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CLOSED FORMULA FOR POLY-BERNOULLI NUMBERS

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1. INTRODUCTION AND BACKGROUND

In the present note we shall give two proofs of a property of the poly-Bernoulli numbers, the closed formula for negative index poly-Bernoulli numbers given by Arakawa and Kaneko [1]. The first proof uses weighted Stirling numbers of the second kind (see [2], [3]). The second, much simpler, proof is due to Zeilberger.

In Kaneko's paper, "On Poly-Bernoulli Numbers" [5], the poly-Bernoulli numbers, which generalize the classical Bernoulli numbers, are defined and studied. For every integer k, called the index, we define a sequence of rational numbers B_n^k $(n=0,1,2,\ldots)$, which we refer to as poly-Bernoulli numbers, by

$$\frac{1}{z}\mathrm{Li}_k(z)\Big|_{z=1-e^{-x}}=\sum_{n=0}^{\infty}\mathrm{B}_n^k\frac{x^n}{n!}. \tag{1}$$

Here, for any integer k, $\mathrm{Li}_k(z)$ denotes the formal power series $\sum_{m=1}^{\infty}z^m/m^k$, which is the k^{th} polylogarithm if $k\geq 1$ and a rational function if $k\leq 0$. When $k=1$, B_n^1 is the usual Bernoulli number (with $\mathrm{B}_1^1=1/2$). In [4] Kaneko obtained an explicit formula for B_n^k:

$$\mathrm{B}_n^k=(-1)^n\sum_{m=0}^{n}\frac{(-1)^m m!}{(m+1)^k}\left\{{n \atop m}\right\}, \tag{2}$$

where $\left\{{n \atop m}\right\}$ is an integer referred to as a Stirling number of the second kind [6].

2. CLOSED FORMULA

Theorem 2.1 (Closed Formula): For any $n,k\geq 0$, we have

$$\mathrm{B}_n^{-k}=\sum_{j=0}(j!)^2\left\{{n+1 \atop j+1}\right\}\left\{{k+1 \atop j+1}\right\}. \tag{3}$$

We need two lemmas. We use the notation and numeration of the equations in Carlitz's paper [3].

Lemma 2.1:

$$\sum_{m=0}^{n}(-1)^m m!\binom{m}{\ell}\left\{{n \atop m}\right\}=(-1)^n\ell!\left\{{n+1 \atop \ell+1}\right\}=(-1)^n\ell!\,\mathrm{R}(n,\ell,1), \tag{4}$$

where

$$\mathrm{R}(n,k,\lambda)=\sum_{m=0}^{n-k}\binom{n}{m}\left\{{n-m \atop k}\right\}\lambda^m.$$

Proof: In order to prove this lemma, we calculate the generating function:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}(-1)^m m!\binom{m}{\ell}\left\{{n \atop m}\right\}\frac{z^n}{n!}=\sum_{m=0}^{\infty}(-1)^m\binom{m}{\ell}m!\sum_{n=0}^{\infty}\left\{{n \atop m}\right\}\frac{z^n}{n!}$$

$$= \sum_{m=0}^{\infty}(-1)^m\binom{m}{\ell}m!\,\frac{(e^z-1)^m}{m!} = \frac{(1-e^z)^\ell}{(1-(1-e^z))^{\ell+1}},\ \text{by the generalized binomial theorem,}$$

$$= e^{-z}(e^{-z}-1)^\ell = \sum_{n=0}^{\infty}\ell!\,\mathrm{R}(n,\ell,1)(-1)^n\frac{z^n}{n!},\quad \text{by [3], (3.9).}$$

Lemma 2.2:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\mathrm{B}_n^{-k}x^n y^k = \sum_{j=0}^{\infty}p_j(x)p_j(y) \tag{5}$$

where $p_j(x) = j!\sum_{n=0}^{\infty}\mathrm{R}(n,j,1)x^n$.

Proof: By (2), we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\mathrm{B}_n^{-k}x^n y^k = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\left((-1)^n(-1)^m m!\left\{{n \atop m}\right\}(m+1)^k\right)x^n y^k,\ \text{by [3], (3.4),}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{\infty}\ell!\binom{m}{\ell}\mathrm{R}(k,\ell,1)\left((-1)^n(-1)^m m!\left\{{n \atop m}\right\}\right)x^n y^k$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{\infty}\ell!\binom{m}{\ell}\left((-1)^n(-1)^m m!\left\{{n \atop m}\right\}\right)\frac{p_\ell(y)}{\ell!}x^n$$

$$= \sum_{\ell=0}^{\infty}\sum_{n=0}^{\infty}p_\ell(y)(-1)^n\sum_{m=0}^{\infty}\binom{m}{\ell}(-1)^m m!\left\{{n \atop m}\right\}x^n,\ \text{by Lemma 2.1,}$$

$$= \sum_{\ell=0}^{\infty}\sum_{n=0}^{\infty}p_\ell(y)(-1)^n(-1)^n\ell!\,\mathrm{R}(n,\ell,1)\,x^n = \sum_{\ell=0}^{\infty}p_\ell(y)\ell!\sum_{n=0}^{\infty}\mathrm{R}(n,\ell,1)\,x^n = \sum_{\ell=0}^{\infty}p_\ell(x)p_\ell(y).$$

Proof of (3): To prove (3), we compare the coefficients on both sides of (5). In the course of Arakawa and Kaneko's proof they prove the following proposition.

Proposition 2.1: For $n > 0$,

$$\sum_{\ell=0}^{n}(-1)^\ell\mathrm{B}_{n-\ell}^{-\ell} = 0.$$

Proof: We offer a more direct proof:

$$\sum_{\ell=0}^{n}(-1)^\ell\mathrm{B}_{n-\ell}^{-\ell} = \sum_{\ell=0}^{n}(-1)^\ell(-1)^{n-\ell}\sum_{m=0}^{n-\ell}(-1)^m m!(m+1)^\ell\left\{{n-\ell \atop m}\right\}$$

$$= (-1)^n\sum_{m=0}^{n}\sum_{\ell=0}^{n}(-1)^m m!(m+1)^\ell\left\{{n-\ell \atop m}\right\},\ \text{by [4], (6.20),}$$

$$= (-1)^n\sum_{m=0}^{n}(-1)^m m!\left\{{n+1 \atop m+1}\right\} = (-1)^n\delta_{1\,n+1} = 0.$$

3. ANOTHER PROOF

In Kaneko's paper [4], he obtained the *symmetric* formula:

$$\sum_{k\ge0}\sum_{n\ge0}\mathrm{B}_n^{-k}\frac{x^n}{n!}\frac{y^k}{k!} = \frac{e^{x+y}}{e^x+e^y-e^{x+y}}. \tag{6}$$

By using (6), D. Zeilberger gives a much simpler proof of (3) as follows:

$$\sum_{k\geq 0}\sum_{n\geq 0} B_n^{-k}\frac{x^n}{n!}\frac{y^k}{k!} = \frac{e^{x+y}}{e^x+e^y-e^{x+y}} = e^{x+y}\sum_{j\geq 0}(1-e^x)^j(1-e^y)^j$$

$$= \sum_{j\geq 0}\frac{1}{(1+j)^2}(j+1)(1-e^x)^j(-e^x)(j+1)(1-e^y)^j(-e^y)$$

$$= \sum_{j\geq 0}\frac{1}{(j+1)^2}D_x[(1-e^x)^{j+1}]D_y[(1-e^y)^{j+1}].$$

Now using the usual generating function for the Stirling numbers of the second kind $\left\{ {n \atop k} \right\}$, i.e.,

$$\sum_{n\geq k}\left\{ {n \atop k} \right\}\frac{u^n}{n!} = \frac{(e^u-1)^k}{k!},$$

he obtains:

$$\sum_{n\geq 0}\sum_{k\geq 0} B_n^{-k}\frac{x^n}{n!}\frac{y^k}{k!} = \sum_{j\geq 0}\frac{1}{(j+1)^2}D_x\left[(-1)^{j+1}(j+1)!\sum_{n\geq j+1}\left\{ {n \atop j+1} \right\}\frac{x^n}{n!}\right]$$

$$\times D_y\left[(-1)^{j+1}(j+1)!\sum_{k\geq j+1}\left\{ {k \atop j+1} \right\}\frac{y^k}{k!}\right]$$

$$= \sum_{j\geq 0} j!^2 \sum_{n\geq j}\left\{ {n+1 \atop j+1} \right\}\frac{x^n}{n!}\sum_{k\geq j}\left\{ {k+1 \atop j+1} \right\}\frac{y^k}{k!}$$

$$= \sum_{n\geq 0}\sum_{k\geq 0}\frac{x^n}{n!}\frac{y^k}{k!}\sum_{j\geq 0} j!^2\left\{ {n+1 \atop j+1} \right\}\left\{ {k+1 \atop j+1} \right\}.$$

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