# A Short Proof of Carlitz's Bernoulli Number Identity 

Helmut Prodinger<br>Department of Mathematical Sciences<br>Stellenbosch University<br>7602 Stellenbosch<br>South Africa<br>hproding@sun.ac.za


#### Abstract

For an identity related to Bernoulli numbers, stated by Carlitz, rediscovered and reproved by various researchers, an extremely short and direct proof is provided which uses a bivariate exponential function.


## 1 Introduction and result

The very recent paper [2] deals with the remarkable identity

$$
(-1)^{m} \sum_{k=0}^{m}\binom{m}{k} B_{n+k}=(-1)^{n} \sum_{k=0}^{n}\binom{n}{k} B_{m+k}
$$

involving the Bernoulli numbers $\left(B_{n}\right)$. We learn that it originated as a problem of Carlitz [1], with a solution by Shannon [3], which uses induction. It was rediscovered by Vassilev and Vassilev-Missana [4]. The paper by Gould and Quaintance [2] introduces and uses a binomial transform to prove it.

In this quick note, I would like to present a proof by (exponential) generating functions,
which is perhaps the most direct argument. It goes like this:

$$
\begin{aligned}
F(z, x) & :=\sum_{m \geq 0} \frac{z^{m}}{m!} \sum_{n \geq 0} \frac{x^{n}}{n!}(-1)^{m} \sum_{k=0}^{m}\binom{m}{k} B_{n+k} \\
& =\sum_{n \geq 0} \frac{x^{n}}{n!} \sum_{k \geq 0} \frac{B_{n+k}}{k!} \sum_{m \geq k}(-z)^{m} \frac{1}{(m-k)!} \\
& =\sum_{n \geq 0} \frac{x^{n}}{n!} \sum_{k \geq 0} \frac{B_{n+k}}{k!}(-z)^{k} e^{-z} \\
& =e^{-z} \sum_{N \geq 0} \frac{B_{N}}{N!} \sum_{k=0}^{N}\binom{N}{k}(-z)^{k} x^{N-k} \\
& =e^{-z} \sum_{N \geq 0} \frac{B_{N}}{N!}(x-z)^{N} \\
& =e^{-z} \frac{x-z}{e^{x-z}-1}=\frac{x-z}{e^{x}-e^{z}}=F(x, z) .
\end{aligned}
$$

This symmetry proves the identity.

## References

[1] L. Carlitz. Problem 795. Math. Mag. 44 (1971), 107.
[2] H. W. Gould and J. Quaintance. Bernoulli numbers and a new binomial transform identity. J. Integer Sequences 17 (2014), Article 14.2.2.
[3] A. G. Shannon. Solution of Problem 795. Math. Mag. 45 (1972), 55-56.
[4] P. Vassilev and M. Vassilev-Missana. On one remarkable identity involving Bernoulli numbers. Notes on Number Theory and Discrete Mathematics 11 (2005), 22-24.

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