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A Short Proof of Carlitz's Bernoulli Number Identity

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Abstract

For an identity related to Bernoulli numbers, stated by Carlitz, rediscovered and reproved by various researchers, an extremely short and direct proof is provided which uses a bivariate exponential function.

1 Introduction and result

The very recent paper [2] deals with the remarkable identity

$$(-1)^m \sum_{k=0}^m \binom{m}{k} B_{n+k} = (-1)^n \sum_{k=0}^n \binom{n}{k} B_{m+k}$$

involving the Bernoulli numbers (B_n) . We learn that it originated as a problem of Carlitz [1], with a solution by Shannon [3], which uses induction. It was rediscovered by Vassilev and Vassilev-Missana [4]. The paper by Gould and Quaintance [2] introduces and uses a binomial transform to prove it.

In this quick note, I would like to present a proof by (exponential) generating functions,

which is perhaps the most direct argument. It goes like this:

$$\begin{split} F(z,x) &:= \sum_{m \ge 0} \frac{z^m}{m!} \sum_{n \ge 0} \frac{x^n}{n!} (-1)^m \sum_{k=0}^m \binom{m}{k} B_{n+k} \\ &= \sum_{n \ge 0} \frac{x^n}{n!} \sum_{k \ge 0} \frac{B_{n+k}}{k!} \sum_{m \ge k} (-z)^m \frac{1}{(m-k)!} \\ &= \sum_{n \ge 0} \frac{x^n}{n!} \sum_{k \ge 0} \frac{B_{n+k}}{k!} (-z)^k e^{-z} \\ &= e^{-z} \sum_{N \ge 0} \frac{B_N}{N!} \sum_{k=0}^N \binom{N}{k} (-z)^k x^{N-k} \\ &= e^{-z} \sum_{N \ge 0} \frac{B_N}{N!} (x-z)^N \\ &= e^{-z} \frac{x-z}{e^{x-z}-1} = \frac{x-z}{e^x-e^z} = F(x,z). \end{split}$$

m

This symmetry proves the identity.

References

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