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q-Bernoulli numbers and q-Bernoulli polynomials revisited

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Abstract

This paper performs a further investigation on the q-Bernoulli numbers and q-Bernoulli polynomials given by Acikgöz et al. (Adv Differ Equ, Article ID 951764, 9, 2010), some incorrect properties are revised. It is point out that the generating function for the q-Bernoulli numbers and polynomials is unreasonable. By using the theorem of Kim (Kyushu J Math **48**, 73-86, 1994) (see Equation 9), some new generating functions for the q-Bernoulli numbers and polynomials are shown.
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1. Introduction

As well-known definition, the Bernoulli polynomials are given by

$$\frac{t}{e^t-1}e^{xt}=e^{B(x)t}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!},$$

(see [1-4]),

with usual convention about replacing $B^n(x)$ by $B_n(x)$. In the special case, $x = 0$, $B_n(0) = B_n$ are called the nth Bernoulli numbers.

Let us assume that $q \in \mathbb{C}$ with $|q| < 1$ as an indeterminate. The q-number is defined by

$$[x]_q=\frac{1-q^x}{1-q},$$

(see [1-6]).

Note that $\lim_{q\to 1}[x]_q = x$.

Since Carlitz brought out the concept of the q-extension of Bernoulli numbers and polynomials, many mathematicians have studied q-Bernoulli numbers and q-Bernoulli polynomials (see [1,7,5,6,8-12]). Recently, Acikgöz, Erdal, and Araci have studied to a new approach to q-Bernoulli numbers and q-Bernoulli polynomials related to q-Bernstein polynomials (see [7]). But, their generating function is unreasonable. The wrong properties are indicated by some counter-examples, and they are corrected.

It is point out that Acikgöz, Erdal and Araci's generating function for q-Bernoulli numbers and polynomials is unreasonable by counter examples, then the new generating function for the q-Bernoulli numbers and polynomials are given.

2. *q*-Bernoulli numbers and *q*-Bernoulli polynomials revisited

In this section, we perform a further investigation on the q-Bernoulli numbers and q-Bernoulli polynomials given by Acikgöz et al. [7], some incorrect properties are revised.

Definition 1 (Acikgöz et al. [7]). For $q \in \mathbb{C}$ with $|q| < 1$, let us define q-Bernoulli polynomials as follows:

$$D_q(t,x) = -t\sum_{y=0}^{\infty} q^y e^{[x+y]_q t} = \sum_{n=0}^{\infty} B_{n,q}(x)\frac{t^n}{n!}, \quad \text{where} \quad —t + \log q— ¡ \, 2\pi. \tag{1}$$

In the special case, $x = 0$, $B_{n,q}(0) = B_{n,q}$ are called the nth q-Bernoulli numbers.

Let $D_q(t, 0) = D_q(t)$. Then

$$D_q(t) = -t\sum_{y=0}^{\infty} q^y e^{[y]_q t} = \sum_{n=0}^{\infty} B_{n,q}\frac{t^n}{n!}. \tag{2}$$

Remark 1. Definition 1 is unreasonable, since it is not the generating function of q-Bernoulli numbers and polynomials.

Indeed, by (2), we get

$$\begin{aligned} D_q(t,x) &= -t\sum_{y=0}^{\infty} q^y e^{[x+y]_q t} = -t\sum_{y=0}^{\infty} q^y e^{[x]_q t} e^{q^x [y]_q t} \\ &= \left(-\frac{q^x t}{q^x}\sum_{y=0}^{\infty} q^y e^{q^x[y]_q t}\right) e^{[x]_q t} \\ &= \frac{1}{q^x} e^{[x]_q t} D_q(q^x t) \\ &= \left(\sum_{m=0}^{\infty}\frac{[x]_q^m}{m!}t^m\right)\left(\sum_{l=0}^{\infty}\frac{q^{(l-1)x}B_{l,q}}{l!}t^l\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}[x]_q^{n-l}q^{(l-1)x}B_{l,q}\right)\frac{t^n}{n!}. \end{aligned} \tag{3}$$

By comparing the coefficients on the both sides of (1) and (3), we obtain the following equation

$$B_{n,q}(x) = \sum_{l=0}^{n}\binom{n}{l}[x]_q^{n-l}q^{(l-1)x}B_{l,q}. \tag{4}$$

From (1), we note that

$$\begin{aligned} D_q(t,x) &= -t\sum_{y=0}^{\infty} q^y e^{[x+y]_q t} \\ &= \sum_{n=0}^{\infty}\left(-t\sum_{y=0}^{\infty} q^y [x+y]_q^n\right)\frac{t^n}{n!} \\ &= -\sum_{n=0}^{\infty}\left(\frac{n+1}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{lx}\sum_{y=0}^{\infty} q^{(l+1)y}\right)\frac{t^{n+1}}{(n+1)!} \\ &= \sum_{n=1}^{\infty}\left(\frac{-n}{(1-q)^{n-1}}\sum_{l=0}^{n-1}\binom{n-1}{l}(-1)^l q^{lx}\left(\frac{1}{1-q^{l+1}}\right)\right)\frac{t^n}{n!}. \end{aligned} \tag{5}$$

By comparing the coefficients on the both sides of (1) and (5), we obtain the following equation

$$B_{0,q} = 0,$$
$$B_{n,q} = \frac{-n}{(1-q)^{n-1}} \sum_{l=0}^{n-1} \binom{n-1}{l} (-1)^l q^{lx} \left(\frac{1}{1-q^{l+1}} \right) \quad \text{if} \quad n > 0. \tag{6}$$

By (6), we see that Definition 1 is unreasonable because we cannot derive Bernoulli numbers from Definition 1 for any q.

In particular, by (1) and (2), we get

$$qD_q(t,1) - D_q(t) = t. \tag{7}$$

Thus, by (7), we have

$$qB_{n,q}(1) - B_{n,q} = \begin{cases} 1, \text{ if } & n = 1, \\ 0, \text{ if } & n > 1, \end{cases} \tag{8}$$

and

$$B_{n,q}(1) = \sum_{l=0}^{n} \binom{n}{l} q^{l-1} B_{l,q}. \tag{9}$$

Therefore, by (4) and (6)-(9), we see that the following three theorems are incorrect.

Theorem 1 (Acikgöz et al. [7]). For $n \in \mathbb{N}^*$, one has

$$B_{0,q} = 1, \quad q(qB+1)^n - B_{n,q} = \begin{cases} 1, \text{ if } & n = 0, \\ 0, \text{ if } & n > 0. \end{cases}$$

Theorem 2 (Acikgöz et al. [7]). For $n \in \mathbb{N}^*$, one has

$$B_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} q^{lx} B_{l,q} [x]_q^{n-l}.$$

Theorem 3 (Acikgöz et al. [7]). For $n \in \mathbb{N}^*$, one has

$$B_{n,q}(x) = \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{lx} \frac{l+1}{[l+1]_q}.$$

In [7], Acikgöz, Erdal and Araci derived some results by using Theorems 1-3. Hence, the other results are incorrect.

Now, we redefine the generating function of q-Bernoulli numbers and polynomials and correct its wrong properties, and rebuild the theorems of q-Bernoulli numbers and polynomials.

Redefinition 1. For $q \in \mathbb{C}$ with $|q| <1$, let us define q-Bernoulli polynomials as follows:

$$\begin{aligned} F_q(t,x) &= -t \sum_{m=0}^{\infty} q^{2m+x} e^{[x+m]_q t} + (1-q) \sum_{m=0}^{\infty} q^m e^{[x+m]_q t} \\ &= \sum_{n=0}^{\infty} \beta_{n,q}(x) \frac{t^n}{n!}, \quad \text{where} \quad —t + \log q— < 2\pi. \end{aligned} \tag{10}$$

In the special case, $x = 0$, $\beta_{n,q}(0) = \beta_{n,q}$ are called the nth q-Bernoulli numbers. Let $F_q(t, 0) = F_q(t)$. Then we have

$$\begin{aligned} F_q(t) &= \sum_{n=0}^{\infty} \beta_{n,q} \frac{t^n}{n!} \\ &= -t \sum_{m=0}^{\infty} q^{2m} e^{[m]_q t} + (1-q) \sum_{m=0}^{\infty} q^m e^{[m]_q t}. \end{aligned} \tag{11}$$

By (10), we get

$$\begin{aligned} \beta_{n,q}(x) &= -n \sum_{m=0}^{\infty} q^{2m+x} [x+m]_q^{n-1} + (1-q) \sum_{m=0}^{\infty} q^m [x+m]_q^n \\ &= \frac{-n}{(1-q)^{n-1}} \sum_{l=0}^{n-1} \binom{n-1}{l} \frac{(-1)^l q^{(l+1)x}}{(1-q^{l+2})} + (1-q) \sum_{m=0}^{\infty} q^m [x+m]_q^n \\ &= \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{lx} \frac{l+1}{[l+1]_q}. \end{aligned} \tag{12}$$

By (10) and (11), we get

$$\begin{aligned} F_q(t, x) &= e^{[x]_q t} F_q(q^x t) \\ &= \left(\sum_{m=0}^{\infty} [x]_q^m \frac{t^m}{m!} \right) \left(\sum_{l=0}^{\infty} \frac{\beta_{l,q}}{l!} q^{lx} t^l \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \frac{q^{lx} \beta_{l,q} [x]_q^{n-l} n!}{l!(n-l)!} \right) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l} q^{lx} \beta_{l,q} [x]_q^{n-l} \right) \frac{t^n}{n!}. \end{aligned} \tag{13}$$

Thus, by (12) and (13), we have

$$\begin{aligned} \beta_{n,q}(x) &= \sum_{l=0}^{n} \binom{n}{l} q^{lx} \beta_{l,q} [x]_q^{n-l} \\ &= -n \sum_{m=0}^{\infty} q^m [x+m]_q^{n-1} + (1-q)(n+1) \sum_{m=0}^{\infty} q^m [x+m]_q^n. \end{aligned} \tag{14}$$

From (10) and (11), we can derive the following equation:

$$qF_q(t, 1) - F_q(t) = t + (q-1). \tag{15}$$

By (15), we get

$$q\beta_{n,q}(1) - \beta_{n,q} = \begin{cases} q-1, & \text{if } n = 0, \\ 1, & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases} \tag{16}$$

Therefore, by (14) and (15), we obtain

$$\beta_{0,q} = 1, q(q\beta_q + 1)^n - \beta_{n,q} = \begin{cases} 1, \text{ if } & n = 1, \\ 0 \text{ if } & n > 1, \end{cases} \tag{17}$$

with the usual convention about replacing β_q^n by $\beta_{n,q}$.

From (12), (14) and (16), Theorems 1-3 are revised by the following Theorems 1'-3'.

Theorem 1'. For $n \in \mathbb{Z}_+$, we have

$$\beta_{0,q} = 1, \quad \text{and} \quad q(q\beta_q + 1)^n - \beta_{n,q} = \begin{cases} 1, \text{ if } & n = 1, \\ 0 \text{ if } & n > 1. \end{cases}$$

Theorem 2'. For $n \in \mathbb{Z}_+$, we have

$$\beta_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} q^{lx} \beta_{l,q} [x]_q^{n-l}.$$

Theorem 3'. For $n \in \mathbb{Z}_+$, we have

$$\beta_{n,q}(x) = \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{lx} \frac{l+1}{[l+1]_q}.$$

From (10), we note that

$$F_q(t,x) = \frac{1}{[d]_q} \sum_{a=0}^{d-1} q^a F_{q^d}\left([d]_q t, \frac{x+a}{d}\right), \quad d \in \mathbb{N}. \tag{18}$$

Thus, by (10) and (18), we have

$$\beta_{n,q}(x) = [d]_q^{n-1} \sum_{a=0}^{d-1} q^a \beta_{n,q^d}\left(\frac{x+a}{d}\right), \quad n \in \mathbb{Z}_+.$$

For $d \in \mathbb{N}$, let χ be Dirichlet's character with conductor d. Then, we consider the generalized q-Bernoulli polynomials attached to χ as follows:

$$\begin{aligned} F_{q,\chi}(t,x) &= -t \sum_{m=0}^{\infty} \chi(m) q^{2m+x} e^{[x+m]_q t} + (1-q) \sum_{m=0}^{\infty} \chi(m) q^m e^{[x+m]_q t} \\ &= \sum_{n=0}^{\infty} \beta_{n,\chi,q}(x) \frac{t^n}{n!}. \end{aligned}$$

In the special case, $x = 0$, $\beta_{n,\chi,q}(0) = \beta_{n,\chi,q}$ are called the nth generalized Carlitz q-Bernoulli numbers attached to χ (see [8]).

Let $F_{q,\chi}(t, 0) = F_{q,\chi}(t)$. Then we have

$$\begin{aligned} F_{q,\chi}(t) &= -t \sum_{m=0}^{\infty} \chi(m) q^{2m} e^{[m]_q t} + (1-q) \sum_{m=0}^{\infty} \chi(m) q^m e^{[m]_q t} \\ &= \sum_{n=0}^{\infty} \beta_{n,\chi,q} \frac{t^n}{n!}. \end{aligned} \tag{20}$$

From (20), we note that

$$\begin{aligned}
\beta_{n,\chi,q} &= -n\sum_{m=0}^{\infty} q^{2m}\chi(m)[m]_q^{n-1} + (1-q)\sum_{m=0}^{\infty} q^{m}\chi(m)[m]_q^{n} \\
&= -n\sum_{a=0}^{d-1}\sum_{m=0}^{\infty} q^{2a+2dm}\chi(a+dm)[a+dm]_q^{n-1} \\
&\quad + \sum_{a=0}^{d-1}\sum_{m=0}^{\infty} q^{a+dm}\chi(a+dm)[a+dm]_q^{n} \\
&= \sum_{a=0}^{d-1}\chi(a)q^a\left(\frac{-n}{(1-q)^{n-1}}\sum_{l=0}^{n-1}\binom{n-1}{l}\frac{(-1)^l q^{(l+1)a}}{(1-q^{d(l+2)})}\right) \\
&\quad + (1-q)\sum_{a=0}^{d-1}\chi(a)q^a\left(\frac{1}{(1-q)^{n}}\sum_{l=0}^{n}\binom{n}{l}\frac{(-1)^l q^{la}}{(1-q^{d(l+1)})}\right) \\
&= \sum_{a=0}^{d-1}\chi(a)q^a\left(\frac{-n}{(1-q)^{n-1}}\sum_{l=0}^{n-1}\binom{n-1}{l}\frac{(-1)^l q^{(l+1)a}}{(1-q^{d(l+2)})}\right) \\
&\quad + \sum_{a=0}^{d-1}\chi(a)q^a\left(\frac{1}{(1-q)^{n-1}}\sum_{l=0}^{n}\binom{n}{l}\frac{(-1)^l q^{la}}{(1-q^{d(l+1)})}\right) \\
&= \sum_{a=0}^{d-1}\chi(a)q^a\left(\frac{1}{(1-q)^{n-1}}\sum_{l=0}^{n}\binom{n}{l}\frac{(-1)^l q^{la} l}{(1-q^{d(l+1)})}\right) \\
&\quad + \sum_{a=0}^{d-1}\chi(a)q^a\left(\frac{1}{(1-q)^{n-1}}\sum_{l=0}^{n}\binom{n}{l}\frac{(-1)^l q^{la}}{(1-q^{d(l+1)})}\right) \\
&= \sum_{a=0}^{d-1}\chi(a)q^a\frac{1-q}{(1-q)^{n}}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{la}\left(\frac{l+1}{1-q^{d(l+1)}}\right).
\end{aligned}$$

Therefore, by (20) and (21), we obtain the following theorem.

Theorem 4. For $n \in \mathbb{Z}_+$, we have

$$\begin{aligned}
\beta_{n,\chi,q} &= \sum_{a=0}^{d-1}\chi(a)q^a\frac{1}{(1-q)^{n}}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{la}\frac{l+1}{[d(l+1)]_q} \\
&= -n\sum_{m=0}^{\infty}\chi(m)q^m[m]_q^{n-1} + (1-q)(1+n)\sum_{m=0}^{\infty}\chi(m)q^m[m]_q^{n},
\end{aligned}$$

and

$$\beta_{n,\chi,q}(x) = -n\sum_{m=0}^{\infty}\chi(m)q^m[m+x]_q^{n-1} + (1-q)(1+n)\sum_{m=0}^{\infty}\chi(m)q^m[m+x]_q^{n}.$$

From (19), we note that

$$F_{q,\chi}(t,x) = \frac{1}{[d]_q}\sum_{a=0}^{d-1}\chi(a)q^a F_{q^d}\left([d]_q t, \frac{x+a}{d}\right). \tag{22}$$

Thus, by (22), we obtain the following theorem.

Theorem 5. For $n \in \mathbb{Z}_+$, we have

$$\beta_{n,\chi,q}(x) = [d]_q^{n-1} \sum_{a=0}^{d-1} \chi(a) q^a \beta_{n,q^d}\left(\frac{x+a}{d}\right).$$

For $s \in \mathbb{C}$, we now consider the Mellin transform for $F_q(t, x)$ as follows:

$$\frac{1}{\Gamma(s)} \int_0^\infty F_q(-t,x) t^{s-2} \mathrm{d}t = \sum_{m=0}^\infty \frac{q^{2m+x}}{[m+x]_q^s} + \frac{1-q}{s-1} \sum_{m=0}^\infty \frac{q^m}{[m+x]_q^{s-1}}, \tag{23}$$

where $x \neq 0, -1, -2, \ldots$.

From (23), we note that

$$\begin{aligned} &\frac{1}{\Gamma(s)} \int_0^\infty F_q(-t,x) t^{s-2} \mathrm{d}t \\ &= \sum_{m=0}^\infty \frac{q^m}{[m+x]_q^s} + (1-q)\left(\frac{2-s}{s-1}\right) \sum_{m=0}^\infty \frac{q^m}{[m+x]_q^{s-1}}, \end{aligned} \tag{24}$$

where $s \in \mathbb{C}$, and $x \neq 0, -1, -2, \ldots$.

Thus, we define q-zeta function as follows:

Definition 2. For $s \in \mathbb{C}$, q-zeta function is defined by

$$\zeta_q(s,x) = \sum_{m=0}^\infty \frac{q^m}{[m+x]_q^s} + (1-q)\left(\frac{2-s}{s-1}\right) \sum_{m=0}^\infty \frac{q^m}{[m+x]_q^{s-1}}, \quad Re(s) > 1,$$

where $x \neq 0, -1, -2, \ldots$.

By (24) and Definition 2, we note that

$$\zeta_q(1-n,x) = (-1)^{n-1} \frac{\beta_{n,q}(x)}{n}, \quad n \in \mathbb{N}.$$

Note that

$$\lim_{q \to 1} \zeta_q(1-n,x) = -\frac{B_n(x)}{n},$$

where $B_n(x)$ are the nth ordinary Bernoulli polynomials.

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Authors' contributions
All authors contributed equally to the manuscript and read and approved the finial manuscript.

Competing interests
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