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A Short Proof of the Explicit Formula for Bernoulli Numbers

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In the present note we give an easy proof of the explicit formula for the Bernoulli numbers

$$B_{n+1} = \frac{(-1)^{n+1}(n+1)}{2^{n+1}-1} \sum_{k=1}^{n+1} \frac{1}{2^k} \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} (j+1)^n. \qquad (1)$$

As Gould pointed out in his survey article [2], formula (1) appeared for the first time in [3]. Garabedian [1] rediscovered (1) and proved it by using Cesaro sums.

We introduce the set $\{a_{n,k}\}$ of integers given by

$$a_{n,k} = (-1)^n \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} (j+1)^n$$

for $n = 0, 1, 2, \ldots$ and $k = 1, 2, 3, \ldots$. The numbers $a_{n,k}$ can be expressed in terms of finite differences. By the mth difference of a function $g(x)$ we mean the quantity

$$\Delta^m g(x) = \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} g(x+j) \quad (m = 0, 1, 2, \ldots),$$

and it is easily seen that $a_{n,k} = (-1)^{n+k-1} \Delta^{k-1} g(x)$ for $g(x) = x^n$ and $x = 1$.

Lemma 1. *The numbers $a_{n,k}$ satisfy the recurrence formula*

$$k a_{n,k} - (k+1) a_{n,k+1} = a_{n+1,k+1}. \tag{2}$$

Proof. We compute:

$$\begin{aligned} k a_{n,k} - (k+1) a_{n,k+1} &= (-1)^n k \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} (j+1)^n \\ &\quad - (-1)^n (k+1) \sum_{j=0}^{k} (-1)^j \binom{k}{j} (j+1)^n \\ &= (-1)^{n+1} \sum_{j=0}^{k-1} (-1)^j \left[(k+1) \binom{k}{j} - k \binom{k-1}{j} \right] (j+1)^n \\ &\quad + (-1)^{n+1} (k+1) (-1)^k \binom{k}{k} (k+1)^n \\ &= (-1)^{n+1} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (j+1)^{n+1} = a_{n+1,k+1}. \end{aligned}$$