

## Research Article

# Identities on the Bernoulli and Genocchi Numbers and Polynomials

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We give some interesting identities on the Bernoulli numbers and polynomials, on the Genocchi numbers and polynomials by using symmetric properties of the Bernoulli and Genocchi polynomials.

## 1. Introduction

Let  $p$  be a fixed odd prime number. Throughout this paper  $\mathbb{Z}_p$ ,  $\mathbb{Q}_p$ , and  $\mathbb{C}_p$  will denote the ring of  $p$ -adic rational integers, the field of  $p$ -adic rational numbers, and the completion of the algebraic closure of  $\mathbb{Q}_p$ . Let  $\mathbb{N}$  be the set of natural numbers and  $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ . The  $p$ -adic norm on  $\mathbb{C}_p$  is normalized so that  $|p|_p = p^{-1}$ . Let  $C(\mathbb{Z}_p)$  be the space of continuous functions on  $\mathbb{Z}_p$ . For  $f \in C(\mathbb{Z}_p)$ , the fermionic  $p$ -adic integral on  $\mathbb{Z}_p$  is defined by Kim as follows:

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N \rightarrow \infty} \sum_{x=0}^{p^N-1} f(x) (-1)^x \quad (1.1)$$

(see [1–16]). From (1.1), we have

$$I_{-1}(f_1) = -I_{-1}(f) + 2f(0) \quad (1.2)$$

(see [1–16]), where  $f_1(x) = f(x+1)$ .

Let us take  $f(x) = e^{xt}$ . Then, by (1.2), we get

$$t \int_{\mathbb{Z}_p} e^{xt} d\mu_{-1}(x) = \frac{2t}{e^t + 1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}, \quad (1.3)$$

where  $G_n$  are the  $n$ th ordinary Genocchi numbers (see [8, 15]).

From the same method of (1.3), we can also derive the following equation:

$$t \int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_{-1}(y) = \frac{2t}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}, \quad (1.4)$$

where  $G_n(x)$  are called the  $n$ th Genocchi polynomials (see [14, 15]).

By (1.3), we easily see that

$$G_n(x) = \sum_{l=0}^n \binom{n}{l} G_l x^{n-l} \quad (1.5)$$

(see [15]). By (1.3) and (1.4), we get Witt's formula for the  $n$ th Genocchi numbers and polynomials as follows:

$$\int_{\mathbb{Z}_p} x^n d\mu_{-1}(x) = \frac{G_{n+1}}{n+1}, \quad \int_{\mathbb{Z}_p} (x+y)^n d\mu_{-1}(y) = \frac{G_{n+1}(x)}{n+1}, \quad \text{for } n \in \mathbb{Z}_+. \quad (1.6)$$

From (1.2), we have

$$\int_{\mathbb{Z}_p} (x+1)^n d\mu_{-1}(x) + \int_{\mathbb{Z}_p} x^n d\mu_{-1}(x) = 2\delta_{0,n}, \quad (1.7)$$

where the symbol  $\delta_{0,n}$  is the Kronecker symbol (see [4, 5]).

Thus, by (1.5) and (1.7), we get

$$(G+1)^n + G_n = 2\delta_{1,n} \quad (1.8)$$

(see [15]). From (1.4), we can derive the following equation:

$$\int_{\mathbb{Z}_p} (1-x+y)^n d\mu_{-1}(y) = (-1)^n \int_{\mathbb{Z}_p} (x+y)^n d\mu_{-1}(y). \quad (1.9)$$

By (1.6) and (1.9), we see that

$$\frac{G_{n+1}(1-x)}{n+1} = (-1)^n \frac{G_{n+1}(x)}{n+1}. \quad (1.10)$$

Thus, by (1.10), we get  $G_{n+1}(2)/(n+1) = (-1)^n (G_{n+1}(-1)/(n+1))$ .

From (1.5) and (1.8), we have

$$\frac{G_{n+1}(2)}{n+1} = 2 - \frac{G_{n+1}(1)}{n+1} = 2 + \frac{G_{n+1}}{n+1} - 2\delta_{1,n+1}. \quad (1.11)$$

The Bernoulli polynomials  $B_n(x)$  are defined by

$$\frac{t}{e^t - 1} e^{xt} = e^{B(x)t} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \quad (1.12)$$

(see [6, 9, 12]) with the usual convention about replacing  $B^n(x)$  by  $B_n(x)$ .

In the special case,  $x = 0$ ,  $B_n(0) = B_n$  is called the  $n$ -th Bernoulli number. By (1.12), we easily see that

$$B_n(x) = \sum_{l=0}^n \binom{n}{l} x^{n-l} B_l = (B+x)^n \quad (1.13)$$

(see [6]). Thus, by (1.12) and (1.13), we get reflection symmetric formula for the Bernoulli polynomials as follows:

$$B_n(1-x) = (-1)^n B_n(x), \quad (1.14)$$

$$B_0 = 1, \quad (B+1)^n - B_n = \delta_{1,n} \quad (1.15)$$

(see [6, 9, 12]). From (1.14) and (1.15), we can also derive the following identity:

$$(-1)^n B_n(-1) = B_n(2) = n + B_n(1) = n + B_n + \delta_{1,n}. \quad (1.16)$$

In this paper, we investigate some properties of the fermionic  $p$ -adic integrals on  $\mathbb{Z}_p$ . By using these properties, we give some new identities on the Bernoulli and the Euler numbers which are useful in studying combinatorics.

## 2. Identities on the Bernoulli and Genocchi Numbers and Polynomials

Let us consider the following fermionic  $p$ -adic integral on  $\mathbb{Z}_p$  as follows:

$$\begin{aligned} I_1 &= \int_{\mathbb{Z}_p} B_n(x) d\mu_{-1}(x) = \sum_{l=0}^n \binom{n}{l} B_{n-l} \int_{\mathbb{Z}_p} x^l d\mu_{-1}(x) \\ &= \sum_{l=0}^n \binom{n}{l} B_{n-l} \frac{G_{l+1}}{l+1}, \quad \text{for } n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}. \end{aligned} \quad (2.1)$$

On the other hand, by (1.14) and (1.15), we get

$$\begin{aligned}
 I_1 &= (-1)^n \int_{\mathbb{Z}_p} B_n(1-x) d\mu_{-1}(x) \\
 &= (-1)^n \sum_{l=0}^n \binom{n}{l} B_{n-l} \int_{\mathbb{Z}_p} (1-x)^l d\mu_{-1}(x) \\
 &= (-1)^n \sum_{l=0}^n \binom{n}{l} B_{n-l} (-1)^l \frac{G_{l+1}(-1)}{l+1} \\
 &= (-1)^n \sum_{l=0}^n \binom{n}{l} B_{n-l} \left( 2 + \frac{G_{l+1}}{l+1} - 2\delta_{1,l+1} \right) \\
 &= 2(-1)^n (B_n + \delta_{1,n}) + (-1)^n \sum_{l=0}^n \binom{n}{l} B_{n-l} \frac{G_{l+1}}{l+1} + 2(-1)^{n+1} B_n.
 \end{aligned} \tag{2.2}$$

Equating (2.1) and (2.2), we obtain the following theorem.

**Theorem 2.1.** For  $n \in \mathbb{Z}_+$ , one has

$$\left( 1 + (-1)^{n+1} \right) \sum_{l=0}^n \binom{n}{l} B_{n-l} \frac{G_{l+1}}{l+1} = 2(-1)^n \delta_{1,n}. \tag{2.3}$$

By using the reflection symmetric property for the Euler polynomials, we can also obtain some interesting identities on the Euler numbers.

Now, we consider the fermionic  $p$ -adic integral on  $\mathbb{Z}_p$  for the polynomials as follows:

$$\begin{aligned}
 I_2 &= \int_{\mathbb{Z}_p} G_n(x) d\mu_{-1}(x) \\
 &= \sum_{l=0}^n \binom{n}{l} G_{n-l} \int_{\mathbb{Z}_p} x^l d\mu_{-1}(x) \\
 &= \sum_{l=0}^n \binom{n}{l} G_{n-l} \frac{G_{l+1}}{l+1}, \quad \text{for } n \in \mathbb{Z}_+.
 \end{aligned} \tag{2.4}$$

On the other hand, by (1.8), (1.10), and (1.11), we get

$$\begin{aligned}
 I_2 &= (-1)^{n-1} \int_{\mathbb{Z}_p} G_n(1-x) d\mu_{-1}(x) \\
 &= (-1)^{n-1} \sum_{l=0}^n \binom{n}{l} G_{n-l} \int_{\mathbb{Z}_p} (1-x)^l d\mu_{-1}(x) \\
 &= (-1)^{n-1} \sum_{l=0}^n \binom{n}{l} G_{n-l} (-1)^l \frac{G_{l+1}(-1)}{l+1}
 \end{aligned}$$

$$\begin{aligned}
 &= (-1)^{n-1} \sum_{l=0}^n \binom{n}{l} G_{n-l} \left( 2 + \frac{G_{l+1}}{l+1} - 2\delta_{1,l+1} \right) \\
 &= 2(-1)^{n-1} (2\delta_{1,n} - G_n) + 2(-1)^n G_n \\
 &\quad + (-1)^{n-1} \sum_{l=0}^n \binom{n}{l} G_{n-l} \frac{G_{l+1}}{l+1}.
 \end{aligned} \tag{2.5}$$

Equating (2.4) and (2.5), we obtain the following theorem.

**Theorem 2.2.** For  $n \in \mathbb{Z}_+$ , one has

$$(1 + (-1)^n) \sum_{l=0}^n \binom{n}{l} G_{n-l} \frac{G_{l+1}}{l+1} = 4(-1)^n G_n + 4(-1)^{n+1} \delta_{1,n}. \tag{2.6}$$

Let us consider the fermionic  $p$ -adic integral on  $\mathbb{Z}_p$  for the product of  $B_n(x)$  and  $G_n(x)$  as follows:

$$\begin{aligned}
 I_3 &= \int_{\mathbb{Z}_p} B_m(x) G_n(x) d\mu_{-1}(x) \\
 &= \sum_{k=0}^m \sum_{l=0}^n \binom{m}{k} \binom{n}{l} B_{m-k} G_{n-l} \int_{\mathbb{Z}_p} x^{k+l} d\mu_{-1}(x) \\
 &= \sum_{k=0}^m \sum_{l=0}^n \binom{m}{k} \binom{n}{l} B_{m-k} G_{n-l} \frac{G_{k+l+1}}{k+l+1}.
 \end{aligned} \tag{2.7}$$

On the other hand, by (1.10) and (1.14), we get

$$\begin{aligned}
 I_3 &= \int_{\mathbb{Z}_p} B_m(x) G_n(x) d\mu_{-1}(x) \\
 &= (-1)^{n+m-1} \int_{\mathbb{Z}_p} B_m(1-x) G_n(1-x) d\mu_{-1}(x) \\
 &= (-1)^{n+m-1} \sum_{k=0}^m \sum_{l=0}^n \binom{m}{k} \binom{n}{l} B_{m-k} G_{n-l} \int_{\mathbb{Z}_p} (1-x)^{k+l} d\mu_{-1}(x) \\
 &= 2(-1)^{n+m-1} B_m(1) G_n(1) + 2(-1)^{m+n} B_m G_n \\
 &\quad + (-1)^{n+m-1} \sum_{k=0}^m \sum_{l=0}^n \binom{m}{k} \binom{n}{l} B_{m-k} G_{n-l} \frac{G_{k+l+1}}{k+l+1}.
 \end{aligned} \tag{2.8}$$

By (2.7) and (2.8), we easily see that

$$\begin{aligned}
 & \left(1 + (-1)^{n+m+1}\right) \sum_{k=0}^m \sum_{l=0}^n \binom{m}{k} \binom{n}{l} B_{m-k} G_{n-l} \frac{G_{k+l+1}}{k+l+1} \\
 &= 2(-1)^{m+n-1} (\delta_{1,m} + B_m) (2\delta_{1,n} - G_n) + 2(-1)^{m+n} B_m G_n \\
 &= 4(-1)^{m+n-1} B_m \delta_{1,n} + 2(-1)^{m+n} B_m G_n + 4(-1)^{m+n-1} \delta_{1,m} \delta_{1,n} \\
 &\quad + 2(-1)^{m+n} \delta_{1,m} G_n + 2(-1)^{m+n} B_m G_n.
 \end{aligned} \tag{2.9}$$

Therefore, by (2.9), we obtain the following theorem.

**Theorem 2.3.** For  $n, m \in \mathbb{Z}_+$ , one has

$$\begin{aligned}
 & \left(1 + (-1)^{n+m+1}\right) \sum_{k=0}^m \sum_{l=0}^n \binom{m}{k} \binom{n}{l} B_{m-k} \frac{G_{n-l+1}}{n-l+1} \frac{G_{k+l+1}}{k+l+1} \\
 &= 4(-1)^{m+n} B_m G_n + 4(-1)^{m+n-1} B_m \delta_{1,n} + 4(-1)^{m+n-1} \delta_{1,m} \delta_{1,n} \\
 &\quad + 2(-1)^{m+n} \delta_{1,m} G_n.
 \end{aligned} \tag{2.10}$$

**Corollary 2.4.** For  $n, m \in \mathbb{N}$ , one has

$$\sum_{k=0}^{2m} \sum_{l=0}^{2n} \binom{2m}{k} \binom{2n}{l} B_{2m-k} G_{2n-l} \frac{G_{k+l+1}}{k+l+1} = 2B_{2m} G_{2n}. \tag{2.11}$$

Let us consider the fermionic  $p$ -adic integral on  $\mathbb{Z}_p$  for the product of the Bernoulli polynomials and the Bernstein polynomials. For  $n, k \in \mathbb{Z}_+$ , with  $0 \leq k \leq n$ ,  $B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}$  are called the Bernstein polynomials of degree  $n$ , see [11]. It is easy to show that  $B_{k,n}(x) = B_{n-k,n}(1-x)$ ,

$$\begin{aligned}
 I_4 &= \int_{\mathbb{Z}_p} B_m(x) B_{k,n}(x) d\mu_{-1}(x) \\
 &= \binom{n}{k} \sum_{l=0}^m \binom{m}{l} B_{m-l} \int_{\mathbb{Z}_p} x^{k+l} (1-x)^{n-k} d\mu_{-1}(x) \\
 &= \binom{n}{k} \sum_{l=0}^m \sum_{j=0}^{n-k} \binom{m}{l} \binom{n-k}{j} (-1)^j B_{m-l} \int_{\mathbb{Z}_p} x^{k+l+j} d\mu_{-1}(x) \\
 &= \binom{n}{k} \sum_{l=0}^m \sum_{j=0}^{n-k} \binom{m}{l} \binom{n-k}{j} (-1)^j B_{m-l} \frac{G_{k+l+j+1}}{k+l+j+1}.
 \end{aligned} \tag{2.12}$$

On the other hand, by (1.14) and (2.12), we get

$$\begin{aligned}
 I_4 &= (-1)^m \int_{\mathbb{Z}_p} B_m(1-x)B_{n-k,n}(1-x)d\mu_{-1}(x) \\
 &= (-1)^m \binom{n}{k} \sum_{l=0}^m \sum_{j=0}^k \binom{m}{l} \binom{k}{j} (-1)^j B_{m-l} \int_{\mathbb{Z}_p} (1-x)^{n-k+l+j} d\mu_{-1}(x) \\
 &= (-1)^m \binom{n}{k} \sum_{l=0}^m \sum_{j=0}^k \binom{m}{l} \binom{k}{j} (-1)^j B_{m-l} \\
 &\quad \times \left( 2 - 2\delta_{1,n-k+l+j+1} + \frac{G_{n-k+l+j+1}}{n-k+l+j+1} \right) \\
 &= 2(-1)^m \binom{n}{k} B_m(1)\delta_{0,k} + 2(-1)^{m+1} \binom{n}{k} B_m \delta_{k,n} \\
 &\quad + (-1)^m \binom{n}{k} \sum_{l=0}^m \sum_{j=0}^k \binom{m}{l} \binom{k}{j} (-1)^j B_{m-l} \frac{G_{n-k+l+j+1}}{n-k+l+j+1}.
 \end{aligned} \tag{2.13}$$

Equating (2.12) and (2.13), we see that

$$\begin{aligned}
 &\sum_{l=0}^m \sum_{j=0}^{n-k} \binom{m}{l} \binom{n-k}{j} (-1)^j B_{m-l} \frac{G_{k+l+j+1}}{k+l+j+1} \\
 &= 2(-1)^m B_m(1)\delta_{0,k} + 2(-1)^{m+1} B_m \delta_{k,n} \\
 &\quad + (-1)^m \sum_{l=0}^m \sum_{j=0}^k \binom{m}{l} \binom{k}{j} (-1)^j B_{m-l} \frac{G_{n-k+l+j+1}}{n-k+l+j+1}.
 \end{aligned} \tag{2.14}$$

Thus, from (2.14), we obtain the following theorem.

**Theorem 2.5.** For  $n, m \in \mathbb{N}$ , one has

$$\sum_{l=0}^{2m} \sum_{j=0}^n \binom{2m}{l} \binom{n}{j} (-1)^j B_{2m-l} \frac{G_{l+j+1}}{l+j+1} = 2B_{2m}(1) + \sum_{l=0}^{2m} \binom{2m}{l} B_{2m-l} \frac{G_{n+l+1}}{n+l+1}. \tag{2.15}$$

Finally, we consider the fermionic  $p$ -adic integral on  $\mathbb{Z}_p$  for the product of the Euler polynomials and the Bernstein polynomials as follows:

$$\begin{aligned}
 I_5 &= \int_{\mathbb{Z}_p} G_m(x)B_{k,n}(x)d\mu_{-1}(x) \\
 &= \binom{n}{k} \sum_{l=0}^m \binom{m}{l} G_{m-l} \int_{\mathbb{Z}_p} x^{k+l}(1-x)^{n-k} d\mu_{-1}(x)
 \end{aligned}$$

$$\begin{aligned}
&= \binom{n}{k} \sum_{l=0}^m \sum_{j=0}^{n-k} \binom{m}{l} \binom{n-k}{j} (-1)^j G_{m-l} \int_{\mathbb{Z}_p} x^{k+l+j} d\mu_{-1}(x) \\
&= \binom{n}{k} \sum_{l=0}^m \sum_{j=0}^{n-k} \binom{m}{l} \binom{n-k}{j} (-1)^j G_{m-l} \frac{G_{k+l+j+1}}{k+l+j+1}.
\end{aligned} \tag{2.16}$$

On the other hand, by (1.10) and (2.12), we get

$$\begin{aligned}
I_5 &= (-1)^{m-1} \int_{\mathbb{Z}_p} G_m(1-x) B_{n-k,n}(1-x) d\mu_{-1}(x) \\
&= (-1)^{m-1} \binom{n}{k} \sum_{l=0}^m \binom{m}{l} G_{m-l} \sum_{j=0}^k \binom{k}{j} (-1)^j \int_{\mathbb{Z}_p} (1-x)^{n-k+l+j} d\mu_{-1}(x) \\
&= (-1)^{m-1} \binom{n}{k} \sum_{l=0}^m \sum_{j=0}^k \binom{m}{l} \binom{k}{j} (-1)^j G_{m-l} \\
&\quad \times \left( 2 + \frac{G_{n-k+l+j+1}}{n-k+l+j+1} - 2\delta_{1,n-k+l+j+1} \right) \\
&= 2(-1)^{m-1} \binom{n}{k} G_m(1)\delta_{0,k} + 2(-1)^m \binom{n}{k} G_m \delta_{k,n} \\
&\quad + (-1)^{m-1} \binom{n}{k} \sum_{l=0}^m \sum_{j=0}^k \binom{m}{l} \binom{k}{j} (-1)^j G_{m-l} \frac{G_{n-k+l+j+1}}{n-k+l+j+1}.
\end{aligned} \tag{2.17}$$

Equating (2.16) and (2.17), we obtain

$$\begin{aligned}
&\sum_{l=0}^m \sum_{j=0}^{n-k} \binom{m}{l} \binom{n-k}{j} (-1)^j G_{m-l} \frac{G_{k+l+j+1}}{k+l+j+1} \\
&= 2(-1)^{m-1} G_m(1)\delta_{0,k} + 2(-1)^m G_m \delta_{k,n} \\
&\quad + (-1)^{m-1} \sum_{l=0}^m \sum_{j=0}^k \binom{m}{l} \binom{k}{j} (-1)^j G_{m-l} \frac{G_{n-k+l+j+1}}{n-k+l+j+1}.
\end{aligned} \tag{2.18}$$

Therefore, by (2.18), we obtain the following theorem.

**Theorem 2.6.** For  $n, m \in \mathbb{N}$ , one has

$$\sum_{l=0}^{2m} \sum_{j=0}^n \binom{2m}{l} \binom{n}{j} (-1)^j G_{2m-l} \frac{G_{l+j+1}}{l+j+1} = -2G_{2m}(1) - \sum_{l=0}^{2m} \binom{2m}{l} G_{2m-l} \frac{G_{n+l+1}}{n+l+1}. \tag{2.19}$$



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## References

- [1] T. Kim, " $q$ -Volkenborn integration," *Russian Journal of Mathematical Physics*, vol. 9, no. 3, pp. 288–299, 2002.
- [2] L. Carlitz, "Note on the integral of the product of several Bernoulli polynomials," *Journal of the London Mathematical Society*, vol. 34, pp. 361–363, 1959.
- [3] T. Kim, "A note on  $q$ -Volkenborn integration," *Proceedings of the Jangjeon Mathematical Society*, vol. 8, no. 1, pp. 13–17, 2005.
- [4] T. Kim, "On the multiple  $q$ -Genocchi and Euler numbers," *Russian Journal of Mathematical Physics*, vol. 15, no. 4, pp. 481–486, 2008.
- [5] T. Kim, "On the  $q$ -extension of Euler and Genocchi numbers," *Journal of Mathematical Analysis and Applications*, vol. 326, no. 2, pp. 1458–1465, 2007.
- [6] T. Kim, "Symmetry  $p$ -adic invariant integral on  $\mathbb{Z}_p$  for Bernoulli and Euler polynomials," *Journal of Difference Equations and Applications*, vol. 14, no. 12, pp. 1267–1277, 2008.
- [7] T. Kim, "Symmetry of power sum polynomials and multivariate fermionic  $p$ -adic invariant integral on  $\mathbb{Z}_p$ ," *Russian Journal of Mathematical Physics*, vol. 16, no. 1, pp. 93–96, 2009.
- [8] A. Bayad and T. Kim, "Identities for the Bernoulli, the Euler and the Genocchi numbers and polynomials," *Advanced Studies in Contemporary Mathematics*, vol. 20, no. 2, pp. 247–253, 2010.
- [9] H. M. Srivastava, T. Kim, and Y. Simsek, " $q$ -Bernoulli numbers and polynomials associated with multiple  $q$ -zeta functions and basic  $L$ -series," *Russian Journal of Mathematical Physics*, vol. 12, no. 2, pp. 241–268, 2005.
- [10] T. Kim, "Barnes-type multiple  $q$ -zeta functions and  $q$ -Euler polynomials," *Journal of Physics A*, vol. 43, no. 25, Article ID 255201, 11 pages, 2010.
- [11] T. Kim, "A note on  $q$ -Bernstein polynomials," *Russian Journal of Mathematical Physics*, vol. 18, no. 1, pp. 73–82, 2011.
- [12] L. C. Jang, W. J. Kim, and Y. Simsek, "A study on the  $p$ -adic integral representation on  $\mathbb{Z}_p$  associated with Bernstein and Bernoulli polynomials," *Advances in Difference Equations*, vol. 2010, Article ID 163217, 6 pages, 2010.
- [13] Y. Simsek, "Special functions related to Dedekind-type DC-sums and their applications," *Russian Journal of Mathematical Physics*, vol. 17, no. 4, pp. 495–508, 2010.
- [14] L.-C. Jang, T. Kim, D.-H. Lee, and D.-W. Park, "An application of polylogarithms in the analogs of Genocchi numbers," *Notes on Number Theory and Discrete Mathematics*, vol. 7, no. 3, pp. 65–70, 2001.
- [15] S.-H. Rim, K.-H. Park, and E.-J. Moon, "On Genocchi numbers and polynomials," *Abstract and Applied Analysis*, vol. 2008, Article ID 898471, 7 pages, 2008.
- [16] T. Kim, J. Choi, Y. H. Kim, and C.-S. Ryoo, "On the fermionic  $p$ -adic integral representation of Bernstein polynomials associated with Euler numbers and polynomials," *Journal of Inequalities and Applications*, vol. 2010, Article ID 864247, 12 pages, 2010.