1 + q_2 + \cdots + q_s = k = m, \qquad 0 \leqslant q_i \leqslant p_i, \end{cases}$$

imply that for one and only one q_i, $q_i = p_i - 1$ and for the other q_j's, $q_j = p_j$. We have (see, e.g., [1, p. 824], [6, p. 264] or [11, p. 149])

$$S_n^{n-1} = -\binom{n}{2} = -\frac{n(n-1)}{2}. \quad (2.2)$$

Since $S_n^n = 1$, we have, under the condition of $p_1 + p_2 + \cdots + p_s = m + 1$,

$$\sum_{\substack{q_1 + q_2 + \cdots + q_s = m \\ q_i \geqslant 0}} S_{p_1+1}^{q_1+1} S_{p_2+1}^{q_2+1} \cdots S_{p_s+1}^{q_s+1} = \sum_{i=1}^{s} S_{p_i+1}^{p_i} = -\sum_{i=1}^{s} \frac{(p_i+1)\,p_i}{2}.$$

Therefore letting $k = m$ in (2) of Theorem 3, we get

$$\frac{B_m}{m} \frac{m(m-1)}{2}$$

$$= m! \sum_{s=1}^{m+1} (-1)^s \sum_{\substack{p_1 + p_2 + \cdots + p_s = m+1 \\ p_i \geqslant 1}} \frac{p_1^2 + p_2^2 + \cdots + p_s^2 + m + 1}{2(p_1+1)! \, (p_2+1)! \cdots (p_s+1)!}.$$

Using (2.1), we obtain the first part of the following

THEOREM 4.   (1)   *Let $m \geq 1$. Then*

$$(m-1) B_m - B_{m+1}$$
$$= m! \sum_{s=1}^{m+1} (-1)^s \sum_{\substack{p_1 + p_2 + \cdots + p_s = m+1 \\ p_i \geq 1}} \frac{p_1^2 + p_2^2 + \cdots + p_s^2}{(p_1+1)! \, (p_2+1)! \cdots (p_s+1)!}. \tag{2.3}$$

(2)   *Let $m \geq 2$. Then*

$$m(m-2)(3m-1) B_{m-1} - 6m(m-1) B_m + (3m-1) B_{m+1}$$
$$= m! \sum_{s=1}^{m+1} (-1)^s$$
$$\times \sum_{\substack{p_1 + p_2 + \cdots + p_s = m+1 \\ p_i \geq 1}} \frac{3(p_1^2 + p_2^2 + \cdots + p_s^2)^2 - 4(p_1^3 + p_2^3 + \cdots + p_s^3)}{(p_1+1)! \, (p_2+1)! \cdots (p_s+1)!}.$$

To prove the second part of this theorem, we deal with the case $k = m-1 \ (m \geq 2)$. We have (see, e.g., [11, p. 149] or [12, p. 144])

$$S_n^{n-2} = 3 \binom{n}{4} + 2 \binom{n}{3} = \frac{1}{24} n(n-1)(n-2)(3n-1). \tag{2.4}$$

Let $k = m-1 \geq 1$ in (2) of Theorem 3. Then the conditions

$$\begin{cases} p_1 + p_2 + \cdots + p_s = m+1, \\ q_1 + q_2 + \cdots + q_s = k = m-1, \qquad 0 \leq q_i \leq p_i, \end{cases}$$

yield the following two cases:

(A)   For one and only one q_i, $q_i = p_i - 2$ and for all other q_j's, $q_j = p_j$. In this case, we have, by (2.4),

$$S_{p_i+1}^{q_i+1} = S_{p_i+1}^{p_i-1} = \tfrac{1}{24}(p_i+1) \, p_i(p_i-1)(3p_i+2),$$
$$S_{p_j+1}^{q_j+1} = S_{p_j+1}^{p_j+1} = 1.$$

(B)   There are only two q_i, q_j such that $q_i = p_i - 1$, $q_j = p_j - 1$, and for all other q_k's, $q_k = p_k$. Then by (2.2), we have

$$S_{p_i+1}^{q_i+1} S_{p_j+1}^{q_j+1} = S_{p_i+1}^{p_i} S_{p_j+1}^{p_j} = \tfrac{1}{4}(p_i^2 + p_i)(p_j^2 + p_j) \qquad \text{for} \quad i \neq j,$$
$$S_{p_k+1}^{q_k+1} = S_{p_k+1}^{p_k+1} = 1.$$

From (A) and (B), we obtain, under the condition of $p_1 + p_2 + \cdots + p_s = m + 1$,

$$
\begin{aligned}
X &:= \sum_{\substack{q_1 + q_2 + \cdots + q_s = m-1 \\ q_i \geq 0}} S_{p_1+1}^{q_1+1} S_{p_2+1}^{q_2+1} \cdots S_{p_s+1}^{q_s+1} \\
&= \tfrac{1}{24} \sum_{i=1}^{s} (p_i + 1)\, p_i(p_i - 1)(3p_i + 2) + \tfrac{1}{4} \sum_{i \neq j} (p_i^2 + p_i)(p_j^2 + p_j).
\end{aligned}
$$

Since the last summation can be evaluated as

$$
\tfrac{1}{2}\left(\sum_{i=1}^{s} p_i^2 \right)^2 + (m+1) \sum_{i=1}^{s} p_i^2 + \tfrac{1}{2}(m+1)^2 - \tfrac{1}{2} \sum_{i=1}^{s} p_i^2(p_i+1)^2,
$$

we have

$$
\begin{aligned}
X &= \tfrac{1}{8}(p_1^2 + p_2^2 + \cdots + p_s^2)^2 + \tfrac{1}{4}m(p_1^2 + p_2^2 + \cdots + p_s^2) + \tfrac{1}{8}(m+1)^2 \\
&\quad - \tfrac{1}{6}(p_1^3 + p_2^3 + \cdots + p_s^3) - \tfrac{1}{12}(m+1).
\end{aligned}
$$

Putting this X in (2) of Theorem 3 with $k = m - 1$, we obtain by (2.4),

$$
\begin{aligned}
\frac{B_{m-1}}{m-1} &\frac{1}{24} m(m-1)(m-2)(3m-1) \\
&= \frac{1}{8} m! \sum_{s=1}^{m+1} (-1)^s \sum_{\substack{p_1 + p_2 + \cdots + p_s = m+1 \\ p_i \geq 1}} \frac{(p_1^2 + p_2^2 + \cdots + p_s^2)^2}{(p_1 + 1)!\,(p_2 + 1)! \cdots (p_s + 1)!} \\
&\quad + \frac{1}{4} m \cdot m! \sum_{s=1}^{m+1} (-1)^s \\
&\quad \times \sum_{\substack{p_1 + p_2 + \cdots + p_s = m+1 \\ p_i \geq 1}} \frac{p_1^2 + p_2^2 + \cdots + p_s^2}{(p_1 + 1)!\,(p_2 + 1)! \cdots (p_s + 1)!} \\
&\quad - \frac{1}{6} m! \sum_{s=1}^{m+1} (-1)^s \sum_{\substack{p_1 + p_2 + \cdots + p_s = m+1 \\ p_i \geq 1}} \frac{p_1^3 + p_2^3 + \cdots + p_s^3}{(p_1 + 1)!\,(p_2 + 1)! \cdots (p_s + 1)!} \\
&\quad + \left\{ \frac{1}{8}(m+1)^2 - \frac{1}{12}(m+1) \right\} \\
&\quad \times m! \sum_{s=1}^{m+1} (-1)^s \sum_{\substack{p_1 + p_2 + \cdots + p_s = m+1 \\ p_i \geq 1}} \frac{1}{(p_1 + 1)!\,(p_2 + 1)! \cdots (p_s + 1)!}.
\end{aligned}
$$

Summing the first part and the third part on the right side of this equality, and moreover putting (2.3) in the second part and (2.1) in the fourth part, we obtain the second part of Theorem 4.

3. VARIOUS REPRESENTATIONS OF BERNOULLI NUMBERS

In the preceding section, we have calculated the right side of (2) in Theorem 3 for a few values of k. In this section, we evaluate, in general, the term

$$Y := \sum_{\substack{p_1 + p_2 + \cdots + p_s = m+1 \\ p_i \geqslant 1}} \sum_{\substack{q_1 + q_2 + \cdots + q_s = k \\ q_i \geqslant 0}} \frac{S_{p_1+1}^{q_1+1} S_{p_2+1}^{q_2+1} \cdots S_{p_s+1}^{q_s+1}}{(p_1+1)! \, (p_2+1)! \cdots (p_s+1)!}$$

$$(1 \leqslant k \leqslant m+1, \, 1 \leqslant s \leqslant m+1)$$

appearing on the right side of (2) in Theorem 3.

By the well-known expansion of $[\log(1+t)]^m$ (see, e.g., [1, p. 824], [10, p. 272, (76)] or [11, p. 202, (7)]), we have

$$\frac{[\log(1+t)]^{q_i+1}}{(q_i+1)!} = \sum_{n=q_i+1}^{\infty} \frac{S_n^{q_i+1}}{n!} t^n.$$

Hence

$$\prod_{i=1}^{s} \frac{[\log(1+t)]^{q_i+1}}{(q_i+1)!} = \sum_{c=k+s}^{\infty} \left(\sum_{\substack{n_1 + n_2 + \cdots + n_s = c \\ n_i \geqslant q_i+1}} \frac{S_{n_1}^{q_1+1} S_{n_2}^{q_2+1} \cdots S_{n_s}^{q_s+1}}{n_1! \, n_2! \cdots n_s!} \right) t^c,$$

because of $q_1 + 1 + q_2 + 1 + \cdots + q_s + 1 = k + s$. The left side becomes

$$\frac{[\log(1+t)]^{k+s}}{(q_1+1)! \, (q_2+1)! \cdots (q_s+1)!}$$

$$= \frac{(k+s)!}{(q_1+1)! \, (q_2+1)! \cdots (q_s+1)!} \sum_{c=k+s}^{\infty} \frac{S_c^{k+s}}{c!} t^c.$$

Since $S_{p_i+1}^{q_i+1} = 0$ for $q_i > p_i$, we may limit the ranges of q_i's to $0 \leqslant q_i \leqslant p_i$. Then $k = q_1 + \cdots + q_s \leqslant p_1 + \cdots + p_s = m+1$ and $k + s \leqslant m+1+s$. Comparing coefficients of t^{m+1+s} and setting $n_i = p_i + 1$, we have

$$\frac{(k+s)!}{(q_1+1)! \, (q_2+1)! \cdots (q_s+1)!} \cdot \frac{S_{m+1+s}^{k+s}}{(m+1+s)!}$$

$$= \sum_{\substack{p_1 + p_2 + \cdots + p_s = m+1 \\ p_i \geqslant 0}} \frac{S_{p_1+1}^{q_1+1} S_{p_2+1}^{q_2+1} \cdots S_{p_s+1}^{q_s+1}}{(p_1+1)! \, (p_2+1)! \cdots (p_s+1)!}.$$

In order to determine the value of Y, we have to sum over all $q_i \geqslant 0$ with $q_1 + q_2 + \cdots + q_s = k$. We make use of the identity (see [11, p. 176, (5)])

$$\sum_{\substack{q_1 + q_2 + \cdots + q_s = k \\ q_i \geqslant 0}} \frac{1}{(q_1+1)!\,(q_2+1)!\cdots(q_s+1)!} = \frac{s!}{(k+s)!}\,\mathfrak{S}_{k+s}^s,$$

where $\mathfrak{S}_n^m$ denotes the Stirling number of the second kind generated by

$$x^n = \sum_{m=1}^{n} x(x-1)(x-2)\cdots(x-m+1)\,\mathfrak{S}_n^m.$$

Thus from the above, we get

$$\frac{s!}{(m+1+s)!}\,\mathfrak{S}_{k+s}^s S_{m+1+s}^{k+s}$$

$$= \sum_{\substack{p_1 + p_2 + \cdots + p_s = m+1 \\ p_i \geqslant 0}} \sum_{\substack{q_1 + q_2 + \cdots + q_s = k \\ q_i \geqslant 0}} \frac{S_{p_1+1}^{q_1+1} S_{p_2+1}^{q_2+1} \cdots S_{p_s+1}^{q_s+1}}{(p_1+1)!\,(p_2+1)!\cdots(p_s+1)!}.$$

Y is equal to the sum of the terms with all $p_i \geqslant 1$ on the right side. If some p_i's are 0, say, $p_{r+1} = p_{r+2} = \cdots = p_s = 0$, then the second summation on the right side of the above equality reduces to

$$\sum_{\substack{q_1 + q_2 + \cdots + q_r = k \\ q_i \geqslant 0}} \frac{S_{p_1+1}^{q_1+1} S_{p_2+1}^{q_2+1} \cdots S_{p_r+1}^{q_r+1}}{(p_1+1)!\,(p_2+1)!\cdots(p_r+1)!},$$

because, since $S_1^1 = 1$ and $S_1^m = 0$ if $m \neq 1$, only the terms with $q_{r+1} = q_{r+2} = \cdots = q_s = 0$ remain.

Now fix m, k, s, and for $1 \leqslant r \leqslant s$, define

$$F(r) := \sum_{\substack{p_1 + p_2 + \cdots + p_r = m+1 \\ p_i \geqslant 1}} \sum_{\substack{q_1 + q_2 + \cdots + q_r = k \\ q_i \geqslant 0}} \frac{S_{p_1+1}^{q_1+1} S_{p_2+1}^{q_2+1} \cdots S_{p_r+1}^{q_r+1}}{(p_1+1)!\,(p_2+1)!\cdots(p_r+1)!},$$

and $F(0) := 0$. Then $Y = F(s)$, and by the argument mentioned above, we have

$$\frac{s!}{(m+1+s)!}\,\mathfrak{S}_{k+s}^s S_{m+1+s}^{k+s} = \sum_{r=0}^{s} \binom{s}{r} F(r).$$

In view of the well-known binomial coefficient inversion theorem (see, e.g., [6, p. 192, (5.48)] or [9, p. 22])

$$G(n) = \sum_{k=0}^{n} \binom{n}{k} F(k) \Leftrightarrow F(n) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} G(k),$$

we find an expression for $Y = F(s)$, and with $\mathfrak{S}_k^0 = 0$ $(k \geqslant 1)$ we obtain

THEOREM 5. *Let $1 \leqslant k \leqslant m+1$ and $1 \leqslant s \leqslant m+1$. Then we have*

$$\sum_{\substack{p_1 + p_2 + \cdots + p_s = m+1 \\ p_i \geqslant 1}} \sum_{\substack{q_1 + q_2 + \cdots + q_s = k \\ q_i \geqslant 0}} \frac{S_{p_1+1}^{q_1+1} S_{p_2+1}^{q_2+1} \cdots S_{p_s+1}^{q_s+1}}{(p_1+1)! \, (p_2+1)! \cdots (p_s+1)!}$$
$$= \sum_{j=1}^{s} (-1)^{s-j} \binom{s}{j} \frac{j!}{(m+1+j)!} \, \mathfrak{S}_{k+j}^{j} S_{m+1+j}^{k+j}.$$

With Theorem 3, (2) this gives for $1 \leqslant k \leqslant m+1$,

$$\frac{B_k}{k} S_m^{k-1} = m! \sum_{s=1}^{m+1} \sum_{j=1}^{m+1} (-1)^j \binom{s}{j} \frac{j!}{(m+1+j)!} \, \mathfrak{S}_{k+j}^{j} S_{m+1+j}^{k+j},$$

and using the binomial identity $\sum_{s=1}^{m+1} \binom{s}{j} = \binom{m+2}{j+1}$, we obtain

THEOREM 6. *Let $1 \leqslant k \leqslant m+1$. Then*

$$\frac{B_k}{k} S_m^{k-1} = \frac{1}{m+1} \sum_{j=1}^{m+1} (-1)^j \frac{\binom{m+2}{j+1}}{\binom{m+j+1}{m+1}} \, \mathfrak{S}_{k+j}^{j} S_{m+j+1}^{k+j}.$$

Letting $k = m+1$ in this theorem, we get

$$B_{m+1} = \sum_{j=1}^{m+1} (-1)^j \frac{\binom{m+2}{j+1}}{\binom{m+j+1}{m+1}} \, \mathfrak{S}_{m+j+1}^{j} \qquad (m \geqslant 0),$$

which is a known representation of Bernoulli numbers (see [4, p. 48, (11)] or [11, p. 219]). We can proceed further.

COROLLARY 7. *For $m \geqslant 2$, we have*

$$(1) \qquad B_m = \frac{m}{m-1} \sum_{j=1}^{m+1} (-1)^j \frac{\binom{m+2}{j+1}}{\binom{m+j-1}{m-1}} \mathfrak{S}_{m+j}^j,$$

$$(2) \qquad B_m = \frac{m}{(m-1)(3m+2)} \sum_{j=1}^{m+2} (-1)^j$$

$$\times \frac{(3m+3j+5)\binom{m+3}{j+1}}{\binom{m+j-1}{m-1}} \mathfrak{S}_{m+j}^j.$$

Proof. (1) follows immediately from Theorem 6 with $k = m \geqslant 1$, if we use (2.2). Similarly, for (2) we set $k = m - 1 \geqslant 1$; then we use (2.4) and simplify. ∎

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