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A note on q-Bernoulli numbers and polynomials

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Abstract

In this paper we give the generating functions of q-Bernoulli numbers and q-Bernoulli polynomials. Next, we consider the q-zeta function which interpolates the q-Bernoulli numbers and q-Bernoulli polynomials. Finally we investigate the roots of the q-Bernoulli polynomials $B_{n,q^r}(x)$ for values of the index n by using a computer.

Keywords: Bernoulli polynomials; Bernoulli numbers; Zeta function; q-Bernoulli polynomials; q-Bernoulli numbers; q-Zeta function

1. Introduction

Bernoulli polynomials and Bernoulli numbers are of significant importance in mathematics and physics. The reason is that Bernoulli polynomials and Bernoulli numbers arise in many applications. q-Bernoulli polynomials and q-Bernoulli numbers possess many interesting properties and arise in many areas of mathematics and physics (see [1–4, 6–9]). Many mathematicians have studied q-Bernoulli polynomials and q-Bernoulli numbers. In the case of Bernoulli polynomials and Bernoulli numbers, there are several results, such as those of Whittaker and Waston [11], and Erdelyi [5]. For q-Bernoulli polynomials and q-Bernoulli numbers, several results have been studied by Carlitz [4], Kim [6,7], Kobilitz [8,9], and Todorov [10]. First, we introduce the ordinary Bernoulli numbers and Bernoulli polynomials. For any complex number x, it is well known that the familiar Bernoulli polynomials $B_n(x)$ are defined by means of the following generating function:

$$F(x,t) := \frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad |t| < 1. \tag{1}$$

Note that, by substituting $x = 0$ into (1), $B_n(0) = B_n$ is the familiar nth Bernoulli number defined by

$$e^{Bt} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = \frac{t}{e^t - 1}, \quad |t| < 1$$

where the symbol B_k is interpreted to mean that B^k must be replaced by B_k when we expand the one on the left. This relation can be written as

$$e^{(B+1)t} - e^{Bt} = t.$$

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Hence we obtain

$$B_0 = 1, \qquad (B+1)^k - B^k = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{if } k > 1, \end{cases}$$

with the usual convention about replacing B^k by B_k, $(i \geq 0)$. The Hurwitz zeta function

$$\zeta(s, x) = \sum_{k=0}^{\infty} \frac{1}{(k+x)^s} \tag{2}$$

is a meromorphic function of s. We give the generating functions of q-Bernoulli numbers and q-Bernoulli polynomials. Next, we consider the q-analogue of this Hurwitz zeta function. The paper is organized as follows. In the following section, we define the q-zeta functions, q-Hurwitz zeta functions, and we consider the q-zeta function which interpolates the q-Bernoulli numbers and q-Bernoulli polynomials. In Section 3, we describe the beautiful zeros of the $B_{n,q^r}(x)$ using a numerical investigation.

2. q-Bernoulli numbers and polynomials

In this section we define the q-Bernoulli numbers β_{n,q^r} and polynomials $\beta_{n,q^r}(x)$ and investigate their properties. Throughout this paper we use the following notations. By $\mathbb{Z}$ we denote the ring of rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{C}$ denotes the complex number field,

$$[x]_q = \frac{1-q^x}{1-q} \quad \text{for any real } x.$$

We assume that $q \in \mathbb{C}$ with $|q| < 1$. First, we introduce the q-Bernoulli polynomials using a generating function (cf. [6,7]). Let

$$F_{q^r}(t) = \frac{q^r - 1}{r \log q} \mathrm{e}^{\frac{t}{1-q^r}} - t \sum_{n=0}^{\infty} q^{rn} \mathrm{e}^{[n]_{q^r} t}, \quad |t| < 1. \tag{3}$$

Consider the Taylor expansion at $t = 0$.

$$F_{q^r}(t) = \beta_{0,q^r} + \beta_{1,q^r} \frac{t}{1!} + \beta_{2,q^r} \frac{t^2}{2!} + \cdots + \beta_{n,q^r} \frac{t^n}{n!} + \cdots.$$

The coefficients β_{n,q^r} are called the nth q-Bernoulli numbers. Note that

$$\frac{1}{t}\left(F_{q^r}(t) - \frac{q^r - 1}{r \log q} \mathrm{e}^{\frac{t}{1-q^r}}\right) = -\sum_{k=0}^{\infty} \left(\sum_{n=0}^{\infty} q^{rn} [n]_{q^r}^k\right) \frac{t^k}{k!}. \tag{4}$$

Now we consider the generating function of the q-Bernoulli polynomials as follows:

$$\sum_{n=0}^{\infty} \beta_{n,q^r}(x) \frac{t^n}{n!} = F_{q^r}(x, t) = \frac{q^r - 1}{r \log q} \mathrm{e}^{\frac{t}{1-q^r}} - t \sum_{n=0}^{\infty} q^{rn+rx} \mathrm{e}^{[n+x]_{q^r} t}. \tag{5}$$

Note that

$$t = \mathrm{e}^t F_{q^r}(q^r t) - F_{q^r}(t) = \sum_{n=0}^{\infty} \left\{(q^r \beta_{q^r} + 1)^n - \beta_{n,q^r}\right\} \frac{t^n}{n!}.$$

By comparing the coefficients on both sides, we obtain

$$(q^r \beta_{q^r} + 1)^k - \beta_{k,q^r} = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{if } k > 1. \end{cases}$$

By simple calculations, we have the following remark.

Remark 1. Note that

(1) $\lim_{q\to 1} F_{q^r}(x,t) = \frac{t}{e^t-1}e^{xt} = F(x,t)$,
(2) $\beta_{n,q^r} = \frac{1}{1-q^{rn}}\sum_{k=0}^{n-1}\frac{n!}{k!(n-k)!}q^{rk}\beta_{k,q^r}$, for $n > 1$,
(3) $\beta_{n,q^r}(0) = \beta_{n,q^r}$,
(4) $\lim_{q\to 1}\beta_{n,q^r} = B_n$, $\lim_{q\to 1}\beta_{n,q^r}(x) = B_n(x)$.
(5) $\beta_{n,q^r}(x) = \frac{1}{(1-q^r)^n}\sum_{j=0}^{n}\binom{n}{j}(-1)^j(q^r)^{xj}\frac{j}{[j]_{q^r}}$.

Now we define the q-Bernoulli numbers B_{n,q^r} as

$$G_{q^r}(t) = F_{q^r}(t) - \frac{q^r-1}{r\log q}e^{\frac{t}{1-q^r}} = \sum_{n=0}^{\infty} B_{n,q^r}\frac{t^n}{n!}, \quad \text{cf. [7]}. \tag{6}$$

We have

$$\frac{q^r-1}{r\log q}\left(\frac{1}{1-q^r}\right)^n + B_{n,q^r} = \beta_{n,q^r}.$$

We also consider the q-Bernoulli polynomials $B_{n,q^r}(x)$ given by

$$G_{q^r}(x,t) = F_{q^r}(x,t) - \frac{q^r-1}{r\log q}e^{\frac{t}{1-q^r}} = \sum_{n=0}^{\infty} B_{n,q^r}(x)\frac{t^n}{n!}.$$

Then we obtain

$$B_{n,q^r}(x) = \beta_{n,q^r}(x) - \frac{q^r-1}{r\log q}\left(\frac{1}{1-q^r}\right)^n. \tag{7}$$

Note that

$$G_{q^r}(-q^r t) = q^r t\sum_{n=0}^{\infty} q^{rn}e^{[n]_{q^r}(-q^r t)}. \tag{8}$$

By (6), we see that

$$\sum_{n=0}^{\infty} B_{n,q^r}(x)\frac{t^n}{n!} = -t\sum_{n=0}^{\infty} q^{rn+rx}e^{([x]_{q^r}+q^{rx}[n]_{q^r})t} = e^{[x]_{q^r}t}G_{q^r}(q^{rx}t).$$

Thus we have

$$\sum_{n=0}^{\infty} B_{n,q^r}(x)\frac{t^n}{n!} = G_{q^r}(x,t) = e^{[x]_{q^r}t}G_{q^r}(q^{rx}t) = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}q^{rmx}B_{m,q^r}[x]_{q^r}^{n-m}\right)\frac{t^n}{n!}.$$

Hence, we obtain the following theorem:

Theorem 2. *For $n \geq 0$,*

$$B_{n,q^r}(x) = \sum_{j=0}^{n}\binom{n}{j}q^{rjx}B_{j,q^r}[x]_{q^r}^{n-j}$$

$B_{n,q^r}(x)$ are called the nth q-Bernoulli polynomials. Note that $B_{n,q^r}(0) = B_{n,q^r}$.

Let $\Gamma(s)$ be the gamma function. By (8), for $s \in \mathbb{C}$, we obtain

$$\frac{1}{\Gamma(s)}\int_0^{\infty} G_{q^r}(-q^r t)e^{-t}dt = \sum_{n=0}^{\infty} q^{rn+r}\frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-2}e^{-[n+1]_{q^r}t}t^{s-1}dt = \sum_{n=1}^{\infty}\frac{q^{rn}}{[n]_{q^r}^s}.$$

Using (8), we define the functions $\zeta_q(s,x)$ and $\zeta_q(s)$ as follows.

Definition 3. For $x \in \mathbb{R}$, $s \in \mathbb{C}$, we define the Hurwitz q-zeta function as

$$\zeta_q(s,x) = \sum_{n=0}^{\infty}\frac{q^{rn+rx}}{[n+x]_{q^r}^s}.$$

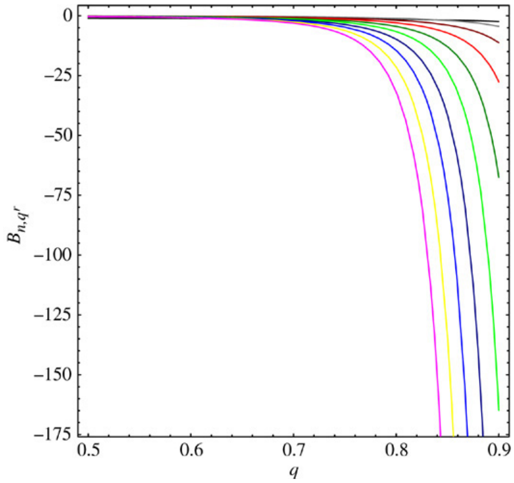

Fig. 1. Shape of B_{n,q^5}.

Remark 4. Note that $\zeta_q(s, x)$ has an analytic continuation on $\mathbb{C}$ with only one simple pole at $s = 1$. Let us define the q-zeta function as $\zeta_q(s) = \zeta_q(s, 1)$.

Using (7), we have

$$\frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-2} G_{q^r}(x, -t)\mathrm{d}t = \sum_{n=0}^{\infty} q^{rn+rx} \frac{1}{\Gamma(s)}\int_0^{\infty} \mathrm{e}^{-[n+x]_{q^r} t} t^{s-1}\mathrm{d}t = \zeta_q(s, x).$$

We also obtain

$$\zeta_q(s, x) = \sum_{n=0}^{\infty} \frac{(-1)^n B_{n,q^r}(x)}{n!} \frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-2+n}\mathrm{d}t.$$

Hence, we have the following theorem.

Theorem 5. *For $n \in \mathbb{N}$, we have*

$$\zeta_q(1-n, x) = -\frac{B_{n,q^r}(x)}{n}, \qquad \zeta_q(1-n) = -\frac{B_{n,q^r}}{n}.$$

3. Beautiful zeros of the q-Bernoulli numbers and polynomials

Over the years, there has been increasing interest in solving mathematical problems with the aid of computers. Recently, Woon [2] and Veselov and Ward [3] observed the regular behaviour of the real roots of Bernoulli polynomials using a numerical investigation. Using computer experiments, Woon [2] verifies a remarkably regular structure of the complex roots of Bernoulli polynomials. Also, Veselov and Ward [3] proved the regular lattice behaviour of almost all of the real roots of the Bernoulli polynomials. However, to this point there have been no such investigations for q-Bernoulli polynomials $B_{n,q^r}(x)$ and q-Bernoulli numbers B_{n,q^r}. In this section, we display the shapes of the q-Bernoulli numbers and polynomials. Next, we investigate the zeros of the q-Bernoulli polynomials by using a computer.

For $n = 1, \ldots, 10$, $\frac{5}{10} \le q \le \frac{9}{10}$, we can draw a plot of the q-Bernoulli numbers B_{n,q^r}, respectively. This shows the ten plots combined into one. We display the shapes of the q-Bernoulli numbers B_{n,q^r} (Figs. 1 and 2).

For $n = 1, \ldots, 10$, we can draw a plot of the q-Bernoulli polynomials $B_{n,q^r}(x)$, respectively. This shows the ten plots combined into one. We describe the shapes of the q-Bernoulli polynomials $B_{n,q^r}(x)$ for $n = 1, \ldots, 10$, $0 \le x \le 1$, $q = \frac{1}{2}$ (Figs. 3 and 4).

We plot the zeros of the q-Bernoulli polynomials $B_{20,q^r}(x)$, $x \in \mathbb{C}$, $q = \frac{1}{2}$ (Figs. 5–8).

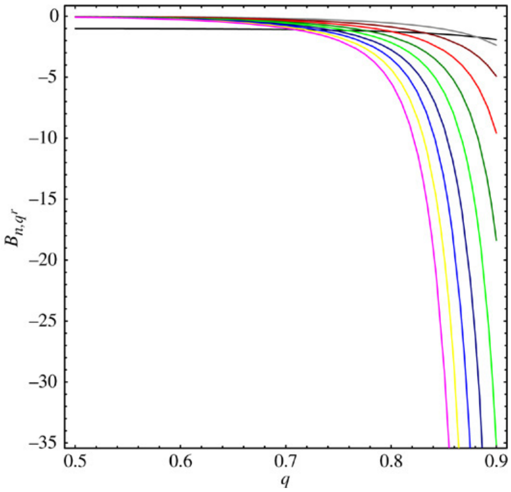

Fig. 2. Shape of B_{n,q^7}.

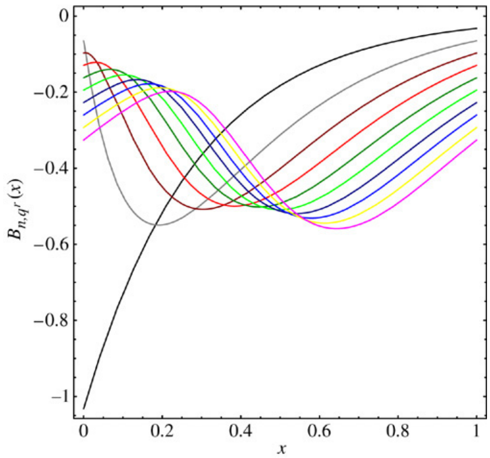

Fig. 3. Shape of $B_{n,q^5}(x)$.

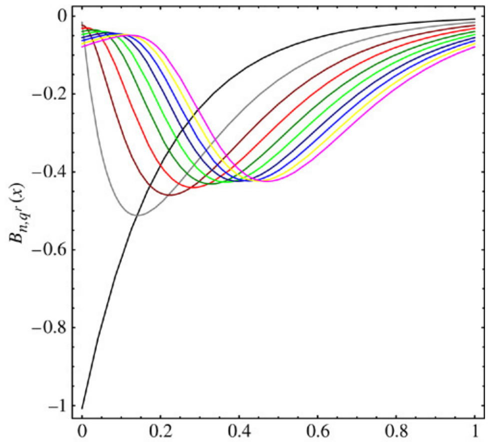

Fig. 4. Shape of $B_{n,q^7}(x)$.

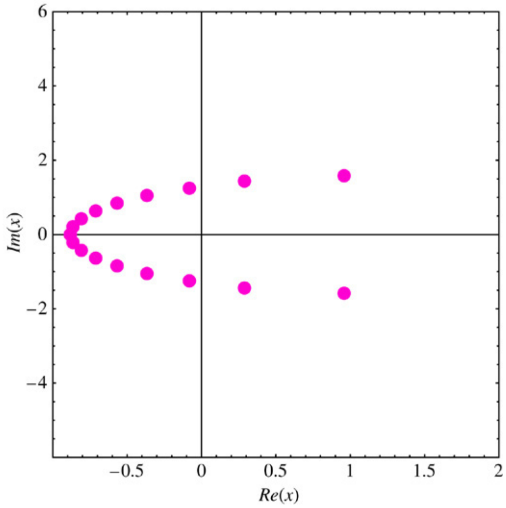

Fig. 5. Zeros of $B_{20,q}(x)$.

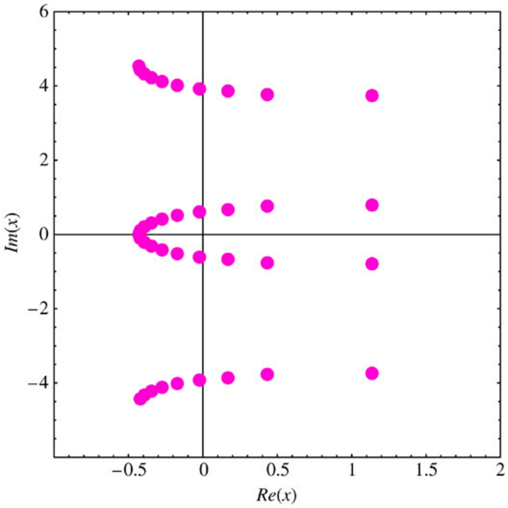

Fig. 6. Zeros of $B_{20,q^2}(x)$.

In Figs. 5 and 8, for $r = 1, 5$, $B_{n,q^r}(x), x \in \mathbb{C}$, has $\text{Im}(x) = 0$ reflection symmetry. This translates to the following open problem. Prove that $B_{n,q^r}(x), x \in \mathbb{C}$, has $\text{Im}(x) = 0$ reflection symmetry, $r \in \mathbb{N}_o$, where $\mathbb{N}_o = \{x \mid x \text{ is a odd number }\}$. Our numerical results for approximate solutions of real zeros of the $B_{n,q^r}(x), r = 1, 5$, $q = \frac{1}{2}$ are displayed in Tables 1 and 2. Using computer experiments, we verify a remarkably regular structure of the real roots of q-Bernoulli polynomials $B_{n,q^r}(x)$ (see Table 1).

Finally, we shall consider the more general problems. Prove or disprove: since n is the degree of the polynomial $B_{n,q^r}(x)$, the number of real zeros $\text{re}_{B_{n,q^r}(x)}$ lying on the real plane $\text{Im}(x) = 0$ is then $\text{re}_{B_{n,q^r}(x)} = r(n-1) - c_{B_{n,q^r}(x)}(n > 1)$, where $c_{B_{n,q^r}(x)}$ denotes complex zeros. See Table 1 for tabulated values of $\text{re}_{B_{n,q^r}(x)}$ and $c_{B_{n,q^r}(x)}$. In general, how many roots does $B_{n,q^r}(x)$ have? Find the numbers of complex zeros $c_{B_{n,q^r}(x)}$ of the $B_{n,q^r}(x)$, the equation of envelope curves bounding the real zeros lying on the plane, and the equation of a trajectory curve running through the complex zeros on any one of the arcs. It would be very interesting to find a mathematical explanation for this. In any case, these calculations are too complicated to compute by hand, so we have to use a computer. The author has no doubt that investigations along this line will lead to a new approach employing numerical methods in

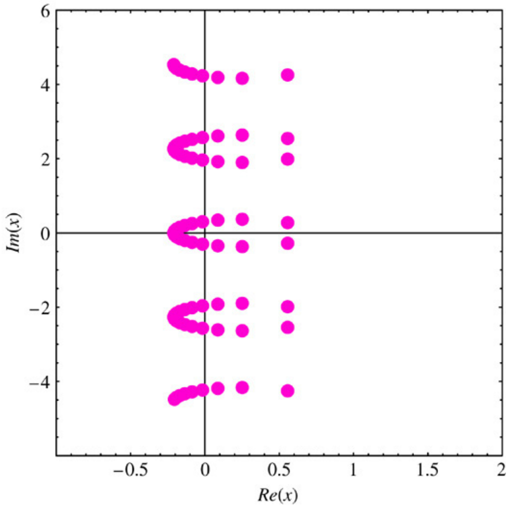

Fig. 7. Zeros of $B_{20,q^4}(x)$.

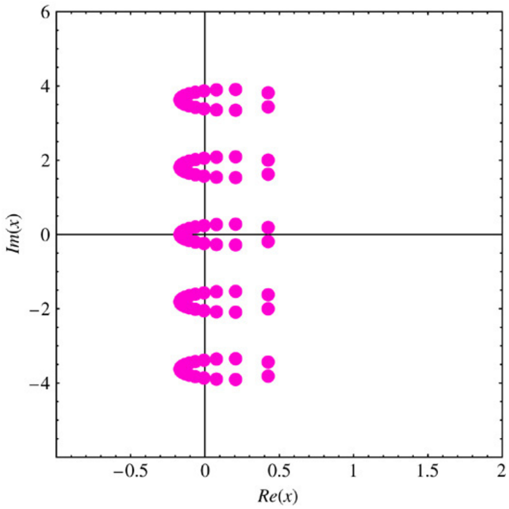

Fig. 8. Zeros of $B_{20,q^5}(x)$.

Table 1
Numbers of real and complex zeros of $B_{n,q^r}(x)$

Degree n	$r = 1$		$r = 5$	
	Real zeros	Complex zeros	Real zeros	Complex zeros
2	1	0	1	0
3	0	2	0	10
4	1	2	1	14
5	0	4	0	20
6	1	5	1	24
7	0	6	0	30
8	1	6	1	34

the field of research of the q-Bernoulli polynomials $B_{n,q^r}(x)$ to appear in mathematics and physics. For related topics the interested reader is referred to [1–3].

Table 2
Approximate solutions of $B_{n,q^5}(x) = 0$, $x \in \mathbb{R}$

Degree n	Real zeros
2	−0.00887882
3	×
4	−0.0769761
5	×
6	−0.113251
7	×
8	−0.132542

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