

A GENERALIZATION OF THE BERNOULLI POLYNOMIALS

PIERPAOLO NATALINI AND ANGELA BERNARDINI

Received 16 April 2002 and in revised form 20 July 2002

A generalization of the Bernoulli polynomials and, consequently, of the Bernoulli numbers, is defined starting from suitable generating functions. Furthermore, the differential equations of these new classes of polynomials are derived by means of the factorization method introduced by Infeld and Hull (1951).

1. Introduction

The Bernoulli polynomials have important applications in number theory and classical analysis. They appear in the integral representation of differentiable periodic functions since they are employed for approximating such functions in terms of polynomials. They are also used for representing the remainder term of the composite Euler-MacLaurin quadrature rule (see [15]).

The Bernoulli numbers [3, 13] appear in number theory, and in many mathematical expressions, such as

- (i) the Taylor expansion in a neighborhood of the origin of the circular and hyperbolic tangent and cotangent functions;
- (ii) the sums of powers of natural numbers;
- (iii) the residual term of the Euler-MacLaurin quadrature rule.

The Bernoulli polynomials $B_n(x)$ are usually defined (see, e.g., [7, page xxix]) by means of the generating function

$$G(x, t) := \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad |t| < 2\pi, \quad (1.1)$$

and the Bernoulli numbers $B_n := B_n(0)$ by the corresponding equation

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}. \quad (1.2)$$

The B_n are rational numbers. We have, in particular, $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, and $B_{2k+1} = 0$, for $k = 1, 2, \dots$,

$$B_0(x) = 1, \quad B_1(x) = x - \frac{1}{2}, \quad B_2(x) = x^2 - x + \frac{1}{6}. \quad (1.3)$$

The following properties are well known:

$$B_n(0) = B_n(1) = B_n, \quad n \neq 1, \\ B_n(x) = \sum_{k=0}^n \binom{n}{k} B_k x^{n-k}, \quad B'_n(x) = nB_{n-1}(x). \quad (1.4)$$

The Bernoulli polynomials are easily computed by recursion since

$$\sum_{k=0}^{n-1} \binom{n}{k} B_k(x) = nx^{n-1}, \quad n = 2, 3, \dots \quad (1.5)$$

Some generalized forms of the Bernoulli polynomials and numbers already appeared in literature. We recall, for example, the generalized Bernoulli polynomials $B_n^\alpha(x)$ recalled in the book of Gatteschi [6] defined by the generating function

$$\frac{t^\alpha e^{xt}}{(e^t - 1)^\alpha} = \sum_{n=0}^{\infty} B_n^\alpha(x) \frac{t^n}{n!}, \quad |t| < 2\pi, \quad (1.6)$$

by means of which, Tricomi and Erdélyi [16] gave an asymptotic expansion of the ratio of two gamma functions.

Another generalized forms can be found in [5, 11], starting from the generating functions

$$\frac{(iz)^\alpha e^{(x-1/2)z}}{2^{2\alpha} \Gamma(\alpha+1) J_\alpha(iz/2)} = \sum_{n=0}^{\infty} B_{n,\alpha}(x) \frac{z^n}{n!}, \quad |z| < 2|j_1|, \quad (1.7)$$

where J_α is the Bessel function of the first kind of order α and $j_1 = j_1(\alpha)$ is the first zero of J_α , or

$$\frac{(ht)^\alpha (1 + \omega t)^{x/\omega}}{[(1 + \omega t)^{h/\omega} - 1]^\alpha} = \sum_{n=0}^\infty B_{n;h,\omega}^\alpha(x) \frac{t^n}{n!}, \quad |t| < \left| \frac{1}{\omega} \right|, \tag{1.8}$$

respectively.

In this paper, we introduce a countable set of polynomials $B_n^{[m-1]}(x)$ generalizing the Bernoulli ones, which can be recovered assuming $m = 1$. To this aim, we consider a class of Appell polynomials [2], defined by using a generating function linked to the so-called Mittag-Leffler function

$$E_{1,m+1}(t) := \frac{t^m}{e^t - \sum_{h=0}^{m-1} t^h/h!}, \tag{1.9}$$

considered in the general form by Agarwal [1] (see also [12]).

Furthermore, exploiting the factorization method introduced in [10] and recalled in [8], we derive the differential equation satisfied by these polynomials. It is worth noting that the differential equation for Appell-type polynomials was derived in [14], and more recently recovered in [9] by exploiting the factorization method. It is easily checked that our differential equation matches with the general form of the above mentioned articles [9, 14]. In particular, when $m = 1$, the differential equation of the classical Bernoulli polynomials is derived again.

We will show in this paper that the differential equation satisfied by the $B_n^{[m-1]}(x)$ polynomials is of order n , so that all the considered families of polynomials can be viewed as solutions of differential operators of infinite order.

This is a quite general situation since the Appell-type polynomials, satisfying a differential operator of finite order, can be considered as an exceptional case (see [4]).

2. A new class of generalized Bernoulli polynomials

The generalized Bernoulli polynomials $B_n^{[m-1]}(x)$, $m \geq 1$, are defined by means of the generating function, defined in a suitable neighborhood of $t = 0$

$$G^{[m-1]}(x,t) := \frac{t^m e^{xt}}{e^t - \sum_{h=0}^{m-1} t^h/h!} = \sum_{n=0}^\infty B_n^{[m-1]}(x) \frac{t^n}{n!}. \tag{2.1}$$

For $m = 1$, we obtain, from (2.1), the generating function $G^{(0)}(x,t) = te^{xt}/(e^t - 1)$ of classical Bernoulli polynomials $B_n^{(0)}(x)$.

Since $G^{[m-1]}(x, t) = A(t)e^{xt}$, the generalized Bernoulli polynomials belong to the class of Appell polynomials.

It is possible to define the generalized Bernoulli numbers assuming

$$B_n^{[m-1]} = B_n^{[m-1]}(0). \tag{2.2}$$

From (2.1), we have

$$e^{xt} = \sum_{h=m}^{\infty} \frac{t^{h-m}}{h!} \sum_{n=0}^{\infty} B_n^{[m-1]}(x) \frac{t^n}{n!}. \tag{2.3}$$

Since $e^{xt} = \sum_{n=0}^{\infty} x^n (t^n/n!)$, (2.3) becomes

$$\sum_{n=0}^{\infty} x^n \frac{t^n}{n!} = \sum_{j=0}^{\infty} \frac{j!}{(j+m)!} \frac{t^j}{j!} \sum_{n=0}^{\infty} B_n^{[m-1]}(x) \frac{t^n}{n!} \tag{2.4}$$

and therefore

$$\sum_{n=0}^{\infty} x^n \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{h=0}^n \binom{n}{h} \frac{h!}{(h+m)!} B_{n-h}^{[m-1]}(x) \frac{t^n}{n!}. \tag{2.5}$$

By comparing the coefficients of (2.5), we obtain

$$x^n = \sum_{h=0}^n \binom{n}{h} \frac{h!}{(h+m)!} B_{n-h}^{[m-1]}(x). \tag{2.6}$$

Inverting (2.6), it is possible to find explicit expressions for the polynomials $B_n^{[m-1]}(x)$. The first ones are given by

$$\begin{aligned} B_0^{[m-1]}(x) &= m!, & B_1^{[m-1]}(x) &= m! \left(x - \frac{1}{m+1} \right), \\ B_2^{[m-1]}(x) &= m! \left(x^2 - \frac{2}{m+1}x + \frac{2}{(m+1)^2(m+2)} \right), \end{aligned} \tag{2.7}$$

and, consequently, the first generalized Bernoulli numbers are

$$B_0^{[m-1]} = m!, \quad B_1^{[m-1]} = -\frac{m!}{m+1}, \quad B_2^{[m-1]} = \frac{2m!}{(m+1)^2(m+2)}. \tag{2.8}$$

3. Differential equation for generalized Bernoulli polynomials

In this section, we prove the following theorem.

THEOREM 3.1. *The generalized Bernoulli polynomials $B_n^{[m-1]}(x)$ satisfy the differential equation*

$$\begin{aligned} \frac{B_n^{[m-1]}}{n!} y^{(n)} + \frac{B_{n-1}^{[m-1]}}{(n-1)!} y^{(n-1)} + \dots + \frac{B_2^{[m-1]}}{2!} y'' \\ + (m-1)! \left(\frac{1}{m+1} - x \right) y' + n(m-1)! y = 0. \end{aligned} \tag{3.1}$$

In order to prove (3.1), we first derive a recurrence relation for $B_n^{[m-1]}(x)$.

LEMMA 3.2. *For any integral $n \geq 1$, the following linear homogeneous recurrence relation for the generalized Bernoulli polynomials holds true:*

$$B_n^{[m-1]}(x) = \left(x - \frac{1}{m+1} \right) B_{n-1}^{[m-1]}(x) - \frac{1}{n(m-1)!} \sum_{k=0}^{n-2} \binom{n}{k} B_{n-k}^{[m-1]} B_k^{[m-1]}(x). \tag{3.2}$$

This relation, starting from $n = 1$, and taking into account the initial value $B_0^{[m-1]}(x) = m!$, allows a recursive formula for the generalized Bernoulli polynomials.

Proof. Differentiation of both sides of (2.1), with respect to t , yields

$$\begin{aligned} \frac{\partial}{\partial t} G^{[m-1]}(x, t) &= \frac{m t^{m-1} \left(e^t - \sum_{h=0}^{m-1} t^h / h! \right) - t^m \left(e^t - \sum_{h=1}^{m-1} t^{h-1} / (h-1)! \right)}{\left(e^t - \sum_{h=0}^{m-1} t^h / h! \right)^2} e^{xt} \\ &\quad + \frac{x t^m}{e^t - \sum_{h=0}^{m-1} t^h / h!} e^{xt} \\ &= \left[\frac{m}{t} \frac{t^m}{e^t - \sum_{h=0}^{m-1} t^h / h!} - \frac{t^m}{e^t - \sum_{h=0}^{m-1} t^h / h!} \right. \\ &\quad \left. - \frac{1}{(m-1)!} \frac{t^{2m-1}}{\left(e^t - \sum_{h=0}^{m-1} t^h / h! \right)^2} \right] e^{xt} \\ &\quad + x G^{[m-1]}(x, t) \end{aligned}$$

$$\begin{aligned}
 &= \frac{m}{t} G^{[m-1]}(x, t) + (x-1) G^{[m-1]}(x, t) \\
 &\quad - \frac{t^{m-1}}{(m-1)! \left(e^t - \sum_{h=0}^{m-1} t^h / h! \right)} \\
 &\quad \times \frac{t^m}{e^t - \sum_{h=0}^{m-1} t^h / h!} e^{xt} \\
 &= \frac{1}{(m-1)! t} \left(m! - \frac{t^m}{e^t - \sum_{h=0}^{m-1} t^h / h!} \right) \\
 &\quad \times G^{[m-1]}(x, t) + (x-1) G^{[m-1]}(x, t) \\
 &= \frac{1}{(m-1)! t} \left(m! - \sum_{n=0}^{\infty} B_n^{[m-1]} \frac{t^n}{n!} \right) \\
 &\quad \times G^{[m-1]}(x, t) + (x-1) G^{[m-1]}(x, t),
 \end{aligned} \tag{3.3}$$

and consequently

$$\begin{aligned}
 (m-1)! t \frac{\partial}{\partial t} G^{[m-1]}(x, t) &= m! G^{[m-1]}(x, t) - \sum_{n=0}^{\infty} B_n^{[m-1]} \frac{t^n}{n!} G^{[m-1]}(x, t) \\
 &\quad + (m-1)! t (x-1) G^{[m-1]}(x, t).
 \end{aligned} \tag{3.4}$$

Recalling (2.1), the left-hand side of (3.4) becomes

$$\begin{aligned}
 (m-1)! t \frac{\partial}{\partial t} G^{[m-1]}(x, t) &= (m-1)! \sum_{n=1}^{\infty} B_n^{[m-1]}(x) \frac{t^n}{(n-1)!} \\
 &= (m-1)! \sum_{n=0}^{\infty} n B_n^{[m-1]}(x) \frac{t^n}{n!}.
 \end{aligned} \tag{3.5}$$

Furthermore, introducing $B_{-1}^{[m-1]}(x) := 0$ (but in principle $B_{-1}^{[m-1]}(x)$ could be chosen as an arbitrary constant), the following equation is obtained:

$$\begin{aligned}
 (m-1)! t (x-1) G^{[m-1]}(x, t) &= (m-1)! \sum_{n=0}^{\infty} (x-1) B_n^{[m-1]}(x) \frac{t^{n+1}}{n!} \\
 &= (m-1)! \sum_{n=0}^{\infty} n (x-1) B_{n-1}^{[m-1]}(x) \frac{t^n}{n!},
 \end{aligned} \tag{3.6}$$

and moreover

$$\begin{aligned} \sum_{n=0}^{\infty} B_n^{[m-1]} \frac{t^n}{n!} G^{[m-1]}(x, t) &= \sum_{n=0}^{\infty} B_n^{[m-1]} \frac{t^n}{n!} \sum_{h=0}^{\infty} \frac{t^h}{h!} B_h^{[m-1]}(x) \\ &= \sum_{n=0}^{\infty} \left[\sum_{k=0}^n \binom{n}{k} B_{n-k}^{[m-1]} B_k^{[m-1]}(x) \right] \frac{t^n}{n!}. \end{aligned} \tag{3.7}$$

Substitution of (3.5), (3.6), and (3.7) into (3.4) yields

$$\begin{aligned} (m-1)! \sum_{n=0}^{\infty} n B_n^{[m-1]}(x) \frac{t^n}{n!} &= m! \sum_{n=0}^{\infty} B_n^{[m-1]}(x) \frac{t^n}{n!} \\ &\quad - \sum_{n=0}^{\infty} \left[\sum_{k=0}^n \binom{n}{k} B_{n-k}^{[m-1]} B_k^{[m-1]}(x) \right] \frac{t^n}{n!} \\ &\quad + (m-1)! \sum_{n=0}^{\infty} n(x-1) B_{n-1}^{[m-1]}(x) \frac{t^n}{n!}. \end{aligned} \tag{3.8}$$

Then the conclusion immediately follows by the identity principle of power series, equating coefficients in the left- and right-hand side of the last equation (3.8). \square

Proof of Theorem 3.1. We now use this recurrence relation to find the operator E_n^+ such that

$$E_n^+ B_n^{[m-1]}(x) = B_{n+1}^{[m-1]}(x), \quad n = 0, 1, \dots \tag{3.9}$$

It is easy to see that, for $k = 0, 1, \dots, n-1$,

$$\frac{d^{n-k}}{dx^{n-k}} B_n^{[m-1]}(x) = \frac{n!}{k!} B_k^{[m-1]}(x). \tag{3.10}$$

By means of (3.10), the recurrence relation can be written as

$$B_{n+1}^{[m-1]}(x) = \left[\left(x - \frac{1}{m+1} \right) - \frac{1}{(m-1)!} \sum_{k=0}^{n-1} \frac{B_{n+1-k}^{[m-1]}}{(n+1-k)!} D_x^{n-k} \right] B_n^{[m-1]}(x), \tag{3.11}$$

and therefore

$$E_n^+ = \left(x - \frac{1}{m+1} \right) - \frac{1}{(m-1)!} \sum_{k=0}^{n-1} \frac{B_{n+1-k}^{[m-1]}}{(n+1-k)!} D_x^{n-k}. \tag{3.12}$$

We are now in a position to determine the differential equation for $B_n^{[m-1]}(x)$. Applying both operators $E_{n+1}^- = (1/(n+1))D_x$ and E_n^+ to $B_n^{[m-1]}(x)$, we have

$$(E_{n+1}^- E_n^+) B_n^{[m-1]}(x) = B_n^{[m-1]}(x). \quad (3.13)$$

That is,

$$\begin{aligned} \frac{1}{n+1} D_x \left[\left(x - \frac{1}{m+1} \right) - \frac{1}{(m-1)!} \sum_{k=0}^{n-1} \frac{B_{n+1-k}^{[m-1]}}{(n+1-k)!} D_x^{n-k} \right] B_n^{[m-1]}(x) \\ = B_n^{[m-1]}(x). \end{aligned} \quad (3.14)$$

This leads to the differential equation with $B_n^{[m-1]}(x)$ as a polynomial solution. \square

References

- [1] R. P. Agarwal, *A propos d'une note de M. Pierre Humbert*, C. R. Acad. Sci. Paris Ser. A-B **236** (1953), 2031–2032 (French).
- [2] P. Appell, *Sur une classe de polynômes*, Ann. Sci. École Norm. Sup. (2) **9** (1880), 119–144 (French).
- [3] J. Bernoulli, *Ars Conjectandi*, Thurnisiorum, Basel, 1713 (Italian).
- [4] G. Dattoli, P. E. Ricci, and C. Cesarano, *Differential equations for Appell type polynomials*, Fract. Calc. Appl. Anal. **5** (2002), no. 1, 69–75.
- [5] C. Frappier, *Representation formulas for entire functions of exponential type and generalized Bernoulli polynomials*, J. Austral. Math. Soc. Ser. A **64** (1998), no. 3, 307–316.
- [6] L. Gatteschi, *Funzioni Speciali*, Unione Tipografico—Editrice Torinese (UTET), Torin, 1973 (Italian).
- [7] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, Academic Press, New York, 1980.
- [8] M. X. He and P. E. Ricci, *Differential equations of some classes of special functions via the factorization method*, to appear in J. Comput. Anal. Appl.
- [9] ———, *Differential equation of Appell polynomials via the factorization method*, J. Comput. Appl. Math. **139** (2002), no. 2, 231–237.
- [10] L. Infeld and T. E. Hull, *The factorization method*, Rev. Modern Phys. **23** (1951), 21–68.
- [11] B. Nath, *A generalization of Bernoulli numbers and polynomials*, Gaṇita **19** (1968), no. 1, 9–12.
- [12] I. Podlubny, *Fractional Differential Equations*, Mathematics in Science and Engineering, vol. 198, Academic Press, California, 1999.
- [13] L. Saalschuetz, *Vorlesungen über die Bernoullischen Zahlen*, Springer, Berlin, 1893 (German).
- [14] I. M. Sheffer, *A differential equation for Appell polynomials*, Bull. Amer. Math. Soc. **41** (1935), 914–923.

- [15] J. Stoer, *Introduzione all'Analisi Numerica*, Zanichelli, Bologna, 1972 (Italian).
- [16] F. G. Tricomi and A. Erdélyi, *The asymptotic expansion of a ratio of gamma functions*, Pacific J. Math. **1** (1951), 133–142.

Pierpaolo Natalini: Dipartimento di Matematica, Università degli Studi Roma III, Largo San Leonardo Murialdo 1, 00146 Roma, Italy
E-mail address: natalini@mat.uniroma3.it

Angela Bernardini: Departamento de Física y Matemática Aplicada, Universidad de Navarra, E-30080 Pamplona, Spain
E-mail address: angela@obelix.fisica.unav.es

Differential Equations & Nonlinear Mechanics

An Open Access Journal

Editor-in-Chief

K. Vajravelu
USA

Associate Editors

N. Bellomo
Italy

J. L. Bona
USA

J. R. Cannon
USA

S.-N. Chow
USA

B. S. Dandapat
India

E. DiBenedetto
USA

R. Finn
USA

R. L. Fosdick
USA

J. Frehse
Germany

A. Friedman
USA

R. Grimshaw
UK

J. Malek
Czech Republic

J. T. Oden
USA

R. Quintanilla
Spain

K. R. Rajagopal
USA

G. Saccomandi
Italy

Y. Shibata
Japan

Ivar Stakgold
USA

Swaroop Darbha
USA

A. Tani
Japan

S. Turek
Germany

A. Wineman
USA

Website: <http://www.hindawi.com/journals/denm/>

Aims and Scope

Differential equations play a central role in describing natural phenomena as well as the complex processes that arise from science and technology. Differential Equations & Nonlinear Mechanics (DENM) will provide a forum for the modeling and analysis of nonlinear phenomena. One of the principal aims of the journal is to promote cross-fertilization between the various subdisciplines of the sciences: physics, chemistry, and biology, as well as various branches of engineering and the medical sciences.

Special efforts will be made to process the papers in a speedy and fair fashion to simultaneously ensure quality and timely publication.

DENM will publish original research papers that are devoted to modeling, analysis, and computational techniques. In addition to original full-length papers, DENM will also publish authoritative and informative review articles devoted to various aspects of ordinary and partial differential equations and their applications to sciences, engineering, and medicine.

Open Access Support

The Open Access movement is a relatively recent development in academic publishing. It proposes a new business model for academic publishing that enables immediate, worldwide, barrier-free, open access to the full text of research articles for the best interests of the scientific community. All interested readers can read, download, and/or print any Open Access articles without requiring a subscription to the journal in which these articles are published.

In this Open Access model, the publication cost should be covered by the author's institution or research funds. These Open Access charges replace subscription charges and allow the publishers to give the published material away for free to all interested online visitors.

Instructions for Authors

Original articles are invited and should be submitted through the DENM manuscript tracking system at <http://www.mstracking.com/denm/>. Only pdf files are accepted. If, for some reason, submission through the manuscript tracking system is not possible, you can contact denm.support@hindawi.com.

Hindawi Publishing Corporation

410 Park Avenue, 15th Floor, #287 pmb, New York, NY 10022, USA

HINDAWI

<http://www.hindawi.com/journals/denm/>